Geometric Reproducing Kernels for Signal Reconstruction

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Abstract:

In this paper we propose a smoothing method for non smooth signals, which control the geometry of a sampled signal. The signal is considered as a geometric object and the smoothing is done using a smoothing kernel function that controls the curvature of the obtained smooth signal in a close neighborhood of a metric curvature measure of the original signal.

1. Introduction

In [11], [12], a sampling scheme for signals that posses Riemannian geometric structure was introduced.It turns out that a variety of signals fall in this setting while gray scale images is just one such example. Rather then some Nyquist rate, the sampling scheme presented in [11], [12], is based on geometric characteristics of the sampled signals. Being precise, the following sampling theorem was proved.

Theorem 1 *Let $\Sigma^n, n \geq 2$ be a connected, not necessarily compact, smooth manifold, with finitely many compact boundary components. Then there exists a sampling scheme of Σ^n, with a proper density $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\Big(\frac{1}{k(p)}\Big)$, where $k(p) = \max\{|k_1|, ..., |k_{2n}|\}$, and where $k_1, ..., k_{2n}$ are the principal (normal) curvatures of Σ^n, at the point $p \in \Sigma^n$.*

While the assumed Riemannian structure relies on the assumption that the signal satisfies $\mathcal{C}^2$ smoothness criteria, the authors presented in [11], an extended version of Theorem 1 also for non smooth geometric signals, where the proposed strategy uses smoothing of the original signal. The following theorem was proved.

Theorem 2 *Let Σ be a connected, non-necessarily compact surface of class $\mathcal{C}^0$. Then, for any $\delta > 0$, there exists a δ-sampling of Σ, such that if $\Sigma_\delta \to \Sigma$, then $\mathcal{D}_\delta \to \mathcal{D}$, where $\mathcal{D}_\delta$ and $\mathcal{D}$ denote the densities of Σ_δ and Σ, respectively.*

In the above Theorem 2 Σ_δ is a smoothing of Σ obtained by a convolution of Σ with a partition of unity kernel. Such a kernel being very common for manifolds smoothing indeed guarantees that the resultant manifold is as smooth as we wish however, in this process we do not have any control on the curvature of the obtained manifold.
Some natural question raise in this context,

1. To what extent can we smooth the original signal, using such a reproducing kernel while assuming a predefined bounds on the curvature of the resultant manifold?
2. Can the reproducing kernel be made local, namely, can we have different kernel characteristics for different areas along the sampled signals, while being able to glue the smoothed signal along common boundaries?
3. In what way if at any, we can give affirmative answers to 1 and 2 that are adaptive to the signal? Meaning, how can we have good prior estimates for the desired curvature bounds?

This paper aims at answering the above questions. Note that answering question 1 is analogous to smoothen a signal to have a predefined frequency band-pass, using a band-pass filter as commonly done in signal processing for decades. Answering $1, 2, 3$ is equivalent to the use of filter banks with different band-pass characteristics. In all, giving affirmative answers to all above questions give rise to an adaptive non uniform sampling scheme for a variety of signals.
We will focus along the paper on signals that are do not admit a Riemannian structure but rather have a more general geometric structure of the so called Alexandrov spaces. We will term such signals as geometric-signals.

2. Preliminaries

In this section we will give some basic preliminary definitions and notations.

2.1 Alexandrov spaces

Definition 3 (Alexandrov - Toponogov) *[[9]] A complete metric space X, satisfies the **triangle comparison condition** w.r.t $\kappa \in \mathbb{R}$ if for every geodesic triangle $\Delta_{pqr} \in X$, there exists a **comparison triangle**, i.e. a triangle, $\Delta_{p'q'r'} \in \mathbb{M}^2_\kappa$, such that*

$$pq = p'q'; \;\; qr = q'r'; \;\; rp = r'p'$$

so that, for every point $s \in pr$ we have that

$$d_X(s, q) > d_{\mathbb{M}^2_\kappa}(s', q')$$

where $s' \in p'r'$ such that

$$ps = p's'; \;\; sr = s'r'$$

Where $\mathbb{M}^2_\kappa$ is a complete simply connected surface of ***constant curvature*** *κ.*

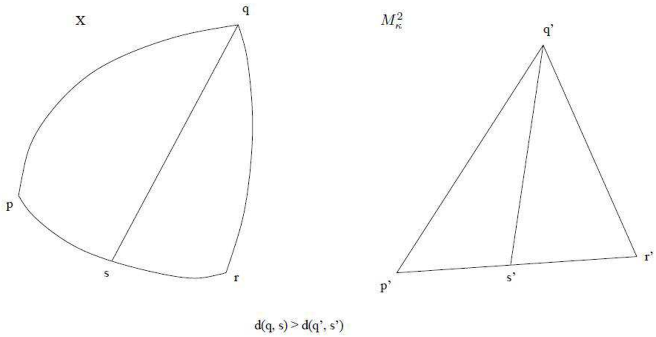

Figure 1: Comparison triangle.

Definition 4 *A complete metric space X, is an* ***Alexandrov space*** *of curvature $> \kappa$ iff*

1. *For all $x, y, \in X$ there exists a length minimizing curve γ joining x and y such that,*

   $$L(\gamma) = d_X(x, y);$$

   where L denotes the arc length of curves in X and d_X stands for the metric given on X. γ is called a ***minimal geodesic***.

2. *X satisfies the triangle comparison condition for κ.*

3. $$dim_H X < \infty;$$

   dim_H = Hausdorff dimension.

Remark 5 *In a similar way, while reversing the direction of inequalities, one can define Alexandrov space of curvature $< \kappa$. For instance, in the comparison triangle condition, we will demand,*

$$d_X(s, q) < d_{\mathbb{M}^2_\kappa}(s', q')$$

Definition 6 (Gromov) *If X is an Alexandrov space of curvature $< \kappa$ and $\kappa \leq 0$ then X is called* ***$CAT(\kappa)$-space***. *CAT = Cartan-Alexandrov-Toponogov.*

2.1.1 Examples:

1. Every complete Riemannian manifold of bounded sectional curvature.

2. The boundary of convex set in $\mathbb{R}^n$ is an Alexandrov space of curvature ≥ 0.

3. If X_i is a sequence of n-dimensional Alexandrov spaces of curv. $\geq \kappa$ then their Gromov-Hausdorff limit, if exists, is an Alexandrov space of curv. $\geq \kappa$ and dimension $\leq n$.

   If the limit of the above sequence is of dimension $< n$ we say the sequence **collapses**.

If X is an Alexandrov space then there exists a self-adjoint operator Δ, called the **Laplacian** defined on $\mathbb{L}^2(X)$ so that,

$$\int_X < \nabla u, \nabla v > d\mathcal{H}^n = \int_X v \nabla u d\mathcal{H}^n$$

where $\mathcal{H}^n$ is the n^{th} Hausdorff measure of X, $u \in \mathcal{D}(\Delta), v \in \mathcal{W}^{1,2}(X)$.

Theorem 7 ([6]) 1. *If X is compact then the spectrum of Δ is discrete.*

2. *There exists a continuous heat kernel $h_t(x, y)$ on X so that,*

   $$e^{-t\Delta}u(x) = \int_X h_t(x, y)u(y)d\mathcal{H}^n(y)$$

2.2 Approximations of manifolds

Let M be a complete Riemannian manifold of bounded sectional curvature. Let $p \in M$ be some point and let ϕ_i be some $\mathcal{C}^\infty$ kernel function supported on some ϵ_i-neighborhood of p. For example one can take ϕ to be partition of unity, heat kernel and others. Let M_i be the manifold obtained by convolution,

$$M_i = \int_M \phi_i * M d\mu;$$

Note that M_i is smooth in a δ_i neighborhood of p even if M fails to be smooth at p. Well known results (see for instance, [7]) in differential topology assert that,

$$\epsilon_i \rightarrow 0 \Rightarrow M_j \rightarrow M;$$

where convergence of manifolds is considered in the Gromov-Hausdorff topology. While the above result concerns the convergence on a topological level, in order to have curvature control we have to account for geometric convergence as well. This is guaranteed from the studies in [3], [4] and [10]. In [3], [4] it is proved that similar convergence to the above also exist for Betti numbers which are generalizations of Euler characteristic to all dimensions and are related to curvature through higher dimensional of Gauss-Bonnet type theorems [2]. In [10] the question of proper gluing of approximations in adjacent neighborhoods is addressed. It is shown that one can obtain geometric convergence in different neighborhoods V, U of the points p, q resp. so that, on the common boundary $\partial V \cap \partial U$ the approximations coincide. In addition, if we write the heat operator on a manifold, $\mathcal{N}$, as

$$e^{-t\Delta_\mathcal{N}} f(x),$$

where $f \in L^2(\mathcal{N})$ and $t > 0, x \in \mathcal{N}$, and $\Delta_\mathcal{N}$, denotes the Laplace-Beltrami operator associated with $\mathcal{N}$, then there is a smooth kernel function $K_\mathcal{N}$, such that,

$$e^{-t\Delta_\mathcal{N}} f(x) = \int_\mathcal{N} K_\mathcal{N}(t, x, y) f(y) dy;$$

In [3] convergence of the heat kernel is also achieved,

$$e^{-t\Delta_{M_i}} \rightarrow e^{-t\Delta_M}$$

3. Smoothing geometric signals with curvature control

In this section we present the results concerning questions $1, 2$ and 3 posed in the introduction. These results give us the ability to smoothen a geometric signal while having an adaptive control on obtained curvatures.

Definition 8 *We say that a signal is a* ***geometric signal*** *iff it admits a structure of an Alexandrov space for some* $\kappa \in \mathbb{R}$.

Let Σ be a geometric signal of sectional curvature bounded from below (above). Let $p \in \Sigma$ be a point, and $U(p) \subset \Sigma$ some compact neighborhood of p. Let

$$\kappa = \limsup K$$

such that $U(p)$ is an Alexandrov space of curvature $> K$.

3.1 Approximations of geometric signals

Theorem 9 ([1]) *Given a point* p *on* Σ, *there exists smooth local kernel* ϕ_i *as above, yielding a sequence of manifolds* M_i, *smooth inside an* ϵ_i *neighborhoods of* p, *such that*

1. $$M_i = \int_{\Sigma} \phi_i * \Sigma d\mu \to \Sigma,$$
   as $\epsilon \to 0$.

2. *If we further assume that while the Riemannian manifolds* M_i *converge to* Σ, ***no collapse occurs*** *i.e. the Hausdorff dimension of* Σ *is the same as of* M_i, *then, the sectional curvature* $K_i(p)$ *of* M_i *at* p *satisfies,*
   $$\lim_{\epsilon \to 0} K_i(p) = \kappa;$$

The theorem above answers both questions 1 and 2. We can control the curvature of the obtained smooth signals in an adaptive way by making it converge to the $\limsup$ of Alexandrov curvature of the signal Σ.

3.2 Gluing

By arguments similar to those in [10] we have,

Theorem 10 ([1]) *Let the above smooth approximations of* Σ *be given in neighborhoods of two points* p, q. *Then they coincide as well as their sectional curvatures* K_{i,V_i}, K_{i,U_i} *on the common boundary, if non empty.*

4. Sampling of geometric signals

We propose the following scheme for sampling of a geometric signals.

1. Consider the signal as an Alexandrov space. This requires the representation of the signal as a tame metric space in a meaningful manner.

2. Assess the appropriate Alexandrov curvature bound. This can be done by the use of discrete metric curvature measures.

3. Smooth the signal while controlling the curvature of the smoothed signal to suitably approximate the estimated curvature.

4. Sample the smoothed signal according to Theorem 1

4.1 Special case - images

It is common to regard images as surfaces embedded in some $\mathbb{R}^n$. For gray scale images $\mathbb{R}^3$ is considered while for color images it is usual to take $\mathbb{R}^5$. Figure 2 shows image re-sampled according to the geometric sampling proposed in Theorem1. In this example no smoothing was applied prior to sampling and artifacts of this can be seen in the reconstructed image. "Flat areas" of the image have 20 times reduced sampling resolution with respect to the original resolution.

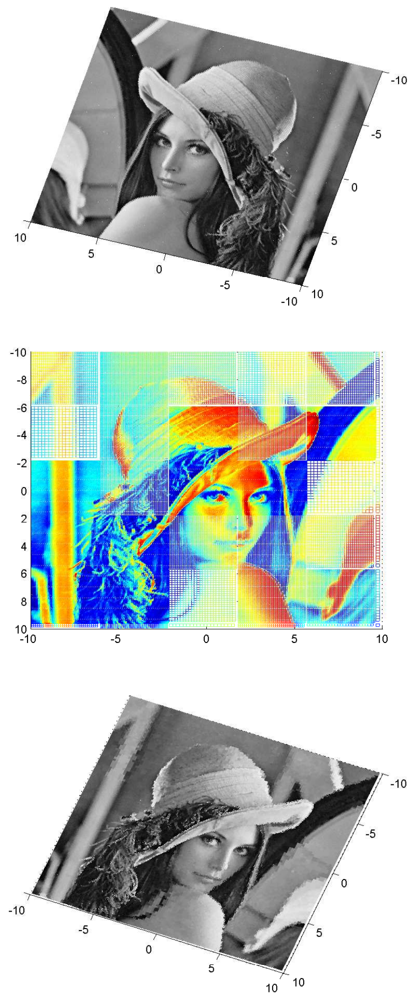

Figure 2: Geometric sampling of a gray scale image. **Top to bottom** - original Lena; Lena resampled. The white dots are the new sampling points. One can see the sparseness w.r.t the original; Lena reconstructed. Reconstruction using linear interpolation over the sampling points. No smoothing was done.

In order to estimate the curvature of an image as an Alexandrov space we can take the set of discrete curvature measures proposed in [5] where such measures are suggested for very general cell-complexes. It is shown in [5] that the one-dimensional curvature measure resembles the **Ricci** curvature of a cell-complex which, in the case of images (since they are 2-dimensional manifolds) coincides with the Gaussian curvature. Figure 3 shows the combinatorial Ricci (= Gauss) curvature of the image in Figure 2, see [13] for details about the adoption of the curvature measures introduced in [5] to images.

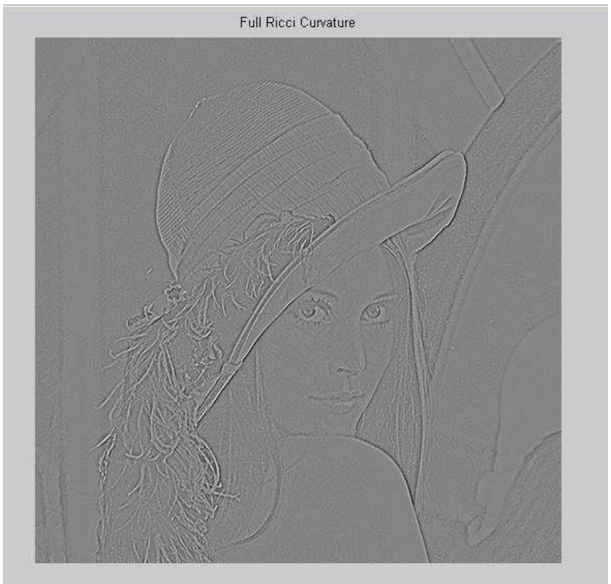

Figure 3: Discrete Ricci curvature of Lena. Apart from giving an assessment for the curvature of the image as an Alexandrov space, it also serves as an excellent edge detector as itself.

5. Further study

Current and future studies of geometric sampling of images and signals, focus on two aspects. First we wish to modify the smoothing process introduced herein so it will be done in the Fourier domain rather than the spatial domain. Namely, we wish to smooth the Fourier transform of the signal while considering curvature in the Fourier plane. This is inspired by the Nash embedding Theorem [8] while the Fourier transform of a manifold is smoothen prior to its embedding thus achieving a higher degree of smoothness with respect to smoothing in the spatial domain.
Another direction of study is devoted to the development of a geometric theory of sparse representations and geometric compress sensing.

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