Curvature Estimation over Smooth Polygonal Meshes Using The Half Tube Formula

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Mathematics of Surfaces XII

Sheffield – September 4, 2007.

The importance of curvature analysis over polygonal meshes can hardly be overestimated for:

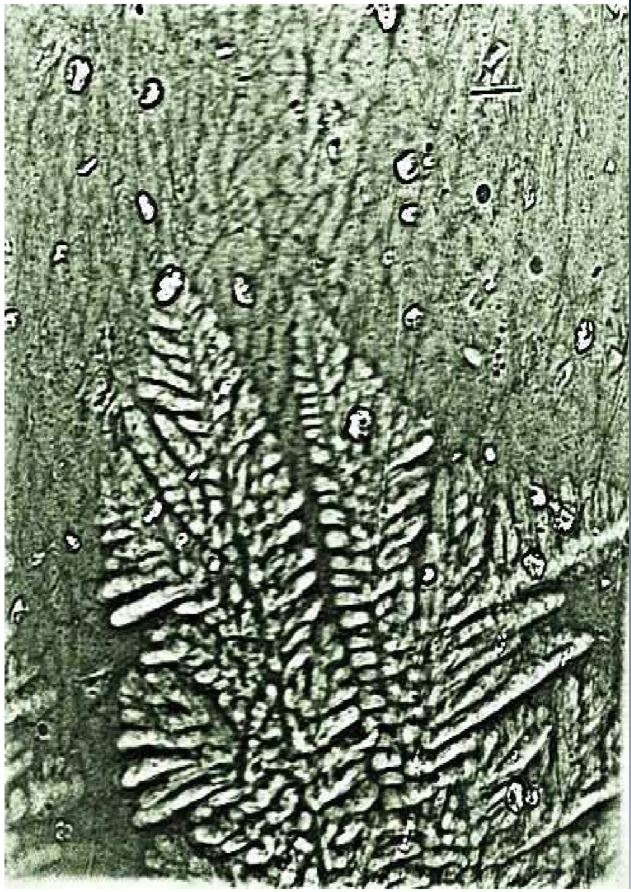
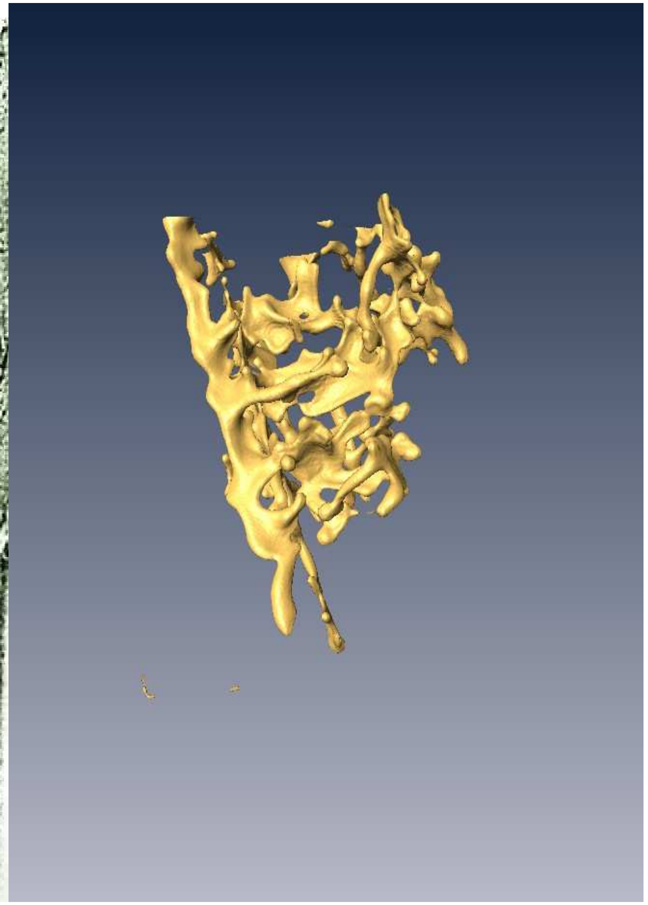
- Reconstruction
- Segmentation
- Recognition
- Non Photorealistic Rendering

with applications in:

- Vision and Image Processing
- Computer Graphics
- Geometric Modelling and Manufacturing
- Biomedical Engineering

and

- Micro-materials (metals solidification)*

*in a Cahn-Hilliard type equation.

Various excellent algorithms for determining Gaussian Curvature:

$$\boxed{K = k_{min} \cdot k_{max}}$$

exist, where:

k_{min}, k_{max} represent the principal curvatures of the surface, e.g. Parabolic Fitting and The Angle Deficiency Methods.

This reflects the fact that, by Gauss's Theorema Egregium K is invariant under bendings, that is Gaussian Curvature is intrinsic to the surface, i.e. it does not depend upon the way in which the surface is embedded in $\mathbf{R}^3$.

However, the Mean Curvature

$$H = \frac{1}{2}(k_{min} + k_{max})$$

is **not** intrinsic (invariant under bending) so it is much less useful an invariant compared to Gaussian Curvature.

Inevitably, one has to ask himself the following immediate and unavoidable question:

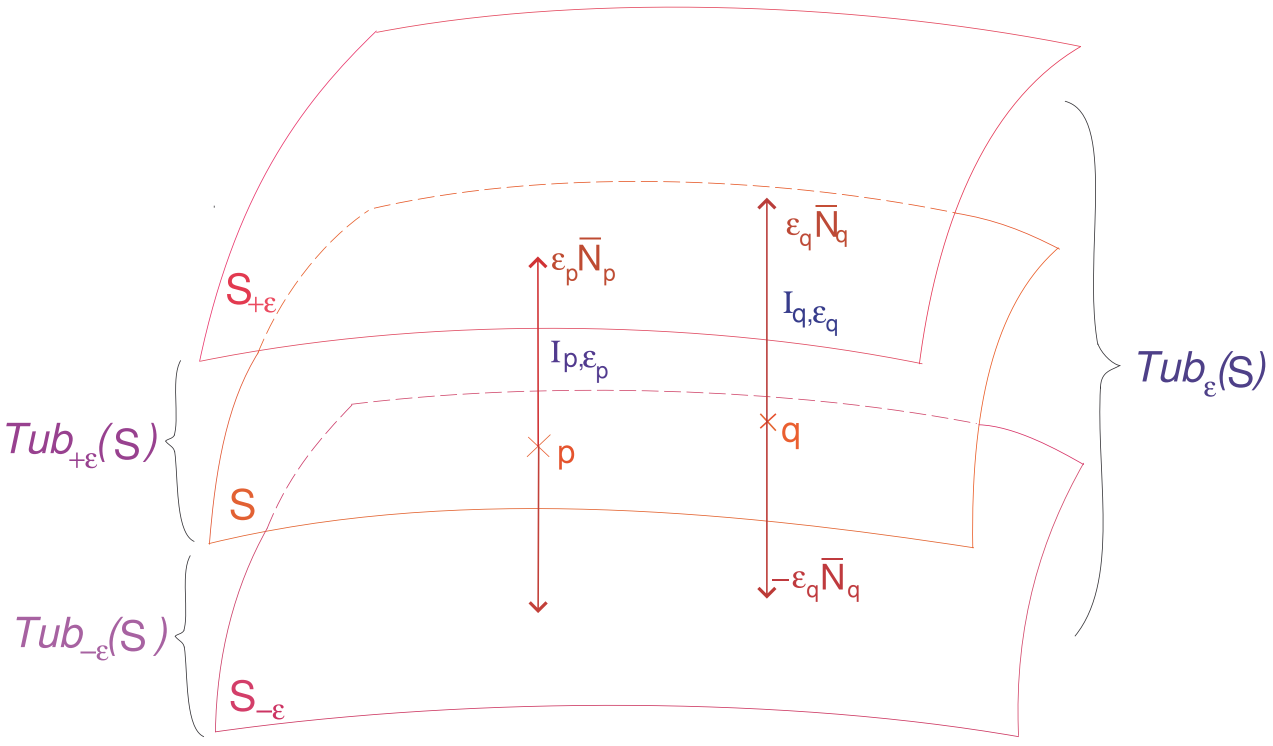
Question 1 *What is the connection between the two curvatures, if any (beyond the arithmetic one)?*

and, on a more practical note:

Question 2 *What is the best way (i.e. geometric way) of computing H?*

Fortunately, a **simultaneous** answer to both our questions exists , via The Tube Formula.

But first, we have to define Tubes:



$$S_{\pm\varepsilon} \stackrel{not}{=} S \pm \varepsilon_p \overline{N}_p \qquad \text{(Offset Surfaces)}$$

Remark 3 *The existence of tubes is assured locally, for any regular, orientable surface, and globally for regular, compact, orientable surfaces.*

Remark 4 *Note the following facts regarding the regularity of $S_{\pm\varepsilon}$:*

- *If S is convex, then $S_{\pm\varepsilon}$ are piecewise $\mathcal{C}^{1,1}$ surfaces (i.e., they admit parameterizations with continuous and bounded derivatives), for all $\varepsilon > 0$.*

- *If S is a smooth enough surface with a boundary (that is, at least piecewise $\mathcal{C}^2$), then $S_{\pm\varepsilon}$ are piecewise $\mathcal{C}^2$ surfaces, for all small enough ε.*

- *For* **any** *compact set $S \subset \mathbb{R}^3$, $S_{\pm\varepsilon}$ are Lipschitz surfaces for a.e. ε.*

We can derive the following formula:

$$(*) \quad \boxed{Vol\big(Tub_{\varepsilon}(S)\big) = 2\,\varepsilon Area(S) + \frac{2\varepsilon^3}{3}\int_S K dA}$$

and, if S is compact, we get – by applying the Gauss-Bonnet Theorem:

$$(**) \quad \boxed{Vol\big(Tub_{\varepsilon}(S)\big) = 2\,\varepsilon Area(S) + \frac{4\pi\varepsilon^3}{3}\chi(S)}$$

where $\chi(S)$ represents the Euler characteristic of S.

Note the absence of H in these formulae. Thus, the Tube Formula cannot be employed to compute the mean curvature.

Moreover, in the case of triangulated surfaces computing K by means of the Tube Formula reduces to approximating $K(p)$ by the angle defect at the point p.

Nevertheless, not everything is lost, since for *Half-Tubes* the following formula holds:

$$(\star)\ \boxed{Vol\big(Tub_{\pm\varepsilon}(S)\big) = \varepsilon Area(S) \mp \varepsilon^2 \int_S HdA + \frac{\varepsilon^3}{3}\int_S KdA}$$

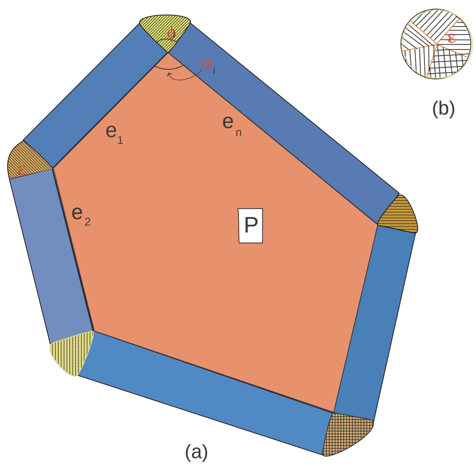
What actually allows one to employ the Half Tube Formula in the computation of H are its roots: The Tube Formula is, in fact, a generalization of the classical Steiner-Minkowski Theorem:

Theorem 5 *Let* $P \subset \mathbf{R}^n$, $n = 2, 3$ *be a compact, convex polyhedron and let* $N_\varepsilon(P) = \{x \in \mathbf{R}^n \,|\, dist(x, P) \leq \varepsilon\}$, $n = 2, 3$.

If $n = 2$, *then*

$$(1) \quad \boxed{Area(N_\varepsilon(P)) = Area(P) + \varepsilon Length(\partial P) + \pi\varepsilon^2}$$

where ∂P *denotes the perimeter of* P.

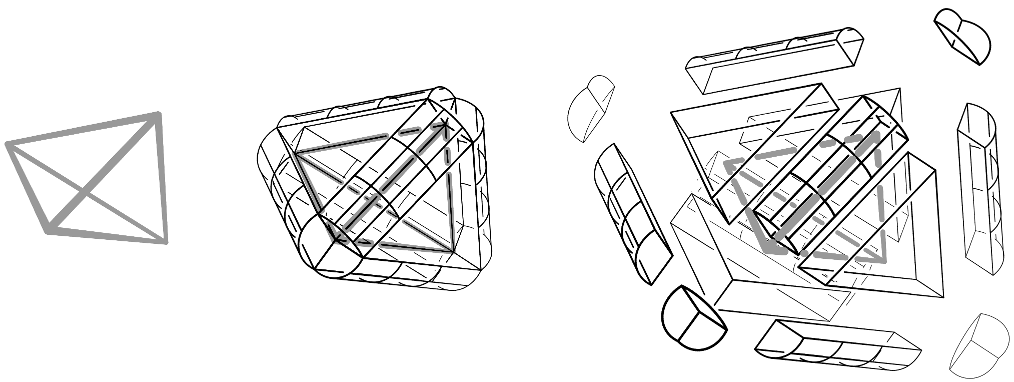


If $n = 3$, *then*

(2) $$Vol(N_\varepsilon(P)) = Vol(P) + \varepsilon Area(\partial_2 P) + C\varepsilon^2 Length(\partial_1 P) + \frac{4\pi\varepsilon^3}{3}$$

where:

- $\partial_2 P$ *denotes the faces of* P
- $\partial_1 P$ *denotes the edges of* P
- *the last term contains the* 0*-dimensional volume contribution of the vertices of* P, *where* • $Vol(\partial_0 P) = |V_P|$
  - *and where* $C = C(P)$ *is a scalar value that encapsulates* $\iint_S H$ *and that essentially depends on the dihedral angles of* P).



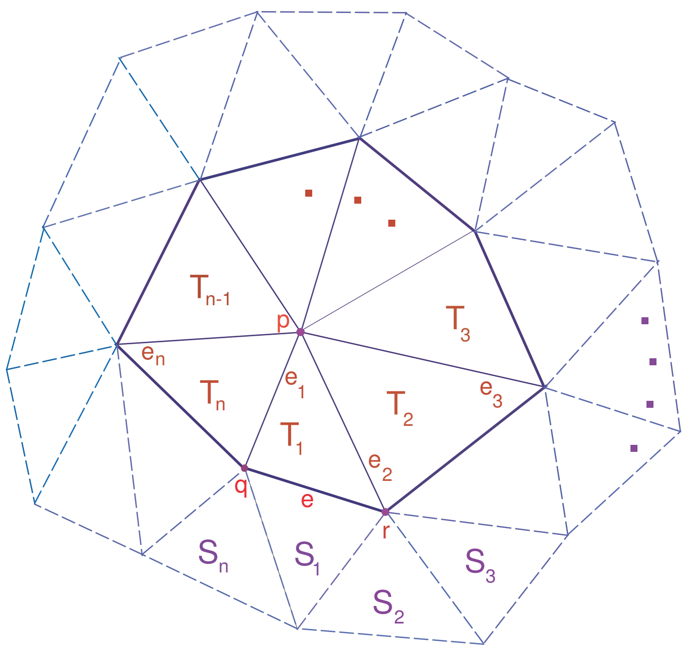
The Algorithm

We begin by making the following:

Remark 6 *It should be emphasized that while in the computation of $H(p)$, the dihedral angles of the edges through p, $e_1, \ldots, e_n$ are computed, this is done only in the sense of measures, this being true for the areas involved in the expression of $K(p)$.*

That is, one should regard, for instance, $\text{Area}(T_i)$ as a weight associated with the triangle T_i and uniformly distributed among its vertices p, q, r. The same uniform distribution is to be considered with respect to the weights naturally associated with edges. Therefore, the measure, i.e., of the dihedral angle associated with the edge e_1, for instance, is to be equally distributed among the vertices adjacent to it, i.e. p and q.

However, the edge $\overline{e_i} = qr$ also contributes to $H(p)$, since it is an element of T_i, which is adjacent to p. Since the boundary edge $\overline{e_i}$ is common to T_i and the second-ring triangle S_i, it's contribution to each of the T_i triangles is half of the associated dihedral angle. Analogous considerations are to be applied in computing the contribution of the boundary vertices (e.g., q), etc.



Therefore, if $\text{Ring}_i(p)$ denotes the i'th ring around p and if $|e_i|$ denotes the length of edge e_i, then, the formula employed for computing the $H(p)$ follows:

$$H(p) = \frac{1}{\text{Area}(\text{Ring}_1(p))}\left[\frac{1}{2}\sum_{i=0}^{n-1}\varphi(T_i, T_{(i+1)\,\text{mod}\,n})|e_i| + \frac{1}{4}\sum_{i=0}^{n-1}\varphi(T_i, S_i)|\overline{e_i}|\right],$$

where $S_i \in \text{Ring}_2(p)$ shares edge $\overline{e_i}$ with $T_i \in \text{Ring}_1(p)$, and $\varphi(T_i, T_{i+1})$ denotes the dihedral angles between adjacent triangles T_i and T_{i+1}.

Experimental Results

We tested the Half Tube Formula based Algorithm on triangular meshes that represent tesselations of the following 5* synthetic models of NURB surfaces:

- An Ellipsoid
- A Hyperbolic Surface of Revolution
- A Torus

and

- The Body

and

- The Spout

of

The Infamous Utah Teapot

*amongst others

The tesselations of each model were produced for several different resolutions:

From ≈ 100 triangles to ≈ 5000 triangles,

allows us to gain some insight into the the convergence rate of the algorithms.

We compared our Algorithm's performance with those of the following previously tested ones:

- Gauss-Bonnet /Angle Deficiency
- Taubin
- Watanabe

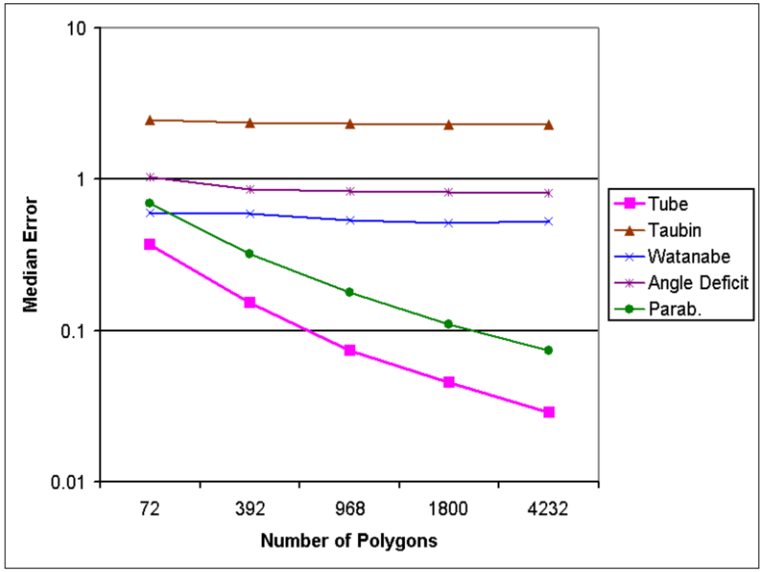
and the classical

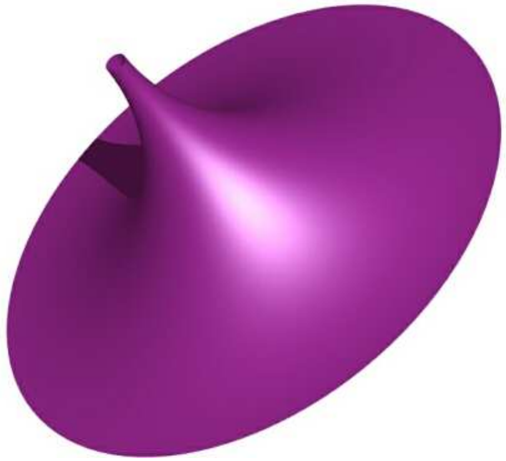
- Parabolic Fit

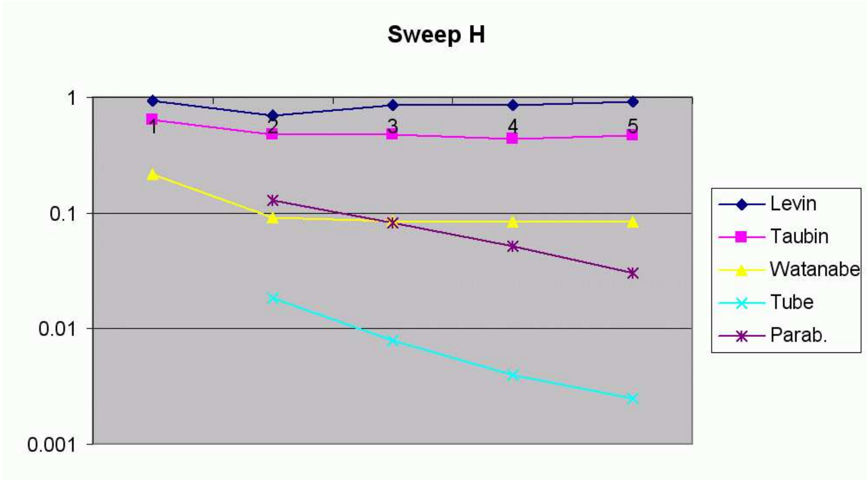
We considered the following mean error value (over all m vertices):

$$\frac{1}{m}\sum_{i=1}^{m}\left| H_i - \overline{H}_i \right| ,$$

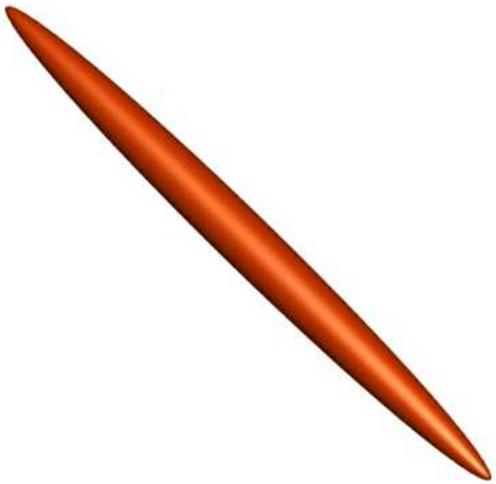
where H_i denotes the analytically computed value of the mean curvature from the smooth NURBs surface $S(r,t)$ at (r_i, s_i) and $\overline{H}_i$ represents the value of the mean curvature that was estimated by one of the comparison methods at the triangular mesh vertex $v_i = S(r_i, t_i)$.

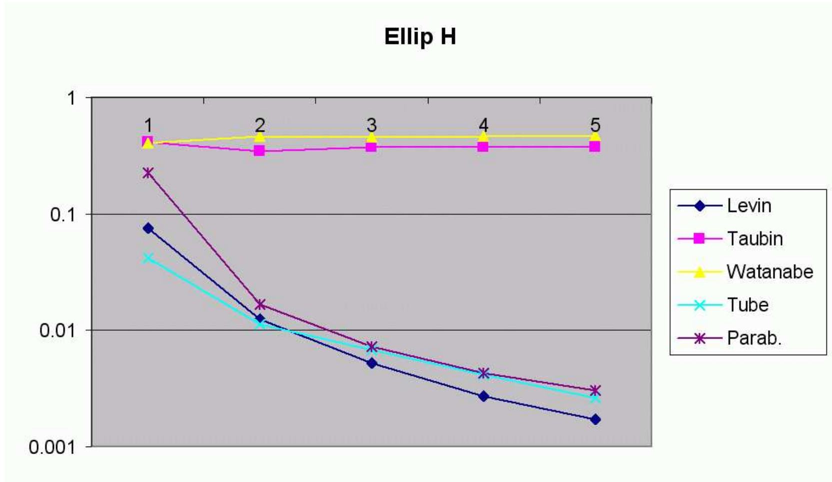

10
1
0.1
0.01
Median Error
72
392
968
1800
4232
Number of Polygons
Tube
Taubin
Watanabe
Angle Deficit
Parab.

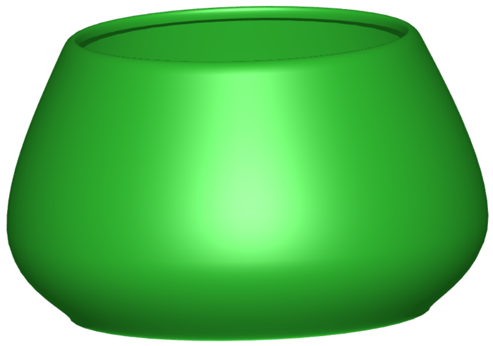


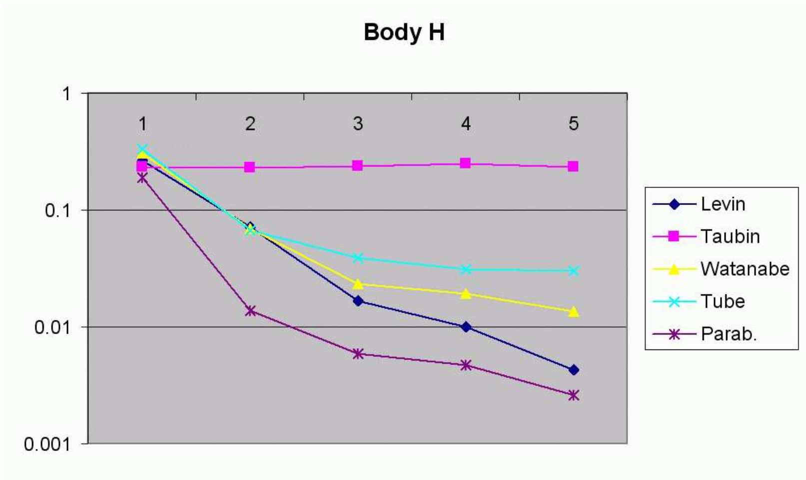


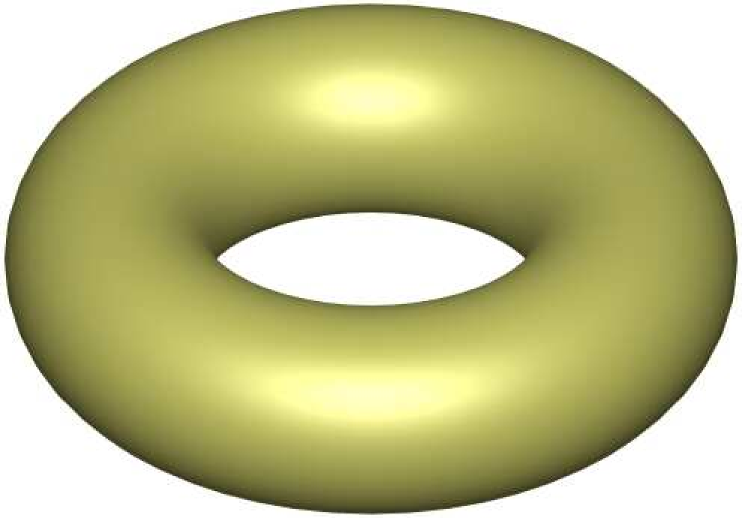
Sweep H
1
0.1
0.01
0.001
1
2
3
4
5
Levin
Taubin
Watanabe
Tube
Parab.

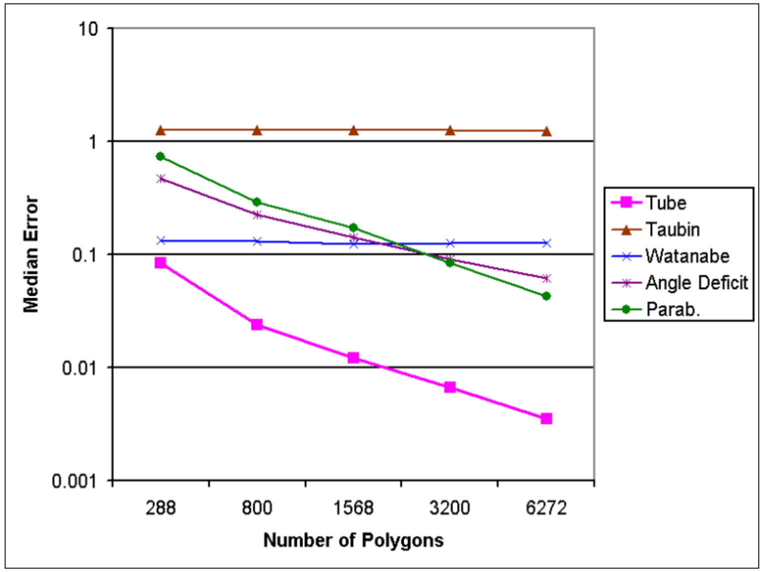



Ellip H
1
0.1
0.01
0.001
1
2
3
4
5
Levin
Taubin
Watanabe
Tube
Parab.




Body H
1
0.1
0.01
0.001
1
2
3
4
5
Levin
Taubin
Watanabe
Tube
Parab.





10
1
0.1
0.01
0.001
Median Error
288
800
1568
3200
6272
Number of Polygons
Tube
Taubin
Watanabe
Angle Deficit
Parab.

- Conclusion:

The Tube Formula method gives the best results among the algorithms for computing mean curvature, both on surfaces of negative Gauss curvature K (i.e., the hyperboloid) and on surfaces where K takes both positive and negative values (e.g., the spout and the torus).

The results obtained closely approach those obtained by the best method (e.g., the Parabolic Fit) in the case of surfaces of positive Gauss curvature of high variance (e.g., the ellipsoid).

It fails to produce very good results only for the Body. The probable reason is that, in the "middle section" of the Body, of almost zero Gauss curvature, the tesselation produces patches of (local) negative Gauss curvature.

- Computational complexity/time performance of our algorithm

The algorithm's complexity is $O(|V|)$ (where, as usual, $|V|$ denotes the number of the vertices of the mesh).

The time performance of the algorithm is concerned, the computing time on a mesh of approximatively $4 \cdot 10^3$ vertices is of mere seconds on a Intel Core 2 machine.

Further Study

• Extend method to compute Gaussian curvature. (Since the computation of K involves ε^3, one expects difficulties to arise, due to numerical instability.)

• Investigate the negative role of “thin” triangulations.

• Extend these ideas to include **general closed sets** $\mathbb{R}^n$, $n \geq 4$ (cf. recent work of Hug et al) – use normale cycles and compute the Weyl curvatures.

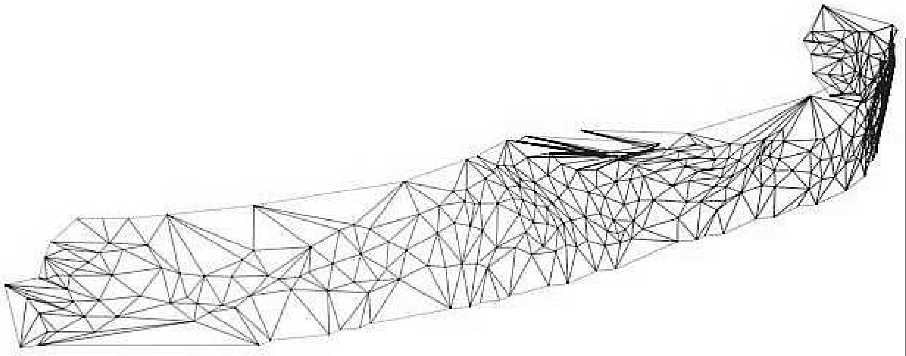
(This is important for evolution equations, s.a. Cahn-Hilliard.)

Further Applications

- Computing the Willmore (elastic) energy

$$W(S) = \int_S H^2 dA\,,$$

where S is a smooth, compact surface S isometrically immersed in $\mathbb{R}^3$, with applications, for instance, in Medical Imaging...



... in micro-materials...

... and, of course, in Computer Graphics