Curvatures, Branched Coverings and Wheeler Foam

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- **Main motivation:** Construction of Branched **Geometric** Coverings of $\mathbb{S}^n$, technically defined as follows:

Definition 1 *Let M^n, N^n be oriented, Reimannian n-manifolds, and let $f : M^n \to N^n$ be a continuous mapping. The mapping f is called quasiregular (qr) iff*

(i) f is locally Lipschitz (and thus differentiable a.e.)

(ii) there exists $K \geq 1$ such that

$$|f'(x)|^n \leq K J_f(x) \ a.e. \tag{1}$$

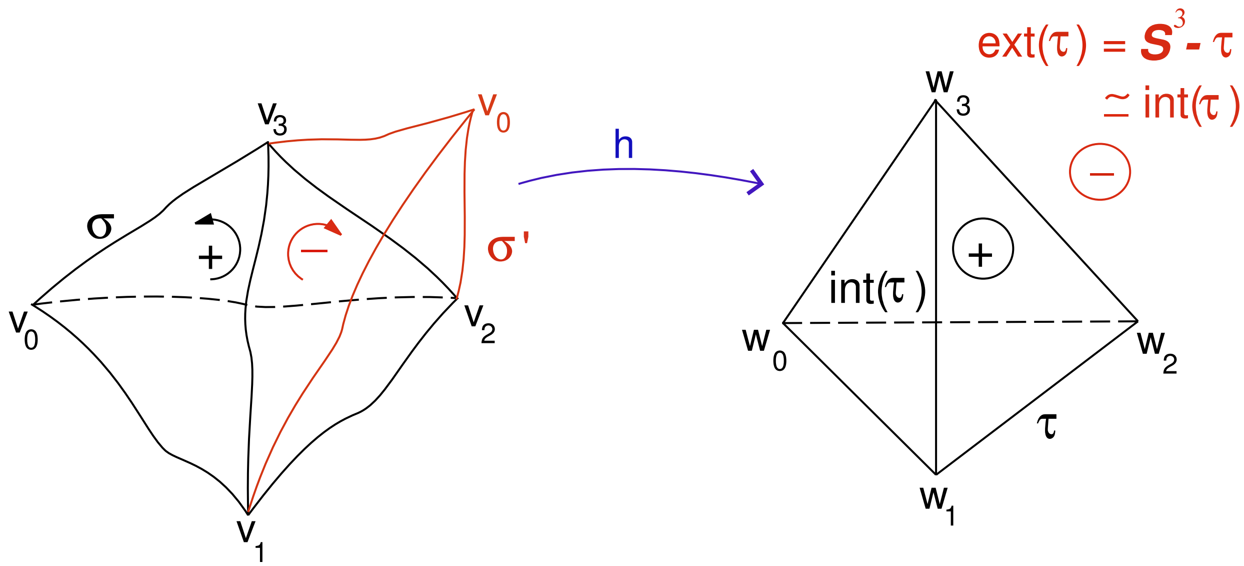
where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = det f'(x)$.

The smallest number K that satisfies (2) is called the outer dilatation of f.

•**Technique:** "Alexander's Trick":

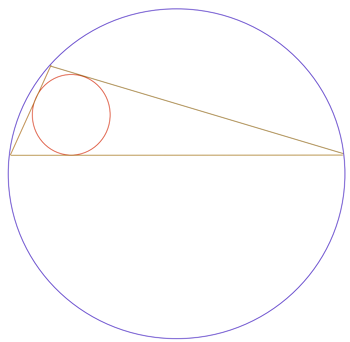
One starts by constructing a chessboard triangulation of an oriented manifold M^n, i.e. a triangulation whose simplices satisfy the condition that every $(n-2)$-face is incident to an even number of n-simplices. Since M^n is orientable, a consistent orientation can be chosen for all the simplices of the triangulation (i.e. such that two given n-simplices having a $(n-1)$-dimensional face in common will have opposite orientations).

Then one quasiregularly and homeomorhically maps the simplices of the triangulation into $\widehat{\mathbb{R}^n} \equiv \mathbb{S}^n$ in a chess-table manner: the positively oriented ones onto the interior of the standard simplex in $\mathbb{R}^n$ and the negatively oriented ones onto its exterior. If the dilatations of the qr-maps constructed above are uniformly bounded, then the resulting map will be a branched geometric covering of $\mathbb{S}^n$.

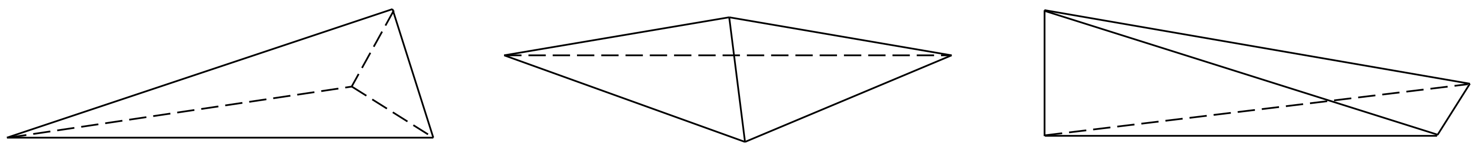


This condition can be assured if the simplices are uniformly *fat*, where the notion of *fatness* is given in the following definition:

Definition 2 *A k-simplex $\tau \subset \mathbb{R}^n$; $2 \leq k \leq n$ is f-fat if there exists $f \geq 0$ such that the ratio $\frac{r}{R} \geq f$; where r denotes the radius of the inscribed sphere of τ (*inradius*) and R denotes the radius of the circumscribed sphere of τ (*circumradius*). A triangulation of a submanifold of $\mathbb{R}^n$ (or $\mathbb{H}^n$) $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is f-fat if all its simplices are f-fat. A triangulation $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ is fat if there exists $f \geq 0$ such that all its simplices are f-fat; $\forall i \in \mathbf{I}$.*



Remark 3 *We want to ensure that "slim" or "flat" simplices such as the ones below do not appear in the triangulation.*



Remark 4 *Fat triangles are precisely those for which the individual simplices considered in Alexander's trick may each be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homotety, with a fixed L.*

- **Main purpose:** to explore some Differential Geometric implications of the following theorem:

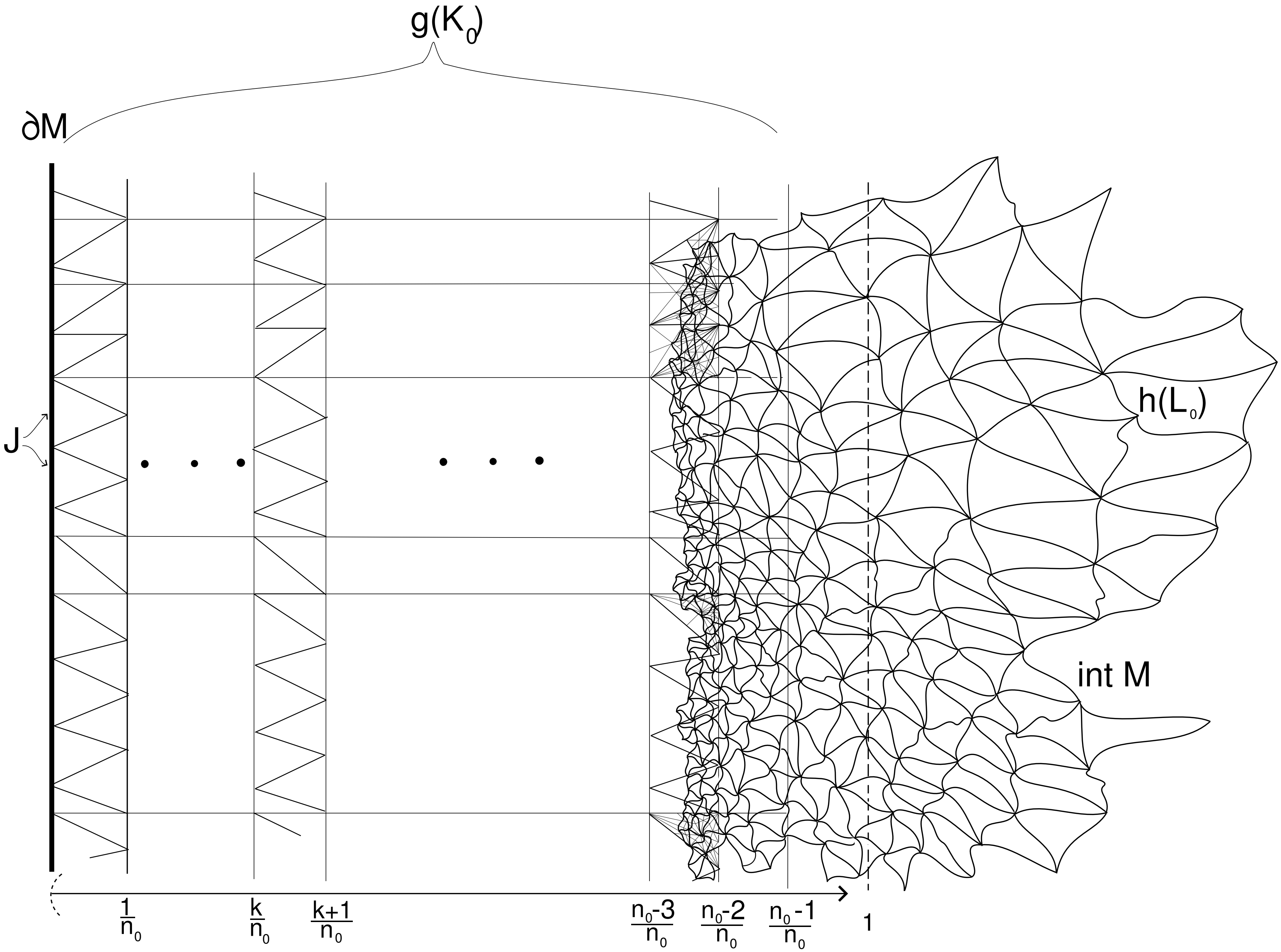
Theorem 5 *Let M^n be a (smooth) manifold with boundary, such that the boundary components admit fat triangulations of fatness $\geq \varphi_0$. Then there exist a global fat triangulation of M^n (hence M^n admits qr-mappings on $\mathbb{S}^n$).*

•The idea of the proof of Theorem 5:

Build two fat triangulations: $\mathcal{T}_1$ of a product neighbourhood N of ∂M^n in M^n and $\mathcal{T}_2$ of $int\, M^n$ (it exists by a result of Peltonen), and then to "mash" the two triangulations into a new triangulation $\mathcal{T}$, while retaining their fatness.

• Basic meshing technique: A modification of a classical result of Munkres.

• Fattening technique: One possibility is to use a fattening method of Cheeger et al.



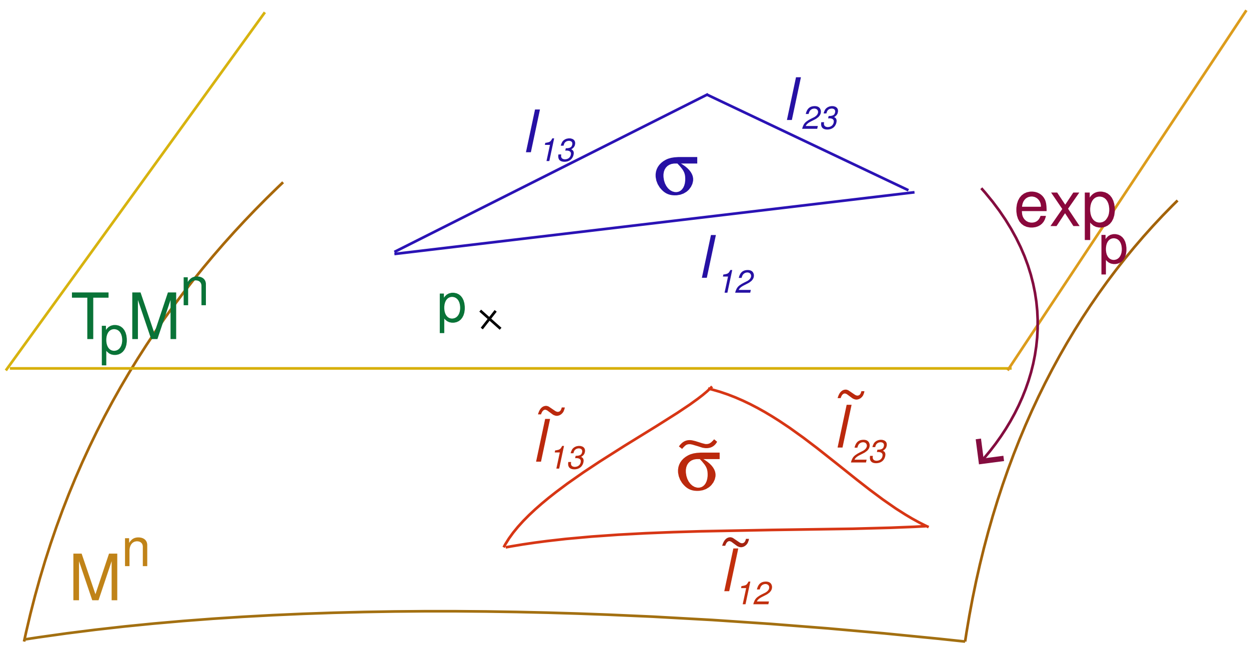
g(K0)
∂M
J
h(L0)
int M
1/n0
k/n0
k+1/n0
n0-3/n0
n0-2/n0
n0-1/n0
1

Cheeger's motivation: to prove the following convergence theorem:

Theorem 6 (Cheeger-Müller-Schrader) *Let M^n be a Riemannian manifold and let M_i^n be a a sequence of* **fat** *PL-manifolds converging to M^n. Denote by $\mathfrak{R}$ and $\mathfrak{R}_i$ respectively, the L.-K. curvatures of M^n, M_i^n. Then $\mathfrak{R}_i \to \mathfrak{R}$* **in the sense of measures**.

The convergence being in the following sense:

- The fatness of the simplices satisfies: $f(\sigma) \geq f_0$;
- Their diameters satisfy: $\eta_i \leq \eta(f_0)$;
- $|l_{ij} - \tilde{l}_{ij}| \leq C\eta_i^2$;
- $f(\tilde{\sigma}) \geq f_0/2$.



Recall that:

For **smooth** manifolds the j-th Lipschitz-Killing curvature R^j is the measure on M^n, such that:

If j is **odd**: $R^j = 0$
and
If j is **even**: R^j is given by integrating the following form:

$$R^j = \frac{1}{(n-j)!2^j\pi^{j/2}(j/2)!}\sum_{\sigma\in S_n}(-1)^{\epsilon(\sigma)}\Omega_{\sigma 1\sigma 2}\wedge\cdots\wedge\Omega_{\sigma(j-1)\sigma(j)}\wedge\omega_{\sigma(j+1)}\wedge\cdots\wedge\omega_{\sigma(n)}$$

where:

$\Omega_{\sigma(j-1)\sigma(j)}$ are the curvature 2-forms

and

ω_{kl} denote the connection 1-forms.

Remark 7 $R^0 \equiv$ volume, $R^1 \equiv$ scalar curvature, $R^n \equiv$ *Gauss-Bonnet-Chern form**, *(for* $n = 2k$*).*

Remark 8 *In a similar manner one defines the mean curvature* H^j *as a* measure *on* ∂M^n*.*

Then: $H^1 \equiv$ *area boundary,* $H^2 \equiv$ *mean curvature for inward normal, etc.*

In view of these intuitive interpretations, the following Remark is immediate:

*See Appendix

Remark 9 *The fatness condition in Theorem 6 is necessary, as the classical example of the Schwarz lantern underlines:*

For PL manifolds the j-th Lipschitz-Killing curvature is defined as follows:

$$R^j = \sum_{\sigma^{n-j}} \Big\{ 1 - \chi\Big(L(\sigma^j)\Big) +$$

$$+ \sum_{l} \measuredangle(\sigma^{n-j}, \sigma^{n-j+i_1}) \cdots \measuredangle(\sigma^{n-j+i_{1-1}}, \sigma^{n-j+i_1}) \cdot \Big[1 - \chi\Big(L(\sigma^{n-j+i_l})\Big) \Big] \Big\} Vol(\sigma^{n-j})$$

where:

- $L(\sigma^j)$ denotes the (*spherical*) link of σ^j;

- $\measuredangle(\sigma^i, \sigma^j)$ is the internal (dihedral) angle of $\sigma^i < \sigma^j$; $\Big(\measuredangle(\sigma^i, \sigma^j) = Vol\Big(L(\sigma^i, \sigma^j)\Big)$, where the volume is normalized s.t. $Vol(\mathbb{S}^n) = 1, \forall\, n.\Big)$

- χ, Vol denote, as usual, the Euler characteristic and volume of σ^k, respectively.

Remark 10

$$R^i = \int_{\{(n-i)-\text{planes}\}} \chi(M^n \cap \pi) d\pi$$

Remark 11

$$R^i(M^n) = \frac{1}{Area(\mathbb{S}^{n-i-1})} \int_{M^{n-1}} S_{n-i-1}(k_1(x), k_2(x), \ldots, k_{n-1}(x)) d\mathcal{H}^{n-1}$$

where $M^{n-1} = \partial(M^n)$ and where the symmetric functions S_i are defined by:

$$S_i\Big(k_1(x), k_2(x), \ldots, k_{n-1}(x)\Big) = \sum_{1 \leq k_{i_1} \leq k_{i_k} n-1} k_{i_1}(x) \cdots k_{i_k}(x)\,,$$

$k_1(x), k_2(x), \ldots, k_{n-1}(x)$ *being the principal curvatures.*

Remark 12 *If M is a submanifold, then:*

$$Vol_n\Big(Tube_\varepsilon(M)\Big) = \sum_0^n R^i(M)\varepsilon^i$$

Remark 13 *One is not restricted to the secant approximation: any triangulation and the intrinsic lengths can be used* (on ∂M or in $N, M \hookrightarrow N$).*

*Regge, C-M-S

Combining C.-M.-S. and S. results immediately produces the following:

Theorem 14 *Let* $N = N^{n-1}$ *be a* **not necessarily connected** *(smooth) manifold, such that* $N = \partial M, M = M^n$*. Than the L.-K. curvature measures of* N *can be extended to those of* M*. More precisely, there exist L.-K. curvature measures* $\mathfrak{R} = \{R^i\}$ *on* $\bar{M} = M \cup N$*, such that* $\mathfrak{R}|_N = \mathfrak{R}_N$ *and* $\mathfrak{R}|_M = \mathfrak{R}_M$*, except on a regular (arbitrarily small) neighbourhood of* N*, where* $\mathfrak{R}_N$*,* $\mathfrak{R}_M$ *denote the curvature measures of* N, M *respectively.*

• Branched Coverings Revisited:

Theorem 5 and Theorem 6 show that spaces that admit **"good" curvature convergence in secant approximation** are geometric branched covers of $\mathbb{S}^n$.

Moreover, Remark 11 readily implies:

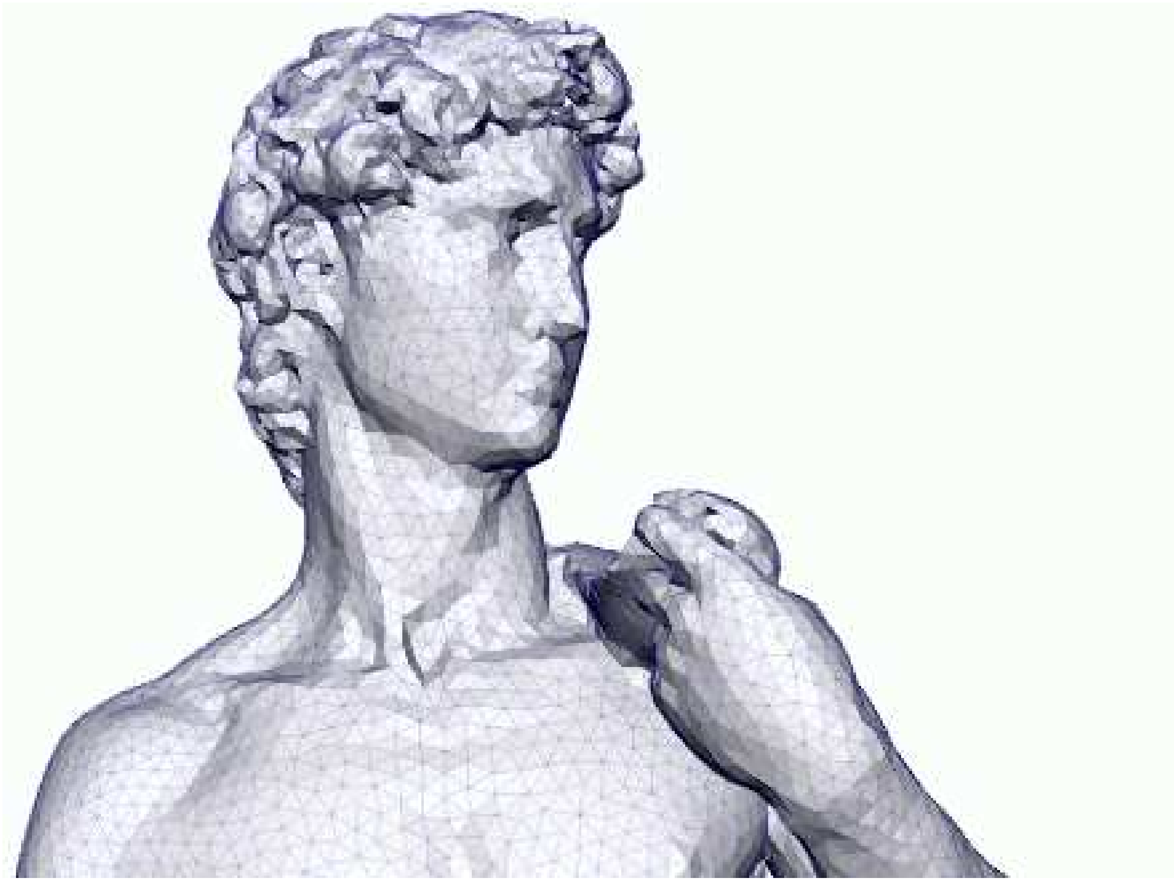
Proposition 15 *Let* $\varphi : M_1^n \to M_2^n$ *be a* $K - qr$ *homeomorphism. Then* $R_2^j = C(K,n)R_1^j$.

Remark 16 *Since all the theorems for geometric branched coverings of* $\mathbb{S}^n$ *were obtained via the Alexander Trick, i.e. by constructing fat chessboard triangulation one is easily conducted to the following:*

Question 17 *Does* M^n *admit a* qr *mapping on* $\mathbb{S}^n$ *iff it admits "good" curvature convergence in secant approximation?.*

- **Some Applications:**

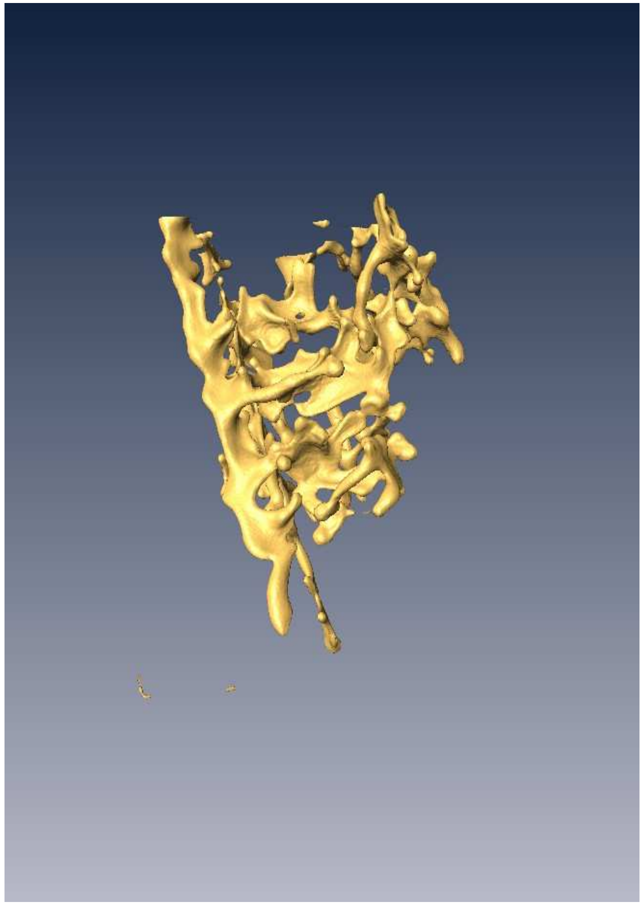
- **Computational Geometry, Computer Graphics, Image Processing:**



Mainly use Theorem 5 for $M^3 \hookrightarrow \mathbb{R}^3$, $(\mathcal{R}_M \equiv \mathbb{R}^3)$ $\partial M^3 = S^2$.

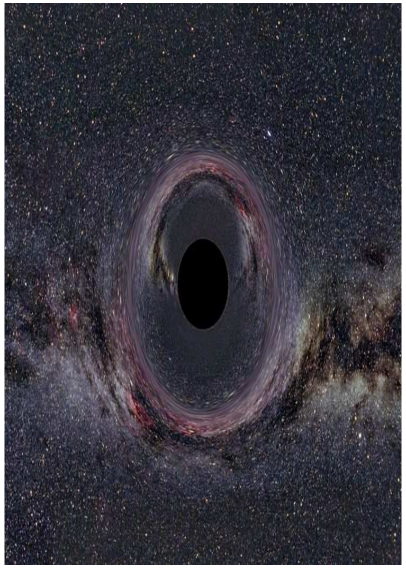
- **Micro-materials:**

Estimation of time evolution of discrete mean-curvatures (for Cahn-Hilliard type equations) e.g. in metal solidification.



- **Regge Calculus:**

Not only for the immediate (trivial) applications i.e. for smooth singularities e.g. "Wheeler wormholes": ($S \simeq \mathbb{S}^2$).



but also for the more interesting case of Wheeler Foam, i.e. non-smooth singularities (for which **topology** and **curvature** change with **scale**).

This point raises a few natural (interrelated) questions:

Question 18 *The PL spaces are still not the discrete metric spaces (lattices) sought for in quantum field theory. Can one discard this restriction?*

Question 19 *Since Regge's drive was to find a purely metric (discrete) formulation of Gravity, the presence of angles in the L.-K. curvatures is a bit "unesthetic". Hence: can one (non-trivially) formulate Theorem 6 (and its consequences) solely in metric terms?*

Candidates:

Definition 20 (The Menger Curvature) *Let* (M, d) *be a metric space, and let* $p, q, r \in M$ *be three distinct points. Then:*

$$K_M(p,q,r) = \frac{\sqrt{(pq+qr+rp)(pq+qr-rp)(pq-qr+rp)(-pq+qr+rp)}}{pq \cdot qr \cdot rp};$$

where $pq = d(p,q)$, *etc. , is called the* Menger Curvature *of the points* p, q, r.

Remark 21 *This approach is based upon two most familiar high school formulas for the aria of the triangle of sides* a, b, c *: (i)* **(Heron's Formula)** $S = \sqrt{p(p-a)(p-b)(p-c)}$ *and (ii)* $S = \frac{abc}{4R}$, *where* $p = (a+b+c)/2$ *and* R *denotes the radius of the circumscribed circle (*circumradius*)*

We can now define the *Menger Curvature* at a given point by passing to the limit:

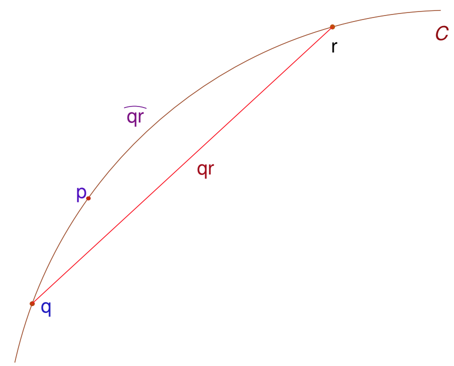
Definition 22 *Let (M,d) be a metric space and let* $p \in M$ *be an accumulation point. Then* M *has at* p Menger Curvature $\kappa_M(p)$ *iff for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *s.t.*

$$\boxed{d(p,p_i) < \delta\,;\ i = 1,2,3 \implies |K(Q) - \kappa_M(p)| < \varepsilon}$$

.

for the PL (and fractal) case, and...

Definition Let (M,d) be a metric space, let $c : I = [0,1] \xrightarrow{\sim} M$ be a homeomorphism, and let $p, q, r \in c(I)$, $q, r \neq p$. Denote by $\widehat{qr}$ the arc of $c(I)$ between q and r, and by qr segment from q to r.



Then c has *Haantjes Curvature* $\kappa_H(p)$ at the point p iff:

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\left(l(\widehat{qr})\right)^3} \;;$$

where "$l(\widehat{qr})$" denotes the length given by the intrinsic metric induced by d of $\widehat{qr}$.

in the general case.

Appendix: Gauss-Bonnet-Chern n-form

$$\int_M \Lambda = \chi(M)$$

where $M = M^n, n = 2k$ and where

$$\Lambda = \frac{(-1)^{n/2}}{2^n \pi^{n/2} (n/2)!} \varepsilon^{A_1 \cdots A_n} F^{A_1 A_2} \cdots F^{A_{n-1} A_n}.$$