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Electron dynamics in the presence of an active medium incorporated in a Penning trap

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Based on an idealized 1D model we demonstrate that electrons oscillating in a Penning trap may get bunched, at the resonant frequency of the active medium. During multiple round trips in the trap, the bunched electrons gain energy and, therefore, they may escape the trap forming a low energy optical injector. © *2011 American Institute of Physics*. [doi:10.1063/1.3559761]

I. INTRODUCTION

A free electron moving in the vicinity of an excited atom may absorb the energy from the bounded electron, this is called collision of the second kind. Contrary to collisions of the first kind where free electrons lose kinetic energy by exciting the atom, in this case, the electron is accelerated and if we assume that all or at least the majority of atoms are excited, then a flow of electrons may be accelerated. Macroscopically, this process resembles, to some extent, friction. In normal conditions, similar to that of collisions of the first kind, friction causes deceleration of the sliding body. Equivalently, in conditions similar to collisions of the second kind, the friction coefficient is negative and the sliding body accelerates. To better envision the process, consider a swing: a child starts to swing with a given amplitude but due to air and joints friction he or she eventually comes to a rest. Now imagine that there is a way to replace the air with a fictitious medium of negative friction. If the acceleration due to this medium is larger than the deceleration due to regular friction at the joints, obviously the motion becomes unstable.

In a previous publication,[1] we harnessed the above-portrayed concept in order to demonstrate conceptually that an ensemble of electrons oscillating in a Penning trap may become bunched, provided an active medium is present in its close vicinity. These bunched electrons are accelerated by the medium and, as a result, they may escape the trap. In this study, we present a detailed analysis of the concept; however, before doing so, a short review of particle acceleration by stimulated emission of radiation (PASER) is necessary. A thorough review was published in Ref. 2.

Historically, collisions of the first kind were first employed by Franck and Hertz (FH)[3] for demonstrating the discrete character of energy states of an electron bounded in an atom. The essence of the experiment was to show that *bounded* electrons can absorb energy from a moving *free* electron only in discrete quanta. Klein and Rosseland[4] were actually those who coined this process the notion of "collision of the first kind." Latyscheff and Leipunsky (LL) demonstrated experimentally the inverse process.[5] Relying on the fact that stimulated absorption of radiation manifests itself as a transition of the atom's outer electron from a low to a higher energy-state, they illuminated vapors of mercury with light from a mercury lamp. A free electron moving near such an excited mercury atom might gain kinetic energy in quanta corresponding to that stored in the atom. In this process, the bounded electron has dropped to the lower energy-state delivering the energy to the free electron, enhancing its kinetic energy. In both FH and LL experiments the vapors' pressure was designed such that, on average, there was only *one collision* of a free electron with a mercury atom and consequently, the average electron's energy gain/loss was of the order of a few electron volts. *Multiple collisions* process was demonstrated by Schawlow and Townes[6] employing photons that were multiplied by excited atoms of ammonia in a consecutive series of collisions. Today, this process is well known as light amplification by stimulated emission of radiation. Recently,[7] we demonstrated that a train of relativistic bunches of electrons may be accelerated by an ensemble of excited atoms provided that the resonant frequency of the medium corresponds to the frequency of the bunches. In the PASER experiment[8,9] a fraction of the moving electrons gained about 200 keV, implying that such an electron has encountered order of 2×10^6 coherent collisions of the second kind.

This proof-of-principle experiment of acceleration at optical wavelengths relies on existing accelerator (45 MeV), wiggler. and high power laser. Motivated by the need for replacing these three components with an optical equivalent, we recently suggested a novel paradigm that relies on the possibility that *nonrelativistic* electrons confined by a Penning trap will experience collisions of the second of kind, leading to bunching of part of the electrons at the resonant frequency of the medium. As the bunched electrons drain energy from the active medium, their kinetic energy increases, allowing them to escape the trap. Consequently, this setup might play a pivotal role in a future optical buncher. It should be emphasized that the nonrelativistic regime is dictated by the facilities available in the author's laboratory and efforts are under way for the design and testing of a relatively much higher energy version.[10]

For a better conceptual understanding of the process, we must bear in mind that, essentially, collisions of the second kind facilitate coupling between two independent processes: storage of charged particles in a Penning trap and storage of electromagnetic energy in active medium. In the absence of the latter, electrons oscillate in the trap for a time duration $(T_{\rm trap})$ determined by the cross section of scattering with the

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remnant atoms present in the vacuum vessel. In the opposite case, when only the active medium is present, the population inversion density decays back to equilibrium with a characteristic time $T_{\rm eq} \ll T_{\rm trap}$. From the electrons' perspective the energy transfer facilitated by collisions of the second kind may be conceived as a monochromatic wave, oscillating at the resonant frequency of the medium, causing electrons to become *bunched*, leading to an enhanced decay rate of the population inversion density.

In the framework of this study, we provide a detailed account of the force exerted by the active medium on the single electron and the corresponding energy exchanged. Similar to self-amplified spontaneous emission (SASE), the buildup is assumed to start from noise. Subject to the assumption that during a single round trip the effect of the active medium on the electron is minuscule, the equations of motion are simplified. Numerical solutions of the system's equations are presented and some analytic relations are developed. Specifically, since collisions of the second kind are represented by a "negative-friction" coefficient, the effect of this parameter on the energy gain and energy spread is investigated in detail.

II. DESCRIPTION OF THE MODEL

Consider a dielectric function representing a material characterized by a series of resonances $\omega_{0,i}$ and spontaneous decay coefficients $T_{2,i}(= Q_i/2\omega_{0,i})$. Further assuming a background ("dc") relative dielectric coefficient ε_r we may formulate this function as

$$\varepsilon(\omega > 0) = \varepsilon_r + \sum_i \frac{\omega_{p,i}^2}{\omega_{0,i}^2 + j\omega/T_{2,i} - \omega^2}, \qquad (1)$$

with $\omega_{p,i}$ representing the "plasma frequency" of the electronic/vibrational/rotational resonances according to the specific medium considered. Explicitly, in case of linear regime, the plasma frequency is related to the population inversion density by $\omega_{p,i}^2 = -\Delta n_i \sigma_{21,i} c \varepsilon_r / T_{2,i}$. In the framework of this notation $\Delta n_i > 0$ represents a population inversion density corresponding to the resonance $\omega_{0,i}$ and similarly, $\sigma_{21,i}$ denotes the cross section for *stimulated emission*. It is important to make two comments at this point: (i) the cross section for stimulated emission is the same for both regular photons and *virtual photons*. (ii) Ionization of the atoms/molecules by the free electrons is ignored, although their kinetic energy may suffice to trigger this process.

In order to get a sense of the typical values involved let us consider a solid state medium namely, Nd:YAG: the dielectric coefficient is $\varepsilon_r = 3.312$, the resonance occurs for $\lambda_0 \simeq 1.06[\mu\text{m}]$, the spontaneous life-time is $T_2 = 240$ μs, the gain is 1% per centimeter provided the population inversion is $\Delta n_{21} \sim 1.1 \times 10^{22}$ m^{-3}. However, much higher values might be available; the cross section for stimulated emission is $\sigma_{21} \sim 1.8 \times 10^{-22}$ m^2.

In this dielectric, if gas, or in its very close vicinity if solid, an ensemble of electrons is oscillating due to the presence of a combination of electric and magnetic fields forming a Penning trap, a schematic of the system is illustrated in Fig. 1.

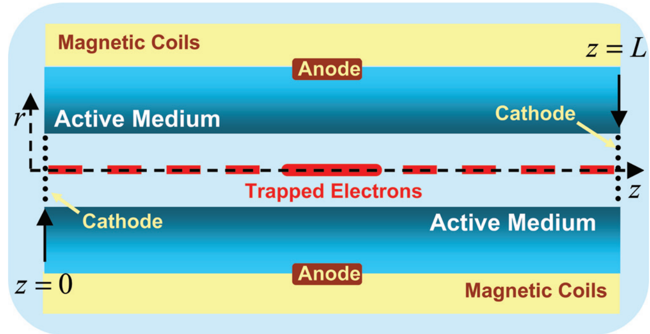

FIG. 1. (Color online) Schematics of a Penning trap of electrons and an active medium. Electrons from both ends are attracted by the anode located off-axis but the longitudinal magnetic field generated by the coils confines the electrons to the close vicinity of the axis. The moving electrons stimulate the photons stored in the medium, resulting in a monochromatic wave that in turn bunches the electron beam. Evidently, the bunched electrons stimulate more efficiently the active medium, thus draining faster the stored energy. As electrons become bunched and acquire energy, they escape the trap.

Each individual electron follows a 2D trajectory represented by $\rho_\nu(t)$ and $\zeta_\nu(t)$ however, in the framework of this study, we consider only the dominant component of the current density namely,

$$J_z(r,z,t) = -q_{\rm mp} \sum_\nu \dot{\zeta}_\nu(t) \frac{1}{2\pi r} \delta[r - \rho_\nu(t)] \delta[z - \zeta_\nu(t)]; \quad (2)$$

$q_{\rm mp}$ represents the charge of one macroparticle representing $N_{\rm el}$ electrons, thus $q_{\rm mp} = eN_{\rm el}$; the charge-to-mass ratio of the macroparticle is identical to that of one electron (e/m) and in the trap there are $N_{\rm mp}$ macroparticles; $[\rho_\nu(t), \zeta_\nu(t)]$ describe the trajectory of the macroparticles. As an example, let us assume that the trap is of length $L \sim 10$ cm and it stores order of 10^{12} electrons. The ensemble can be divided into $2L/\lambda_0 \sim 2 \times 10^5$ segments each one representing one optical wavelength and for adequate description of the dynamics in the range of one segment 500 macroparticles are required, implying that each macroparticle contains order of 10^4 electrons. Correspondingly, the average electrons' density assuming a 1 mm^2 cross section is of the order of 5×10^{12} cm^{-3}. In the absence of the active medium, the trajectory of these electrons follows a damped helical oscillation where the damping is due to scattering of the electrons with remnant gas in the vacuum and eventually they become trapped.

As already stated, for simplicity sake, it is assumed that the contribution of the radial and azimuthal motion to the interaction with the active medium is negligible. Therefore, we consider next the spatial and temporal Fourier transform of the longitudinal current density

$$\begin{aligned} \bar{J}_z(r,\omega,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dz' \exp(jkz') \\ &\quad \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \exp(-j\omega t') J_z(r,z',t') \\ &= \frac{-q_{\rm mp}N_{\rm mp}}{(2\pi)^3 r}\int_{-\infty}^{\infty} dt' \exp(-j\omega t') \\ &\quad \times \left\langle \dot{\zeta}_\nu(t') \exp[jk\zeta_\nu(t')] \delta[r - \rho_\nu(t')] \right\rangle_\nu \end{aligned} \qquad (3)$$

In practice, a macroparticle has a nonzero length, Δ_{mp}, implying that wherever the phase term $\exp[jk\zeta_\nu(t)]$ occurs, the transformation

$$\exp[jk\zeta_\nu(t)] \rightarrow \exp[jk\zeta_\nu(t)]\mathrm{sinc}\left(\frac{1}{2}k\Delta_{\mathrm{mp}}\right) \tag{4}$$

should be kept in mind. In what follows, the notation in Eq. (3) will be preserved.

This current density, Eq. (3), drives the magnetic vector potential, which in the boundless case is given by

$$\bar{A}_z(r,k,\omega) = \mu_0 \int dr'\, r' G(\Lambda r, \Lambda r')\bar{J}_z(r',k,\omega) \tag{5}$$

with $\Lambda^2 = k^2 - \varepsilon(\omega)\omega^2/c^2$ and

$$G(\Lambda r, \Lambda r') = \begin{cases} I_0(\Lambda r)K_0(\Lambda r'), & r < r' \\ K_0(\Lambda r)I_0(\Lambda r'), & r > r' \end{cases} \tag{6}$$

Explicitly, back to the time domain, we get for the longitudinal electric field

$$\begin{aligned} E_z(r,z,t) = {} & \frac{\mu_0 q_{\mathrm{mp}} N_{\mathrm{mp}}}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega\, \exp(j\omega t) \\ & \times \int_{-\infty}^{\infty} dk\, \exp(-jkz)\frac{c^2\Lambda^2}{j\omega\varepsilon(\omega)} \\ & \times \int_{-\infty}^{\infty} dt'\, \exp(-j\omega t') \\ & \times \left\langle G[\Lambda r, \Lambda\rho_\nu(t')]\dot{\zeta}_\nu(t')\exp[jk\zeta_\nu(t')]\right\rangle_\nu, \end{aligned} \tag{7}$$

which in the framework of our 1D model accounts for the energy exchange between the electrons and the active medium.

III. ACTIVE MEDIUM FORCE AND ENERGY EXCHANGE

Subject to the assumptions so far, the force on a single *macroparticle* is

$$\begin{aligned} \mathcal{F}_\nu(t) &= -q_{\mathrm{mp}}E_z[\rho_\nu(t),\zeta_\nu(t),t] \\ &= -\frac{\mu_0 q_{\mathrm{mp}}^2 N_{\mathrm{mp}}}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega\, \exp(j\omega t) \\ &\quad \times \int_{-\infty}^{\infty} dk\, \exp[-jk\zeta_\nu(t)]\frac{c^2\Lambda^2}{j\omega\varepsilon(\omega)}\int_{-\infty}^{\infty} dt'\, \exp(-j\omega t') \\ &\quad \times \left\langle G[\Lambda\rho_\nu(t), \Lambda\rho_\mu(t')]\dot{\zeta}_\mu(t')\exp[jk\zeta_\mu(t')]\right\rangle_\mu \end{aligned} \tag{8}$$

and the total energy exchange is given by

$$\begin{aligned} W_{\mathrm{ex}} &= N_{\mathrm{mp}}\int_{-\infty}^{\infty} dt \left\langle \dot{\zeta}_\nu(t)\mathcal{F}_\nu(t)\right\rangle_\nu \\ &= -\frac{\mu_0 q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2}{(2\pi)^3}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega\, \exp(j\omega t) \\ &\quad \times \int_{-\infty}^{\infty} dk \frac{c^2\Lambda^2}{j\omega\varepsilon(\omega)}\int_{-\infty}^{\infty} dt'\, \exp(-j\omega t') \\ &\quad \times \langle \dot{\zeta}_\nu(t)\exp[-jk\zeta_\nu(t)]G[\Lambda\rho_\nu(t),\Lambda\rho_\mu(t')]\dot{\zeta}_\mu(t') \\ &\quad \times \exp[jk\zeta_\mu(t')]\rangle_{\nu,\mu} \end{aligned} \tag{9}$$

At this stage we proceed by assuming that: (i) the transverse distribution is independent of the longitudinal one, (ii) the transverse dynamics has negligible contribution to the energy exchange process, and (iii) in the radial direction, the electrons are uniformly distributed in the range $0 < r < R_b$. Based on these assumptions it is convenient to define the transverse filling factor,

$$\begin{aligned} F_\perp &\equiv \left\langle G[\Lambda\rho_\nu, \Lambda\rho_\mu]\right\rangle_{\nu,\mu} \\ &= \frac{2}{R_b^2}\int_0^{R_b} dr_1\, r_1 \frac{2}{R_b^2}\int_0^{R_b} dr_2\, r_2\, G(\Lambda r_1, \Lambda r_2) \\ &= \frac{2}{\theta^2}\int_0^{\theta} dx\, x \frac{2}{\theta^2}\int_0^{\theta} dy\, y \begin{cases} I_0(x)K_0(y), & x < y \\ K_0(x)I_0(y), & x > y \end{cases} \\ &= \frac{2}{\theta^2}[1 - 2K_1(\theta)I_1(\theta)] \end{aligned} \tag{10}$$

wherein $\theta = \Lambda R_b$. Consequently, the average (on the radial direction) longitudinal force on the microbunch which has the form of a "pancake" $\left(\Delta_{\mathrm{mp}}\pi R_b^2\right)$ is

$$\begin{aligned} \mathcal{F}_\nu(t) = {} & -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{1}{j\omega\varepsilon(\omega)}\frac{1}{2\pi} \\ & \times \int_{-\infty}^{\infty} dk\, \exp[j\omega t - jk\zeta_\nu(t)][1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)] \\ & \times \left\langle \int_{-\infty}^{\infty} dt'\dot{\zeta}_\mu(t')\exp[-j\omega t' + jk\zeta_\mu(t')]\right\rangle_\mu \\ W_{\mathrm{ex}} = {} & -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{1}{j\omega\varepsilon(\omega)} \\ & \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)] \\ & \times \left|\left\langle \int_{-\infty}^{\infty} dt\dot{\zeta}_\nu(t)\exp[j\omega t - jk\zeta_\nu(t)]\right\rangle_\nu\right|^2. \end{aligned} \tag{11}$$

The second term represents the energy exchange in the framework of the approximations mentioned earlier.

Consider now a Penning trap characterized by an angular frequency Ω and ignoring the damping during *one period* of the oscillation $(T = 2\pi/\Omega)$, the particles' trajectory is assumed to be given by

$$\zeta_\nu(t) = \frac{L}{2}\{1 + \cos[\Omega(t - t_\nu)]\}; \tag{12}$$

t_ν determines the location of the particle at $t = 0$. Consequently, the time integral in Eq. (11) is replaced by

$$\mathcal{L}(k,\omega) \equiv \left\langle \int_{-T/2}^{T/2} dt'\dot{\zeta}_\mu(t')\exp[-j\omega t' + jk\zeta_\mu(t')]\right\rangle_\mu, \tag{13}$$

which may be evaluated analytically during one period by employing the Bessel generating function, $\exp[(u/2)(x - 1/x)] = \sum_{k=-\infty}^{\infty} x^k J_k(u)$,

$$\mathcal{L}(k,\omega) = -\frac{L}{2}\Omega \times \left\langle \begin{array}{c} \exp(-j\omega t_\nu)\exp(jkL/2)\sum_{n=-\infty}^{\infty}\exp(jn\pi/2)J_n\left(k\frac{L}{2}\right) \\ \int_{t_\nu - T/2}^{t_\nu + T/2} dt \sin[\Omega(t - t_\nu)]\exp[j(n\Omega - \omega)(t - t_\nu)] \end{array}\right\rangle_\nu \tag{14}$$

and the identity $J_{n+1}(u) + J_{n-1}(u) = (2n/u)J_n(u)$,

$$\mathcal{L}(k,\omega) = \pi L \langle \exp(-j\omega t_v)\rangle_v \exp(jkL/2) \times \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left[\pi\left(n - \frac{\omega}{\Omega}\right)\right] \times \exp(jn\pi/2)\left[n\frac{J_n(kL/2)}{kL/2}\right]. \quad (15)$$

Consequently, the two quantities of interest $[\mathcal{F}_v(t), W_{\mathrm{ex}}]$ are determined for one period duration. Explicitly, during one period, the force and the energy exchange read

$$\mathcal{F}_v(t) = -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}\pi L}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{1}{j\omega\varepsilon(\omega)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp[j\omega t - jk\zeta_v(t)] \times [1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)]\langle \exp(-j\omega t_\mu)\rangle_\mu \exp(jkL/2) \times \sum_{n=-\infty}^{\infty} \mathrm{sinc}\left[\pi\left(n-\frac{\omega}{\Omega}\right)\right]\exp(jn\pi/2)\left[n\frac{J_n(kL/2)}{kL/2}\right]$$

$$W_{\mathrm{ex}} = -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2 (\pi L)^2}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{1}{j\omega\varepsilon(\omega)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)]\left|\langle \exp(-j\omega t_\mu)\rangle_\mu\right|^2 \times \left|\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left[\pi\left(n-\frac{\omega}{\Omega}\right)\right] \times \exp(jn\pi/2)\left[n\frac{J_n(kL/2)}{kL/2}\right]\right|^2 \quad (16)$$

Further, the infinite summation may be replaced by the resonant term $(n \simeq \omega/\Omega)$

$$\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left[\pi\left(n-\frac{\omega}{\Omega}\right)\right]\exp\left(\frac{1}{2}j\pi n\right)\frac{nJ_n\left(\frac{kL}{2}\right)}{\frac{kL}{2}} \simeq \exp\left(j\frac{\pi}{2}\frac{\omega}{\Omega}\right)\frac{\frac{\omega}{\Omega}J_{\omega/\Omega}\left(\frac{kL}{2}\right)}{\frac{kL}{2}}. \quad (17)$$

Thus the two quantities of interest read

$$\mathcal{F}_v(t) = -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}\pi L}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\frac{1}{j\omega\varepsilon(\omega)}\frac{1}{2\pi} \times \int_{-\infty}^{\infty} dk\exp[j\omega t - jk\zeta_v(t)][1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)] \times \langle\exp(-j\omega t_\mu)\rangle_\mu\exp(jkL/2) \times \exp\left(j\frac{\omega}{\Omega}\frac{\pi}{2}\right)\left[\frac{\omega}{\Omega}\frac{J_{\omega/\Omega}(kL/2)}{kL/2}\right]$$

$$W_{\mathrm{ex}} = -\frac{4q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2(\pi L)^2}{4\pi\varepsilon_0 R_b^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\frac{1}{j\omega\varepsilon(\omega)} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[1 - 2K_1(\Lambda R_b)I_1(\Lambda R_b)] \times \left|\langle\exp(-j\omega t_\mu)\rangle_\mu\right|^2\left[\frac{\omega}{\Omega}\frac{J_{\omega/\Omega}(kL/2)}{kL/2}\right]^2. \quad (18)$$

Before proceeding it is important to determine the conditions for generation of Cerenkov radiation that obviously decelerates the electrons and, therefore, competes with the acceleration process associated with the motion in the presence of an active medium. With this goal in mind, consider only the contribution of the frequency-independent dielectric coefficient. A nonzero time-average radial component of the Poynting vector develops if $\varepsilon_r\omega^2/c^2 - k^2 > 0$ therefore, having in mind that the modified Bessel functions satisfy $K_1(jx) = -(\pi/2)H_1^{(2)}(x)$, $I_1(jx) = -jJ_1(x)$ we get

$$W_{\mathrm{ex}} = \frac{q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2(2\pi L)^2\pi}{4\pi\varepsilon_0\varepsilon_r R_b^2\Omega^2}\frac{1}{\pi}\int_0^{\infty} d\omega\omega\left|\langle\exp(-j\omega t_\mu)\rangle_\mu\right|^2 \times \frac{1}{\pi}\int_0^{|\omega|\sqrt{\varepsilon_r}/c} dk J_1^2\left(R_b\sqrt{\varepsilon_r\frac{\omega^2}{c^2} - k^2}\right)\left[\frac{J_{\omega/\Omega}(kL/2)}{kL/2}\right]^2. \quad (19)$$

For large orders $J_{v\to\infty}(x) \simeq (ex/2v)^v/\sqrt{2\pi v}$, there is a nonzero contribution only if $k > (4/e)(\omega/\Omega)(1/L)$ implying that

$$W_{\mathrm{ex}} = \frac{q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2(2\pi L)^2\pi}{4\pi\varepsilon_0\varepsilon_r R_b^2\Omega^2}\frac{1}{\pi}\int_0^{\infty} d\omega\omega\left|\langle\exp(-j\omega t_\mu)\rangle_\mu\right|^2 \times \frac{1}{\pi}\int_{(4/e)(\omega/\Omega)(1/L)}^{|\omega|\sqrt{\varepsilon_r}/c} dk J_1^2\left(R_b\sqrt{\varepsilon_r\frac{\omega^2}{c^2} - k^2}\right)\left[\frac{J_{\omega/\Omega}(kL/2)}{kL/2}\right]^2. \quad (20)$$

This is to say that the condition for a nonzero contribution the integration limits satisfy $\omega L\sqrt{\varepsilon_r}/c > (4/e)(\omega/\Omega)$. Consequently, for Cerenkov radiation to develop in case of an oscillating particle, the maximum velocity, $\Omega L/2$, must be larger than 0.736 the Cerenkov velocity $(c/\sqrt{\varepsilon_r})$ namely,

$$\Omega\frac{L}{2} > \frac{2}{e}\frac{c}{\sqrt{\varepsilon_r}}. \quad (21)$$

In what follows we assume that Cerenkov condition, Eq. (21), is not satisfied and, as a result, the radiated energy is zero and so is the decelerating force.

Now we are back to the evaluation of the energy exchange as formulated in Eq. (20). The double integration, $\int d\omega \int dk$ is expected to have a maximum at the transition between propagating and evanescent waves, $k^2 - \varepsilon(\omega)\,\omega^2/c^2 \simeq 0$ since the transverse wavelength seems much larger than the radius of the electrons' ensemble or explicitly, $1 - 2K_1(x)I_1(x) \simeq (\pi x/2)^2$ we get for the energy exchange during one round trip

$$W_{\mathrm{ex}} = \frac{q_{\mathrm{mp}}^2 N_{\mathrm{mp}}^2(2\pi L)^2}{4\pi\varepsilon_0 R_b^2\Omega^2}\left(\frac{\pi}{2}\right)^2\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\frac{j\omega}{\varepsilon(\omega)}\left|\langle\exp(-j\omega t_\mu)\rangle_\mu\right|^2 \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left(k^2 - \varepsilon(\omega)\frac{\omega^2}{c^2}\right)\left[\frac{J_{\omega/\Omega}(kL/2)}{kL/2}\right]^2, \quad (22)$$

this being an excellent approximation for $x < 0.02$.

Ignoring the space-charge term, we maintain only the medium-dependent term and as already indicated, the system operates below the Cerenkov condition. Therefore, only the frequency-dependent term of the dielectric coefficient contributes to the energy exchange

$$\begin{aligned}W_{\rm ex} &= \frac{4q_{\rm mp}^2 N_{\rm mp}^2 \pi^4}{4\pi\varepsilon_0 R_b^2 \Omega^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{j\omega}{\varepsilon(\omega)}\left|\left\langle \exp(-j\omega t_\mu)\right\rangle_\mu\right|^2 \\ &\times \frac{1}{\pi}\int_0^\infty dk\left[J_{\omega/\Omega}\left(k\frac{L}{2}\right)\right]^2 \\ &= \frac{-q_{\rm mp}^2 N_{\rm mp}^2 (2\pi)^4}{4\pi\varepsilon_0\varepsilon_r^2\Omega^2(2L)}\sum\nolimits_i \omega_{p,i}^2 \frac{1}{\pi}\int_0^\infty dq\left|J_{\omega_{0,i}/\Omega}(q)\right|^2 \\ &\times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{j\omega\left|\left\langle \exp(-j\omega t_\mu)\right\rangle\right|^2}{\omega_{0,i}^2 - \omega^2 + 2j\omega\omega_{0,i}/Q_i}. \end{aligned} \tag{23}$$

Bearing in mind that $\int_0^\infty dq J_\nu(q) J_{\nu-1}(q) = 1/2$ for *large orders* we get

$$\begin{aligned}W_{\rm ex} &= -\frac{q_{\rm mp}^2 N_{\rm mp}^2 (2\pi)^3}{4\pi\varepsilon_0\varepsilon_r^2\Omega^2(2L)}\sum\nolimits_i \omega_{p,i}^2 \\ &\times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{j\omega\left|\left\langle \exp(-j\omega t_\mu)\right\rangle\right|^2}{\omega_{0,i}^2 - \omega^2 + 2j\omega\omega_{0,i}/Q_i} \\ &= -\frac{q_{\rm mp}^2 N_{\rm mp}^2 (2\pi)^3}{4\pi\varepsilon_0\varepsilon_r^2\Omega^2(2L)}\sum\nolimits_i \omega_{p,i}^2 \\ &\times \left\langle \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{j\omega \exp\left[j\omega(t_\nu - t_\mu)\right]}{\omega_{0,i}^2 - \omega^2 + 2j\omega\omega_{0,i}/Q_i}\right\rangle_{\nu,\mu} \\ &= -\frac{q_{\rm mp}^2 N_{\rm mp}^2 (2\pi)^3}{4\pi\varepsilon_0\varepsilon_r^2\Omega^2(2L)}\sum\nolimits_i \omega_{p,i}^2 \left\langle \cos\left[\omega_{0,i}(t_\nu - t_\mu)\right]\right. \\ &\times \left.\exp\left[-\frac{\omega_{0,i}}{Q_i}(t_\nu - t_\mu)\right] h(t_\nu - t_\mu)\right\rangle_{\nu,\mu}. \end{aligned} \tag{24}$$

Rather than repeating the procedure for evaluating the force on the $\nu^{\rm th}$ macroparticle we consider only the term that may contribute directly to energy exchange

$$\mathcal{F}_\nu(t) = f_\nu \sin[\Omega(t - t_\nu)] \tag{25}$$

and the question is what is f_ν. In the framework of our approximation [Eq. (12)], the velocity of this macroparticle is

$$\dot\zeta_\nu(t) = -\frac{L}{2}\Omega \sin[\Omega(t - t_\nu)] \tag{26}$$

and, therefore, the energy transferred during one period is

$$\begin{aligned}W_{\rm ex} &= N_{\rm mp}\int_{-T/2}^{T/2} dt \left\langle \mathcal{F}_\nu(t)\dot\zeta_\nu(t)\right\rangle_\nu \\ &= -\frac{L}{2}\Omega N_{\rm mp}\left\langle f_\nu \int_{-T/2}^{T/2} dt \sin^2[\Omega(t - t_\nu)]\right\rangle_\nu \\ &= -\frac{L}{2}\Omega N_{\rm mp}\frac{1}{2}T\langle f_\nu\rangle_n. \end{aligned} \tag{27}$$

Comparing the last result with Eq. (24) we get

$$\begin{aligned}f_\nu &= \frac{2q_{\rm mp}^2 N_{\rm mp}(2\pi)^2}{4\pi\varepsilon_0\varepsilon_r^2\Omega^2 L^2}\sum\nolimits_i \omega_{p,i}^2 \\ &\times \left\langle \cos\left[\omega_{0,i}(t_\nu - t_\mu)\right]\exp\left[-\frac{\omega_{0,i}}{Q_i}(t_\nu - t_\mu)\right] h(t_\nu - t_\mu)\right\rangle_\mu. \end{aligned} \tag{28}$$

In principle, there is an additional term that has zero contribution to energy transfer but it may affect the oscillating frequency and includes the space-charge effect. Its form is $g_\nu + h_\nu \cos[\Omega(t - t_\nu)]$, in other words, $\mathcal{F}_\nu(t) \propto \zeta_\nu(t)$ and it will be assumed that this is negligible comparing to the harmonic force $|\Omega^2\zeta_\nu| \gg |\mathcal{F}_\nu|/m_{\rm mp}$ associated with dc field of the trap.

The fact that f_ν is not zero, is one of the important results of this study indicating that the force associated with collision of the second kind is proportional to the *velocity* of the particle $\mathcal{F}_\nu(t) \propto \dot\zeta_\nu(t)$ and in case of population inversion its coefficient is negative, resembling a "negative fiction" force.

IV. EQUATION OF MOTION

The effective impact of the Penning trap on the particles is represented by an ideal harmonic oscillator, its force being proportional to the displacement, $\Omega^2\zeta_\nu$. This is a first order approximation accounting for the fact that the voltage V_0 on the central anode determines the oscillating frequency namely, $\Omega = (2c/L)\sqrt{2eV_0/mc^2}$; L is the length of the trap. In addition, the elastic collisions of the electrons with remnant gas are represented by a decay time $\tau_{\rm scatt}$ implying that the equation of motion of the $\nu^{\rm th}$ macroparticle is

$$\frac{d^2\zeta_\nu}{dt^2} + \left(\frac{2}{\tau_\nu^{(\rm csk)}} + \frac{2}{\tau_{\rm scatt}}\right)\frac{d\zeta_\nu}{dt} + \Omega^2\zeta_\nu = 0, \tag{29}$$

wherein $\tau_\nu^{(\rm csk)}$ represents the "decay time" associated with *collisions of the second kind*,

$$\begin{aligned}\frac{2}{\tau_\nu^{(\rm csk)}} &\equiv -\sum\nolimits_i \frac{1}{\theta_i}\left\langle \cos\left[\omega_{0,i}(t_\nu - t_\mu)\right]\right. \\ &\times \left.\exp\left[-\frac{\omega_{0,i}}{Q_i}(t_\nu - t_\mu)\right] h(t_\nu - t_\mu)\right\rangle_\mu \\ \frac{1}{\theta_i} &\equiv \frac{1}{T_{2,i}}\frac{\Delta n_i \sigma_{21,i} r_e N_{\rm mp} N_{\rm el}}{(\varepsilon_r/2\pi^2)(2eV_0/mc^2)^{3/2}}\,{\rm sinc}^2\left(\frac{1}{2}\omega_{0,i}T_{\rm mp}\right). \end{aligned} \tag{30}$$

$r_e \equiv e^2/4\pi\varepsilon_0 mc^2 \simeq 2.8 \times 10^{-15}$ m is the classical radius of the electron and $T_{\rm mp}$ is the macroparticle duration. Clearly, this parameter is *negative* in case of population inversion. For establishing this quantity in terms of the particles' location in the phase space we must bear in mind that in the framework of one-round-trip approximation we have

$$\left.\begin{aligned}\zeta_\nu &= \frac{L}{2}\{1 + \cos[\Omega(t - t_\nu)]\} \\ \dot\zeta_\nu &= -\frac{\Omega L}{2}\sin[\Omega(t - t_\nu)]\end{aligned}\right\} \Rightarrow t_\nu - t_\mu$$

$$= \frac{1}{\Omega}\left[\arctan\left(\frac{\frac{-\dot\zeta_\mu}{\Omega}}{\zeta_\mu - \langle\zeta\rangle}\right) - \arctan\left(\frac{\frac{-\dot\zeta_\nu}{\Omega}}{\zeta_\nu - \langle\zeta\rangle}\right)\right] \tag{31}$$

wherein $L/2 = \langle\zeta_v\rangle$ is the amplitude of the oscillation. A typical value for the growth rate in case of a single resonance for $T_2 \sim 1$ ms, $\Delta n\,\sigma_{21} \sim 1\ \mathrm{m}^{-1}$, $N_{\rm mp}N_{\rm el} \sim 10^{12}$, $\omega_0 T_{\rm mp} \ll 1$, $V_0 \sim 200$, $\varepsilon_r \sim 3$ is $\tau^{(\rm csk)} \sim 2.5\ \mu$s. Enhancing the population inversion by an order of magnitude and the number of electrons in a bunch by two orders of magnitude, the growth rate increases by three orders of magnitude.

V. SIMULATION RESULTS

For simplicity sake, we define the effective scattering decay coefficient $1/\tau_v \equiv 1/\tau_v^{(\rm csk)} + 1/\tau_{\rm scatt}$ allowing us to solve Eq. (29) analytically subject to the condition that during one round trip in the trap, the decay parameter does not change significantly. Assuming that we know the phase-space distribution at $t=0$ then

$$\zeta_v(t\geq 0) = \left\{\zeta_v(0)\cos(\Omega t) + \frac{1}{\Omega}\left[\dot{\zeta}_v(0) + \frac{1}{\tau_v}\zeta_v(0)\right] \times \sin(\Omega t)\right\}\exp(-t/\tau_v) \qquad (32)$$

Implying that after *one* round trip, $\Delta t = T = 2\pi/\Omega$, the ensemble's phase-space coordinates are given by

$$\zeta_v(T) = \zeta_v(0)\exp(-T/\tau_v),$$
$$\dot{\zeta}_v(T) = \dot{\zeta}_v(0)\exp(-T/\tau_v). \qquad (33)$$

In all the examples that follow, we assume a *single and dominant resonance*.

A. Single-particle effect

To some extent, the analysis so far was biased by the thought that the electrons become bunched and, consequently, collective effects become dominant. Before investigating the collective effects let us briefly investigate the *single-particle* process. According to Eq. (24), the average power exchanged by one electron during one round trip is $P_{\rm ex} = W_{\rm ex}/T$ or explicitly

$$P_{\rm ex,1} \simeq \frac{(mc^2)^2}{eV_0T}\frac{\pi^3 L}{2\varepsilon_r}\frac{\Delta n\sigma_{21}r_e}{cT_2}\frac{1}{Q\omega_0 T} \simeq mc^2\frac{c}{L}\frac{\pi}{2\varepsilon_r}\frac{\Delta n\sigma_{21}r_e}{(\omega_0T_2)^2}. \qquad (34)$$

In case of $N_{\rm el}$ electrons stored in the trap, the "spontaneous" power exchanged between electrons and the active medium is the product of the single power emitted and the number of electrons

$$P_{\rm ex}^{(\rm sp)} \simeq N_{\rm el}P_{\rm ex,1} \simeq N_{\rm el}mc^2\frac{c}{L}\frac{\pi}{2\varepsilon_r}\frac{\Delta n\sigma_{21}r_e}{(\omega_0T_2)^2}. \qquad (35)$$

A typical value for the power for $\omega_0 T_2 \sim 10^2$, $\Delta n\sigma_{21} \sim 10\ \mathrm{m}^{-1}$, $N_{\rm el} \sim 10^{12}$, L=0.1 m is $P_{\rm ex}^{(\rm sp)} \simeq 10$ nW. Note that for this process to become significant we need to excite many regular states; metastable states have small contribution to this mechanism.

B. Bunching process

In the numerical simulations that follow, we trace a fraction of the ensemble that populates *one optical period* and we further assume that there is no significant difference between one optical period to another; therefore, an electron can slip from one optical period to another. This is to say that during one trap period, electrons in various optical periods experience virtually identical conditions—obviously at different instants. Consequently, the macroparticles are advanced in steps of one trap period at a time and during this period, the decay, associated with the quality factor (Q) of the medium, is ignored. Moreover, the transient associated with the buildup of the electrons' ensemble is assumed to have decayed to zero; this implies that the Heaviside step function is taken as unity. Subject to all these assumptions, the equations of motion specified earlier have the same form,

$$\frac{d^2\zeta_v}{dt^2} + \left(\frac{2}{\tau_v^{(\rm csk)}} + \frac{2}{\tau_{\rm scatt}}\right)\frac{d\zeta_v}{dt} + \Omega^2\zeta_v = 0,$$
$$\frac{2}{\tau_v^{(\rm csk)}} \equiv -\left\langle\cos\left[\mathrm{w}_0\left(t_v - t_\mu\right)\right]\right\rangle_\mu \frac{1}{T_2} \times \frac{\Delta n\sigma_{21}r_eN_{\rm mp}N_{\rm el}}{(\varepsilon_r/2\pi^2)(2eV_0/mc^2)^{3/2}}\,\mathrm{sinc}^2\left(\frac{1}{2}\omega_0T_{\rm mp}\right),$$
$$\omega_0\left(t_v - t_\mu\right) = \frac{\omega_0}{\Omega}\left[\arctan\left(\frac{\frac{-\dot{\zeta}_\mu}{\Omega}}{\zeta_\mu - \langle\zeta\rangle}\right) - \arctan\left(\frac{\frac{-\dot{\zeta}_v}{\Omega}}{\zeta_v - \langle\zeta\rangle}\right)\right],$$
$$\zeta_v(T) = \zeta_v(0)\exp(-T/\tau_v),$$
$$\dot{\zeta}_v(T) = \dot{\zeta}_v(0)\exp(-T/\tau_v). \qquad (36)$$

except that we trace only the electrons in *one optical period*. In what follows we present numerical solution of the set of equations in Eq. (36) for the parameters detailed in Table I.

TABLE I. Values of the parameters used in the simulations presented below.

T(ns)	26.5	$\Delta n(\mathrm{m})^{-3}$	1.1×10^{22}	L(cm)	10
T_2(ms)	0.24	$\sigma_{21}(\mathrm{m}^2)$	1.8×10^{-22}	$N_{\rm mp}$	360
$\lambda_0(\mu\mathrm{m})$	1.06	ε_r	3.3	$N_{\rm el}$	10^9
V_0(V)	400	$\tau_{\rm scatt}$(ms)	0.1		

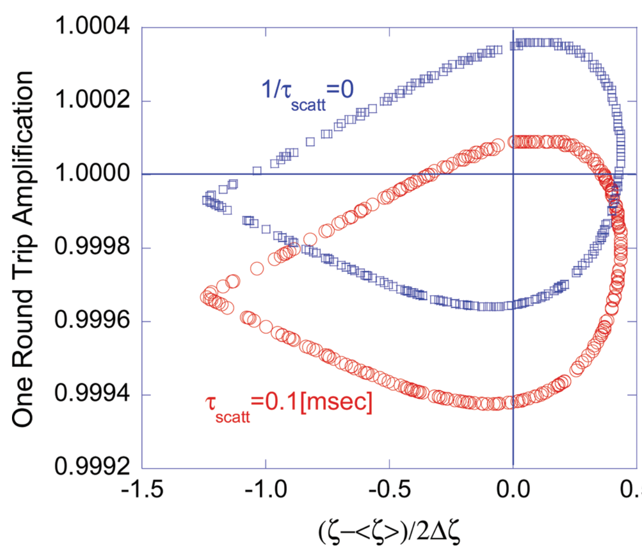

FIG. 2. (Color online) Amplification factor $\exp(-T/\tau_v)$ with and without the normal scattering included.

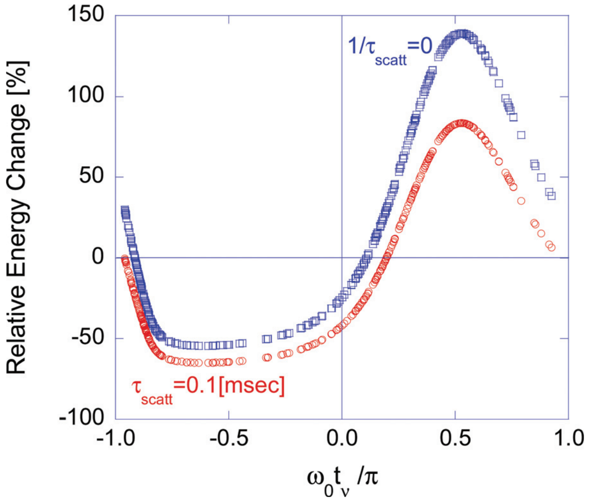

FIG. 3. (Color online) The relative change in the total energy of the electrons ensemble with and without normal scattering after 500 round-trips.

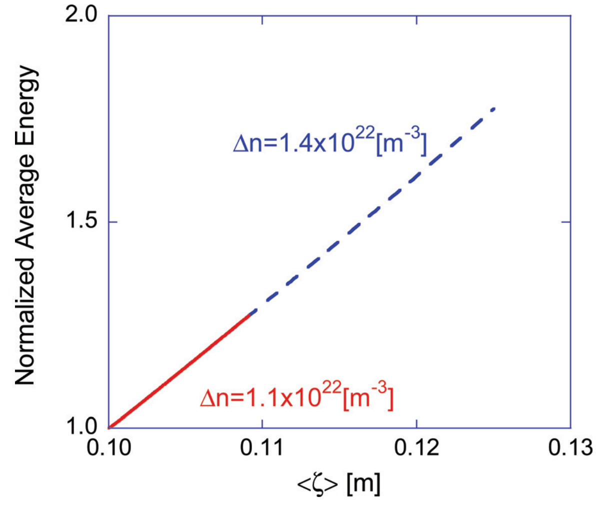

FIG. 5. (Color online) The total average energy gained by the electrons is linear in the amplitude of their oscillation.

As a starting point, we considered the amplification factor $\exp(-T/\tau_v)$ with and without the normal scattering included: in Fig. 2, we observe that in the absence of regular scattering a significant fraction of the electrons absorb energy from the medium. This fraction diminishes as the scattering effect increases. In order to envision the impact of this effect we draw in Fig. 3 the relative change in the total energy of the electrons after 500 round trips. In both cases, part of the electrons gain energy from the active medium—primarily those around phase $\pi/2$. As is expected, the energy gain is strongly dependent on the gain medium.

Figure 4 shows the phase-space for 300, 400 and 500 round trips ignoring normal scattering, as will be assumed to be the case in all simulation results that follow. In the left-frame the population inversion is assumed to be $|\Delta n| = 1.1 \times 10^{22}\ \mathrm{m}^{-3}$, whereas in the right-frame the population inversion density is by 30% higher. This increase leaves the decelerated electrons virtually unchanged but the accelerated electrons more than double their peak energy values. Since originally, these electrons are trapped, this energy excess allows the accelerated electrons to escape the trap. Evidently, their energy at the output is a function of the number of round trips the electrons are undergoing as well as the potential at the output. As collisions of the second kind enhance the energy of a fraction of the electrons, their amplitude of oscillation increases. We have already indicated that for simplicity sake, we assumed an ideal trap (harmonic oscillator) and the escape process is not included in the description of the system. Nevertheless, the energy of the electrons may be inferred by assuming that beyond a given point and at a given instant, the potential well is "turned-off" and electrons are leaving the trap with the energy specified in Fig. 4.

Figure 5 shows that the average energy increases *linearly* with the average amplitude of the electrons indicating that, in zero order, for the chosen parameters the *force* acting on the particles is independent of the oscillation amplitude. Note that the harmonic oscillator describing the trap is assumed to extend beyond $|\zeta| = L/2$. The solid line reveals the linear dependence for $|\Delta n| = 1.1 \times 10^{22}\ \mathrm{m}^{-3}$ whereas the dashed line represents a higher population inversion $|\Delta n| = 1.4 \times 10^{22}[\mathrm{m}^{-3}]$. In both cases, the slope is the same but for the same number of round trips in the trap, the

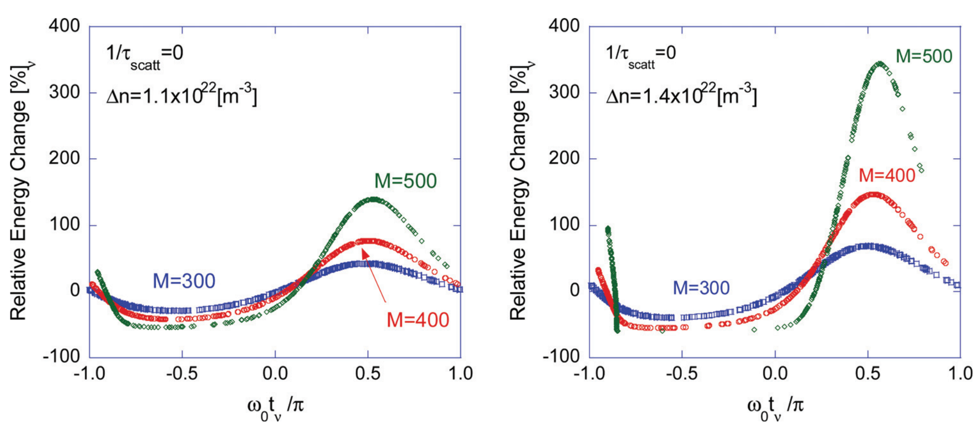

FIG. 4. (Color online) Phase-space for 300, 400 and 500 round-trips ignoring normal scattering. In the left frame, the population inversion is assumed to be $\Delta n = 1.1 \times 10^{22}\ \mathrm{m}^{-3}$, whereas in the right frame the population inversion density is 30% higher.

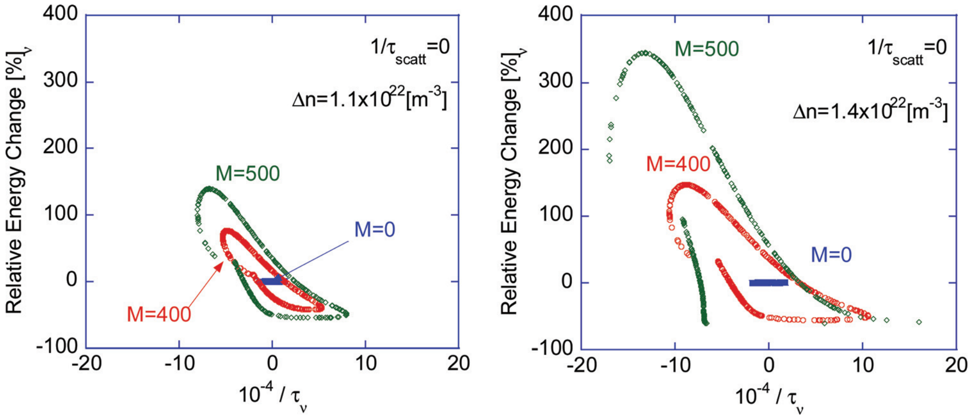

FIG. 6. (Color online) Relative change in the total energy of the ν^{th} macroparticle and the corresponding second kind collision time. The difference between the two frames is the population density.

amplitude of the oscillation is significantly affected by the population inversion as does the total energy (potential and kinetic).

Let us now focus our attention on the phase space, investigating the behavior of each one of the macroparticles. Figure 6 shows the relative change in the total energy of the ν^{th} macroparticle and the corresponding coefficient of collision of the second kind. The difference between the two frames is the population density and we observe that if τ_ν is negative, the relative energy change is positive and may be larger than 300%. However, the system may reach saturation. This to say that electrons that originally have been accelerated, may be decelerated and vice versa. In spite of the fact that overall, the ensemble accelerates, a small fraction of electrons experience a negative friction, yet the relative change in energy is negative.

Finally, we have examined the relation between the average and the standard deviation of the energy of the electrons, as a function of $\langle 1/\tau \rangle$ and $\Delta(1/\tau)$ in each round trip. We illustrate the result of the numerical solution of the system's equations in Fig. 7 and the main trends are summarized in Table II. Two features are evident: first the fact that in zero order, the average energy gained by the electrons is proportional to $\langle 1/\tau \rangle \equiv \langle 1/\tau_\nu^{(csk)} \rangle$ and second, that the standard deviation of the energy is linear with $\Delta(1/\tau) \equiv \sqrt{\langle (1/\tau_\nu^{(csk)})^2 \rangle - \langle 1/\tau_\nu^{(csk)} \rangle^2}$. Less intuitive is the fact that for low standard deviation, the average energy is

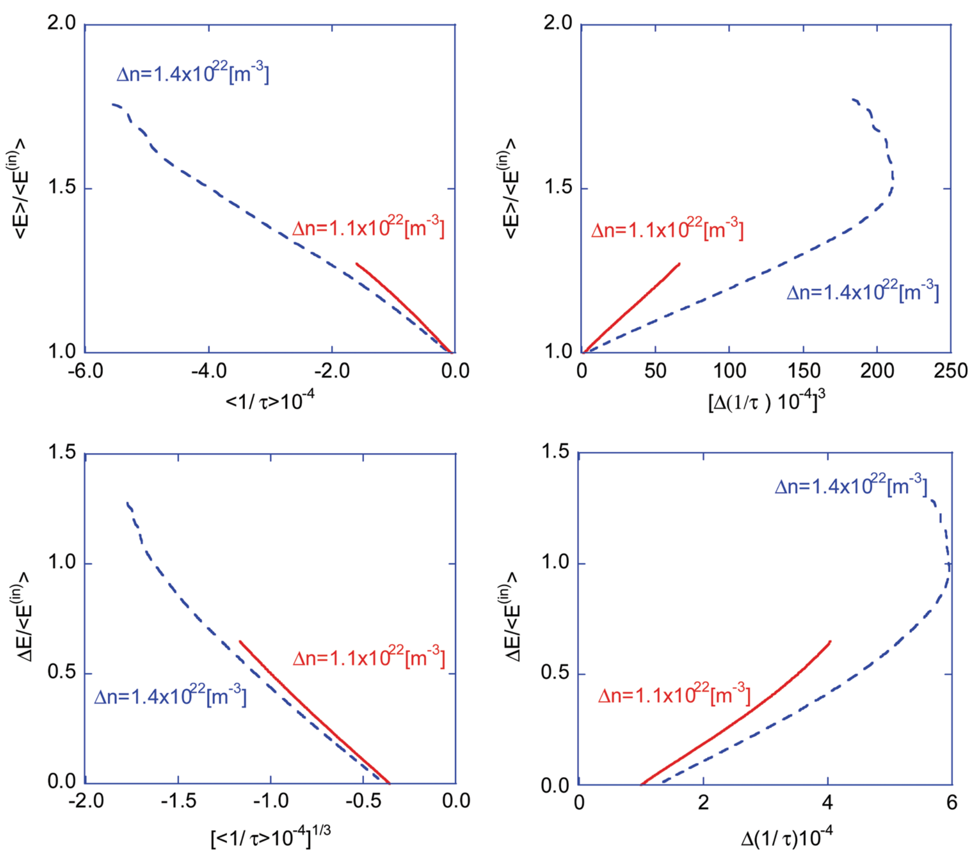

FIG. 7. (Color online) Average and standard deviation of the energy of the electrons, as a function of $\langle 1/\tau \rangle$ and $\Delta(1/\tau)$.

TABLE II. Average and standard deviation energy of the electrons as a function of the collision of the second kind coefficient.

$\frac{\langle E\rangle_n}{\langle E\rangle_1} = \left[1 - ()\langle\frac{1}{\tau}\rangle\right]_n$	$\frac{\langle E\rangle_n}{\langle E\rangle_1} = \left\{1 + ()\left[\Delta(\frac{1}{\tau})\right]^3\right\}_n$
$\frac{\Delta\langle E\rangle_n}{\langle E\rangle_1} = \left[() - ()\langle\frac{1}{\tau}\rangle^{1/3}\right]_n$	$\frac{\Delta\langle E\rangle_n}{\langle E\rangle_1} = \left[() + ()\Delta(\frac{1}{\tau})\right]_n$

proportional to $\Delta^3(1/\tau)$ as well as the fact that the standard deviation of the energy is linear on the term $\langle 1/\tau\rangle^{1/3}$.

VI. CONCLUSIONS AND DISCUSSION

Before summarizing the main results of the present analysis, let us first recapitulate the main assumptions:

(i) The dielectric function representing the material has a single resonance.
(ii) Electron dynamics is confined to one dimension and its velocity is below Cerenkov velocity.
(iii) Linear regime of operation—negligible change in the population inversion.
(iv) Stimulated emission cross section for regular photons is identical to that of virtual photons.
(v) Parasitic effects (e.g., ionization) of electrons on the active medium are ignored; except if otherwise specified, collisions of the first kind are ignored in simulations.
(vi) For the numeric simulations, the characteristic parameters of Nd:YAG are considered.
(vii) The entire system has azimuthal symmetry.

With the model built based on these assumptions, we have demonstrated that electrons oscillating in a Penning trap may become bunched at the resonant frequency of the active medium. During multiple round trips in the trap, the bunched electrons gain energy and therefore, they may escape the trap forming an optical injector. Specifically, the main findings are:

(i) In the absence of regular scattering, a significant fraction of the electrons absorb energy from the medium; this fraction diminishes as the scattering effect increases, Fig. 2.
(ii) After many round trips (500) of the electrons in the Penning trap, electrons get bunched at the resonant frequency of the medium (Fig. 3). Collisions of the first kind suppress somewhat the relative energy change but do not avoid the bunching for typical values of the scattering lifetime (0.1ms).
(iii) The longer the electrons are trapped, they gain more energy from the medium (Fig. 4). At the same time, the energy of the decelerated electrons varies only slowly. For the parameters employed, a rough approximation of *peak* relative energy change is given by $\sim 100\{\exp[3(\Delta n/10^{22})^2(M/10^3)^2] - 1\}$. This result has limited validity since the population inversion is assumed to be unchanged. Obviously, while the energy depletion is accounted for, this peak relative energy change reaches saturation.
(iv) For all practical purposes, if collisions are ignored, the Penning trap may be conceived as a harmonic oscillator; therefore, the total energy (kinetic and potential) is constant. Our analysis has demonstrated that when collisions of the second kind were accounted for, this energy increases *linearly* with average-amplitude the electrons are oscillating. From this result it can be inferred that the average accelerating force exerted by the active medium on the *ensemble* of electrons, is constant.
(v) Similar to the dynamics of electrons in regular accelerators or in traveling wave tubes or free electron lasers, as the interaction reaches saturation, energy flow changes direction. If a specific (macro) particle was originally accelerated, implying that energy was transferred from the medium/radiation to the electrons, once saturation is reached, electrons are decelerated. Therefore, energy is transferred from the electrons to the field or medium.
(vi) During each round trip, the average energy gained is proportional to the population inversion $(\propto \langle 1/\tau_v^{(csk)}\rangle_v)$.

The above-described process resembles SASE in the case of lasers. Starting with a uniformly distributed beam, the active medium causes the beam to become bunched and as they get bunched, the energy extraction is enhanced and so is the bunching. While the current analysis is motivated by a low energy (nonrelativistic) injector, the concept is applicable to relativistic circular machines such as cyclotron or betatron by replacing the (rf) accelerating units with a gaseous active medium. An intermediary energy configuration has been suggested by Kimura *et. al.*[10] and it employs excimer (ArF) as an active medium.

ACKNOWLEDGMENTS

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