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Handout 6 **Codes on Graphs and Iterative Decoding**
Solution of Homework 4 Dr. Igal Sason

Problem 1. This problem suggests a way to construct capacity-achieving sequences of ensembles of LDPC codes, where their transmission takes place on a binary erasure channel (BEC) and iterative message-passing decoding is performed.

(a) Based on the equality

$$\hat{\lambda}_\alpha\Big(1-\rho(1-x)\Big) = x \quad \forall\, x \in [0,1) \tag{1}$$

we obtain that

$$\hat{\lambda}_{\alpha,N}\Big(1-\rho(1-x)\Big) \leq x \quad \forall\, x \in [0,1), \tag{2}$$

where Eq. (2) follows from Eq. (1) and since $\hat{\lambda}_\alpha(\cdot) \in \mathcal{D}$. From the definition of the normalized function $\lambda_{\alpha,N}(x) = \frac{\hat{\lambda}_{\alpha,N}(x)}{\hat{\lambda}_{\alpha,N}(1)}$, then

$$\hat{\lambda}_{\alpha,N}(1) \cdot \lambda_{\alpha,N}\Big(1-\rho(1-x)\Big) \leq x \quad \forall\, x \in [0,1) \tag{3}$$

so it follows that $p^{\text{IT}}(\alpha,N) \geq \hat{\lambda}_{\alpha,N}(1)$. This result provides a lower bound on the threshold under iterative message-passing decoding.

(b) The design rate of the ensemble of $(n, \lambda_{\alpha,N}, \rho_\alpha)$ LDPC codes is

$$\begin{aligned} R(\alpha,N) &= 1 - \frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \lambda_{\alpha,N}(x)\,\mathrm{d}x} \\ &= 1 - \hat{\lambda}_{\alpha,N}(1) \cdot \frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \hat{\lambda}_{\alpha,N}(x)\,\mathrm{d}x}\,. \end{aligned}$$

It follows that the inequality

$$\frac{\hat{\lambda}_{\alpha,N}(1)}{1-\hat{\lambda}_{\alpha,N}(1)} \left[\frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \hat{\lambda}_{\alpha,N}(x)\,\mathrm{d}x} - 1\right] \leq \varepsilon$$

is equivalent to

$$\frac{1-\hat{\lambda}_{\alpha,N}(1) - R(\alpha,N)}{1-\hat{\lambda}_{\alpha,N}(1)} \leq \varepsilon \tag{4}$$

and the latter inequality yields that $R(\alpha,N) \geq (1-\varepsilon)\Big(1-\hat{\lambda}_{\alpha,N}(1)\Big)$. From item (a), since $p^{\text{IT}}(\alpha,N) \geq \hat{\lambda}_{\alpha,N}(1)$, then it implies that $R(\alpha,N) \geq (1-\varepsilon)(1-p^{\text{IT}}(\alpha,N))$. Therefore, the design rate of the ensemble of $(n, \lambda_{\alpha,N}, \rho_\alpha)$ LDPC codes is at least a fraction $1-\varepsilon$ of the capacity of a BEC with erasure probability $p^{\text{IT}}(\alpha,N)$.

(c) Based on Eq. (4), the equality

$$\lim_{N\to\infty}\left(1-\hat{\lambda}_{\alpha,N}(1)\cdot\frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \hat{\lambda}_{\alpha,N}(x)\,\mathrm{d}x}\right)=1-p$$

is equivalent to the requirement that the design rate of the ensemble of $(n, \lambda_{\alpha,N}, \rho_{\alpha,N})$ LDPC codes converges to $1-p$ as $N \to \infty$.
Let $\varepsilon(\alpha, N)$ designate the gap to capacity, i.e., $R(\alpha, N) = (1-p^{\mathrm{IT}}(\alpha, N))\,(1-\varepsilon(\alpha, N))$. From item (a), since $p^{\mathrm{IT}}(\alpha, N) \geq \hat{\lambda}_{\alpha,N}(1)$, then it follows that

$$\varepsilon(\alpha, N) \leq \frac{1-\hat{\lambda}_{\alpha,N}(1)-R(\alpha,N)}{1-\hat{\lambda}_{\alpha,N}(1)} = \frac{\hat{\lambda}_{\alpha,N}(1)}{1-\hat{\lambda}_{\alpha,N}(1)}\left[\frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \hat{\lambda}_{\alpha,N}(x)\,\mathrm{d}x}-1\right].$$

According to the second requirement that

$$\lim_{N\to\infty}\frac{\hat{\lambda}_{\alpha,N}(1)}{1-\hat{\lambda}_{\alpha,N}(1)}\left[\frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \hat{\lambda}_{\alpha,N}(x)\,\mathrm{d}x}-1\right]=0$$

it follows that

$$\lim_{N\to\infty}\varepsilon(\alpha,N)=0,$$

so the gap (in rate) to capacity vanishes as $N \to \infty$.

Problem 2. This problem refers to the best capacity-achieving sequence of ensembles of LDPC codes which is currently known for the BEC.

(a) It can be easily verified that the functions

$$\hat{\lambda}_\alpha(x) = 1-(1-x)^\alpha\,, \quad \rho_\alpha(x) = x^{\frac{1}{\alpha}}\,, \quad \alpha \neq 0 \tag{5}$$

satisfy the equality $\hat{\lambda}_\alpha\Big(1-\rho_\alpha(1-x)\Big) = x$ for $x \in [0,1]$, and $\rho_\alpha(1) = 1$.
If $0 < \alpha < 1$ and k is a positive integer, then

$$\begin{aligned}\binom{\alpha}{k} &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\\[2ex] &= \frac{(-1)^{k-1}\,\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!}\\[2ex] \Rightarrow\ (-1)^{k+1}\binom{\alpha}{k} &= \frac{\alpha(1-\alpha)\cdots(k-1-\alpha)}{k!} > 0.\end{aligned}$$

Therefore, $\hat{\lambda}_\alpha(\cdot)$ has a non-negative power series expansion for $0 < \alpha < 1$ (and clearly this is also the case for $\rho_\alpha(\cdot)$). Since

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \quad \forall\, |x| < 1,$$

then

$$\hat{\lambda}_{\alpha,N}(x) = \sum_{k=1}^{N}(-1)^{k+1}\binom{\alpha}{k}x^k\,. \tag{6}$$

and the corresponding normalized function is

$$\lambda_{\alpha,N}(x) = \frac{\sum_{k=1}^{N}(-1)^{k+1}\binom{\alpha}{k}x^k}{\sum_{k=1}^{N}(-1)^{k+1}\binom{\alpha}{k}}\,. \tag{7}$$

(b) We will prove now two equalities that will be useful later.

LEMMA 1.

$$\sum_{k=1}^{N} \frac{(-1)^{k+1}}{k+1}\binom{\alpha}{k} = \frac{\alpha - \binom{\alpha}{N+1}(-1)^N}{\alpha+1}\ , \quad N = 1, 2, \cdots\ . \tag{8}$$

LEMMA 2.

$$\sum_{k=1}^{N} (-1)^{k+1}\binom{\alpha}{k} = 1 - \frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N\ , \quad N = 1, 2, \cdots\ . \tag{9}$$

Proof. The proof of both lemmas relies on mathematical induction.

– *Poof of Lemma 1*: For $N = 1$, it is easy to verify that both sides of Eq. (8) are equal to $\frac{\alpha}{2}$, so the equality is valid. Assume that Eq. (8) is correct for a certain integer N, then

$$\begin{aligned} & \textstyle\sum_{k=1}^{N+1} \frac{(-1)^{k+1}}{k+1}\binom{\alpha}{k} \\ = \ & \tfrac{\alpha-\binom{\alpha}{N+1}(-1)^N}{\alpha+1} + \tfrac{(-1)^{N+2}}{N+2}\binom{\alpha}{N+1} \\ = \ & \tfrac{\alpha}{\alpha+1} - (-1)^N\binom{\alpha}{N+1}\left(\tfrac{1}{\alpha+1} - \tfrac{1}{N+2}\right) \\ = \ & \tfrac{\alpha}{\alpha+1} + (-1)^N\binom{\alpha}{N+1}\tfrac{\alpha-N-1}{(\alpha+1)(N+2)} \\ = \ & \tfrac{\alpha}{\alpha+1} + (-1)^N\ \tfrac{\alpha(\alpha-1)...(\alpha-N)}{(N+1)!}\tfrac{\alpha-N-1}{(\alpha+1)(N+2)} \\ = \ & \tfrac{\alpha}{\alpha+1} + \tfrac{(-1)^N}{\alpha+1}\tfrac{\alpha(\alpha-1)...(\alpha-N)(\alpha-N-1)}{(N+2)!} \\ = \ & \tfrac{\alpha}{\alpha+1} + \tfrac{(-1)^N}{\alpha+1}\binom{\alpha}{N+2} \\ = \ & \tfrac{\alpha-\binom{\alpha}{N+2}(-1)^{N+1}}{\alpha+1} \end{aligned}$$

so Eq. (8) is also valid for $N + 1$. By mathematical induction, this proves the validity of Eq. (8) for all positive integers.

– *Proof of Lemma 2*: For $N = 1$, it is easily verified that both sides of Eq. (9) are equal to α, so this equality is valid. Assume that Eq. (9) is correct for a certain integer N, then

$$\begin{aligned} & \textstyle\sum_{k=1}^{N+1} (-1)^{k+1}\binom{\alpha}{k} \\ = \ & \left[1 - \tfrac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N\right] + (-1)^{N+2}\binom{\alpha}{N+1} \\ = \ & 1 + (-1)^N\binom{\alpha}{N+1}\left(1 - \tfrac{N+1}{\alpha}\right) \\ = \ & 1 + (-1)^N\ \tfrac{\alpha(\alpha-1)...(\alpha-N)}{(N+1)!}\ \tfrac{\alpha-N-1}{\alpha} \\ = \ & 1 + (-1)^N\ \tfrac{\alpha(\alpha-1)...(\alpha-N)(\alpha-N-1)}{(N+2)!}\ \tfrac{N+2}{\alpha} \\ = \ & 1 - (-1)^{N+1}\binom{\alpha}{N+2}\tfrac{N+2}{\alpha} \end{aligned}$$

so Eq. (9) is also valid for $N + 1$. This proves that Eq. (9) is satisfied for all positive integers.

□

(c) Based on Eqs. (7) and (9), we obtain that

$$\lambda_{\alpha,N}(x) = \frac{\sum_{k=1}^{N}(-1)^{k+1}\binom{\alpha}{k}x^k}{1-\frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N}\,. \tag{10}$$

The design rate of the ensemble of $(n, \lambda_{\alpha,N}, \rho_\alpha)$ LDPC codes is

$$\begin{aligned} R(\alpha,N) &= 1-\frac{\int_0^1 \rho_\alpha(x)\,\mathrm{d}x}{\int_0^1 \lambda_{\alpha,N}(x)\,\mathrm{d}x} \\ &= 1-\frac{\int_0^1 x^{\frac{1}{\alpha}}\,\mathrm{d}x}{\frac{\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k+1}\binom{\alpha}{k}}{1-\frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N}} \\ &\stackrel{(a)}{=} 1-\frac{\alpha}{\alpha+1}\,\frac{1-\frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N}{\frac{\alpha-\binom{\alpha}{N+1}(-1)^N}{\alpha+1}} \\ &= 1-\frac{\alpha-(N+1)\binom{\alpha}{N+1}(-1)^N}{\alpha-\binom{\alpha}{N+1}(-1)^N} \\ &= \frac{N\binom{\alpha}{N+1}(-1)^N}{\alpha-\binom{\alpha}{N+1}(-1)^N} \end{aligned}$$

where equality (a) follows from Eq. (8).

(d) Based on Eqs. (6) and (9), we obtain that

$$\hat{\lambda}_{\alpha,N}(1) = 1-\frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N. \tag{11}$$

From Eq. (11), it can be readily verified that the sufficient condition which was given in item (b) of Problem 1 is equivalent to the inequality

$$\frac{1-\frac{1}{N+1}}{1-\frac{1}{\alpha}\binom{\alpha}{N+1}\,(-1)^N} \geq 1-\varepsilon. \tag{12}$$

(e)

$$\begin{aligned} & (-1)^N\binom{\alpha}{N+1} \\ = \;& (-1)^N\,\frac{\alpha(\alpha-1)\,\ldots\,(\alpha-N+1)(\alpha-N)}{(N+1)!} \\ = \;& \frac{\alpha}{N+1}\;\frac{(1-\alpha)\ldots(N-1-\alpha)(N-\alpha)}{N!} \\ = \;& \frac{\alpha}{N+1}\cdot(1-\alpha)\ldots(1-\frac{\alpha}{N-1})(1-\frac{\alpha}{N}). \end{aligned}$$

$$\Rightarrow\; \ln\left((-1)^N\binom{\alpha}{N+1}\right) = \ln\left(\frac{\alpha}{N+1}\right)+\sum_{k=1}^{N}\ln\left(1-\frac{\alpha}{k}\right).$$

Since

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \qquad -1 \le x < 1$$

and since the latter power series absolutely converges for $|x| < 1$, then the substitution $x = \frac{\alpha}{k}$ for $0 < \alpha < 1$ and for positive integers k yields that

$$\begin{aligned} \ln\left((-1)^N \binom{\alpha}{N+1}\right) &= \ln\left(\tfrac{\alpha}{N+1}\right) + \textstyle\sum_{k=1}^N \left\{-\tfrac{\alpha}{k} - \tfrac{1}{2}\tfrac{\alpha^2}{k^2} - \tfrac{1}{3}\tfrac{\alpha^3}{k^3}\cdots\right\} \\ &= \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha \textstyle\sum_{k=1}^N \tfrac{1}{k} - \tfrac{\alpha^2}{2}\textstyle\sum_{k=1}^N \tfrac{1}{k^2} - \tfrac{\alpha^3}{3}\textstyle\sum_{k=1}^N \tfrac{1}{k^3} - \cdots \\ &= \ln\left(\tfrac{\alpha}{N+1}\right) - \textstyle\sum_{p=1}^\infty \left\{\tfrac{\alpha^p}{p}\textstyle\sum_{k=1}^N \tfrac{1}{k^p}\right\}. \end{aligned}$$

(f) For every positive integer n

$$a_n - a_{n-1} = \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right).$$

Since

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \qquad -1 \le x < 1$$

which implies that

$$a_n - a_{n-1} = -\frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \dots \,. \tag{13}$$

This implies that the sequence $\{a_n\}_{n=1}^\infty$ is monotonic decreasing. Since the function $f(x) = \frac{1}{x}$ is monotonic decreasing and is positive in the interval $[1, \infty)$, then

$$\begin{aligned} \textstyle\sum_{k=1}^n \tfrac{1}{k} &> \textstyle\sum_{k=1}^{n-1} \tfrac{1}{k} \\ &= \textstyle\sum_{k=1}^{n-1} f(k) \\ &> \textstyle\int_1^n f(x)\,\mathrm{d}x \\ &= \textstyle\int_1^n \tfrac{\mathrm{d}x}{x} = \ln(n) \end{aligned}$$

so the sequence $\{a_n\}_{n=1}^\infty$ is also positive. Since the sequence $\{a_n\}_{n=1}^\infty$ is monotonic decreasing and bounded, then it has a limit which we designate by γ (Euler's constant). Moreover, based on Eq. (13), we obtain that

$$\begin{aligned} a_{n-1} - a_n &= \tfrac{1}{2n^2} + \tfrac{1}{3n^3} + \tfrac{1}{4n^4} + \dots \\ &< \tfrac{1}{2n^2} + \tfrac{1}{2n^3} + \tfrac{1}{2n^4} \dots \\ &= \tfrac{1}{2n^2}\left(1 + \tfrac{1}{n} + \tfrac{1}{n^2} + \dots\right) \\ &= \tfrac{1}{2n^2}\,\tfrac{1}{1-\frac{1}{n}} \\ &= \tfrac{1}{2n(n-1)} \end{aligned}$$

so

$$\begin{aligned} a_n &= \lim_{k\to\infty} a_{k+n} + \textstyle\sum_{k=1}^\infty (a_{n+k-1} - a_{n+k}) \\ &= \gamma + \textstyle\sum_{k=1}^\infty (a_{n+k-1} - a_{n+k}) \\ &< \gamma + \textstyle\sum_{k=1}^\infty \tfrac{1}{2(n+k)(n+k-1)} \\ &= \gamma + \tfrac{1}{2}\textstyle\sum_{k=1}^\infty \left(\tfrac{1}{n+k-1} - \tfrac{1}{n+k}\right) \\ &= \gamma + \tfrac{1}{2n}. \end{aligned}$$

Since the sequence $\{a_n\}_{n=1}^{\infty}$ is monotonic decreasing then we obtain that $a_n > \gamma$ for all positive integers n. From the latter two inequalities, it follows that

$$\ln(N) + \gamma < \sum_{k=1}^{N} \frac{1}{k} < \ln(N) + \gamma + \frac{1}{2N} \qquad N = 1, 2, \dots \tag{14}$$

where γ is Euler's constant ($\gamma \approx 0.5772$ which follows from Eq. (14) by calculating a_n for $n = 10^4$).

(g) From Eq. (14) and item (e), it follows that for $0 < \alpha < 1$ and for positive integers N

$$\begin{aligned}
\ln\Bigl((-1)^N \tbinom{\alpha}{N+1}\Bigr) &< \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha \textstyle\sum_{k=1}^{N} \frac{1}{k} \\
&= \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha\Bigl(\textstyle\sum_{k=1}^{N+1} \frac{1}{k} - \frac{1}{N+1}\Bigr) \\
&< \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha\Bigl(\ln(N+1) + \gamma - \tfrac{1}{N+1}\Bigr) \\
&= \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha \ln(N+1) + \alpha\left(\tfrac{1}{N+1} - \gamma\right) \\
&\leq \ln\Bigl(\tfrac{\alpha}{(N+1)^{\alpha+1}}\Bigr) + \alpha\left(\tfrac{1}{2} - \gamma\right) \\
&< \ln\Bigl(\tfrac{\alpha}{(N+1)^{\alpha+1}}\Bigr)
\end{aligned}$$

which yields the inequality

$$(-1)^N \binom{\alpha}{N+1} < \frac{\alpha}{(N+1)^{\alpha+1}} \qquad \forall\, 0 < \alpha < 1,\ N = 1, 2, \dots\ . \tag{15}$$

Moreover, it follows from item (e) that for $0 < \alpha < 1$

$$\begin{aligned}
\ln\Bigl((-1)^N \tbinom{\alpha}{N+1}\Bigr) &\overset{\text{(a)}}{>} \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha \textstyle\sum_{k=1}^{N} \frac{1}{k} - \Bigl(\frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \frac{\alpha^4}{4} + \cdots\Bigr) \sum_{k=1}^{N} \frac{1}{k^2} \\
&\overset{\text{(b)}}{>} \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha \textstyle\sum_{k=1}^{N} \frac{1}{k} - \frac{\pi^2}{6} \sum_{k=2}^{\infty} \frac{\alpha^k}{k} \\
&= \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha\left(\textstyle\sum_{k=1}^{N+1} \frac{1}{k} - \frac{1}{N+1}\right) - \frac{\pi^2}{6} \sum_{k=2}^{\infty} \frac{\alpha^k}{k} \\
&\overset{\text{(c)}}{=} \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha\left(\textstyle\sum_{k=1}^{N+1} \frac{1}{k} - \frac{1}{N+1}\right) + \frac{\pi^2}{6}\left[\ln(1-\alpha) + \alpha\right] \\
&\overset{\text{(d)}}{>} \ln\left(\tfrac{\alpha}{N+1}\right) - \alpha\Bigl[\Bigl(\ln(N+1) + \gamma + \tfrac{1}{2(N+1)}\Bigr) - \tfrac{1}{N+1}\Bigr] + \tfrac{\pi^2}{6}\left[\ln(1-\alpha) + \alpha\right] \\
&= \ln\left(\tfrac{\alpha}{(N+1)^{\alpha+1}}\right) + \alpha\left(\tfrac{\pi^2}{6} - \gamma\right) + \tfrac{\alpha}{2(N+1)} + \tfrac{\pi^2}{6} \ln(1-\alpha) \\
&> \ln\left(\tfrac{\alpha}{(N+1)^{\alpha+1}}\right) + \alpha\left(\tfrac{\pi^2}{6} - \gamma\right) + \tfrac{\pi^2}{6} \ln(1-\alpha)
\end{aligned}$$

where inequality (a) follows directly from item (e) and since $\sum_{k=1}^{N} \frac{1}{k^p} < \sum_{k=1}^{N} \frac{1}{k^2}$ for $p > 2$ and $N \geq 1$, inequality (b) follows from the equality $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, equality (c) follows from the Taylor series of $\ln(1-x)$ for $-1 \leq x < 1$, and inequality (d) is based on (14). By exponentiating the first and the last terms in the above inequality, we obtain that for $0 < \alpha < 1$ and for positive integers N

$$(-1)^N \binom{\alpha}{N+1} > \frac{\alpha\, c(\alpha)}{(N+1)^{\alpha+1}} \qquad 0 < \alpha < 1,\ N = 1, 2, \dots \tag{16}$$

where $c(\alpha) = (1-\alpha)^{\frac{\pi^2}{6}}\, e^{\alpha\left(\frac{\pi^2}{6} - \gamma\right)}$ for $0 < \alpha < 1$. The combination of Eqs. (15) and (16) completes the proof.

(h) From the definition of the function $c(\cdot)$ in item (g), we obtain that for $0 < \alpha < 1$

$$\begin{aligned} & \ln c(\alpha) = \tfrac{\pi^2}{6} \ln(1-\alpha) + \alpha \left(\tfrac{\pi^2}{6} - \gamma \right) \\ \Rightarrow \;& \tfrac{c'(\alpha)}{c(\alpha)} = -\tfrac{\alpha \pi^2}{6(1-\alpha)} - \gamma \\ \Rightarrow \;& c'(\alpha) = -\left(\tfrac{\alpha\pi^2}{6(1-\alpha)} + \gamma \right) \cdot c(\alpha). \end{aligned}$$

This implies that the function $c(\cdot)$ is monotonic decreasing in the interval $(0, 1)$ (since its derivative is negative over this interval), and therefore $c(\alpha) > c(p)$ for $0 < \alpha < p$.

(i) Based on items (c) and (g), we obtain that

$$\begin{aligned} R(\alpha, N) &= \frac{N\binom{\alpha}{N+1}(-1)^N}{\alpha - \binom{\alpha}{N+1}(-1)^N} \\ &\geq \frac{\frac{N\alpha\, c(\alpha)}{(N+1)^{\alpha+1}}}{\alpha - \frac{\alpha}{(N+1)^{\alpha+1}}} \\ &= \frac{N}{N+1} \frac{\frac{c(\alpha)}{(N+1)^\alpha}}{1 - \frac{1}{(N+1)^{\alpha+1}}} \\ &\overset{\text{a}}{=} \frac{N}{N+1} \frac{(1-p)\, c(\alpha)}{1 - \frac{1-p}{N+1}} \end{aligned}$$

where equality (a) relies on the choice of the parameters α and N, so that the equality $\frac{1}{(N+1)^\alpha} = 1-p$ is satisfied (p designates the erasure probability of the BEC). Moreover, from items (c) and (g), we obtain the following upper bound on the design rate of the ensemble of $(n, \lambda_{\alpha,N}, \rho_\alpha)$ LDPC codes

$$\begin{aligned} R(\alpha, N) &= \frac{N\binom{\alpha}{N+1}(-1)^N}{\alpha - \binom{\alpha}{N+1}(-1)^N} \\ &\leq \frac{\frac{N\alpha}{(N+1)^{\alpha+1}}}{\alpha - \frac{\alpha\, c(\alpha)}{(N+1)^{\alpha+1}}} \\ &< \frac{\frac{1}{(N+1)^\alpha}}{1 - \frac{c(\alpha)}{(N+1)^{\alpha+1}}} \\ &= \frac{1-p}{1 - \frac{c(\alpha)\,(1-p)}{N+1}}\,. \end{aligned}$$

In the limit where $N \to \infty$ then $\alpha \to 0$ (since $\frac{1}{(N+1)^\alpha} = 1 - p$ where $0 < p < 1$), and $\lim_{\alpha \to 0} c(\alpha) = 1$. This yields that the upper and lower bounds on $R(\alpha, N)$ tend to $1 - p$, so

$$\lim_{N \to \infty} R(\alpha, N) = 1 - p \quad \text{under the condition that } \frac{1}{(N+1)^\alpha} = 1 - p.$$

(j) Based on Eq. (16), it is possible to replace the requirement in Eq. (12) with the stronger condition

$$\frac{1 - \frac{1}{N+1}}{1 - \frac{c(\alpha)}{(N+1)^{\alpha+1}}} \geq 1 - \varepsilon. \tag{17}$$

If we also require that $0 < \alpha \leq p$, then from item (h) we obtain that $c(\alpha) \geq c(p)$, so the condition in Eq. (17) can be further strengthened to

$$\frac{1 - \frac{1}{N+1}}{1 - \frac{c(p)}{(N+1)^{\alpha+1}}} \geq 1 - \varepsilon \tag{18}$$

where the latter inequality is satisfied for

$$N \geq \left\lfloor \frac{1 - c(p) \cdot (1-p)(1-\varepsilon)}{\varepsilon} \right\rfloor. \tag{19}$$

The choice of the parameters N and α so that $\frac{1}{(N+1)^{\alpha}} = 1-p$ and the assumption above that $0 < \alpha \leq p$, require to choose N so that $\frac{1}{(N+1)^{p}} \leq 1-p$. The latter inequality is satisfied for

$$N \geq \left\lfloor (1-p)^{-\frac{1}{p}} \right\rfloor. \tag{20}$$

By combining Eqs. (19) and (20), it follows that by choosing

$$N = \max\left(\left\lfloor \frac{1 - c(p) \cdot (1-p)(1-\varepsilon)}{\varepsilon} \right\rfloor, \left\lfloor (1-p)^{-\frac{1}{p}} \right\rfloor \right) \tag{21}$$

and α so that $\frac{1}{(N+1)^{\alpha}} = 1-p$, then under iterative message-passing decoding, the ensemble of $(n, \lambda_{\alpha,N}, \rho_{\alpha})$ LDPC codes achieves asymptotically (as $n \to \infty$) a fraction which is at least $1-\varepsilon$ of the channel capacity with vanishing bit erasure probability.

(k) Since $\rho_{\alpha}(x) = x^{\frac{1}{\alpha}}$, then the right degree is constant and its value is $\frac{1}{\alpha} + 1$. Based on the choice of α so that $\frac{1}{(N+1)^{\alpha}} = 1-p$, we obtain that the right degree is

$$\begin{aligned} a_{\mathrm{R}} &= \frac{1}{\alpha} + 1 \\ &= \frac{\ln(N+1)}{\ln\left(\frac{1}{1-p}\right)} + 1 \\ &= \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln\left(\frac{1}{1-p}\right)} + \frac{\ln\left(\frac{\varepsilon(N+1)}{1-p}\right)}{\ln\left(\frac{1}{1-p}\right)}. \end{aligned}$$

Based on Eq. (21), since N scales like $\frac{1}{\varepsilon}$, then it is easy to verify that the function

$$h(\varepsilon, p) = \frac{\ln\left(\frac{\varepsilon(N+1)}{1-p}\right)}{\ln\left(\frac{1}{1-p}\right)}$$

is upper bounded by a positive function which only depends on p (but not on the gap to capacity (ε)). Moreover, in the limit where the gap to capacity vanishes

$$\lim_{\varepsilon \to 0} h(\varepsilon, p) = \frac{\ln\left(\frac{1-c(p)\,(1-p)}{1-p}\right)}{\ln\left(\frac{1}{1-p}\right)}.$$

We note that the coefficient of $\ln\left(\frac{1}{\varepsilon}\right)$ in the expression here for the right degree (a_{R}) is equal to the one which appears in the lower bound on the average right degree for capacity-achieving sequences of LDPC codes on the binary erasure channel (this lower bound was derived in class under iterative message-passing decoding), and in both cases it is equal to $\frac{1}{\ln\left(\frac{1}{R}\right)}$ where $R = 1-p$ is the asymptotic rate in item (h). This indicates on the optimality of the considered ensemble of LDPC codes, where the optimality here refers to the growth rate of the average right degree as a function of the achievable gap to capacity with vanishing bit erasure probability.
Since the iterative message-passing decoding algorithm on the binary erasure channel can be modified so that every edge is used only once (due to the absolute reliability of

the messages which are transmitted through the edges of the graph), then the decoding complexity of this modified algorithm is proportional to the average right degree in the bipartite graph. The comparison of the right degree which was calculated in this item with the lower bound on the average right degree which was derived in class yields that the decoding complexity (as a function of the gap to capacity) for the considered ensemble of LDPC codes is optimal (up to a small constant which only depends on the erasure probability of the channel, but does not depend on the gap to capacity).

(l) The stability condition for an ensemble of $(n, \lambda_{\alpha,N}, \rho_\alpha)$ LDPC codes which are transmitted on a binary erasure channel whose erasure probability is p is

$$\lambda'_{\alpha,N}(0)\rho'_\alpha(1) < \frac{1}{p} \,. \tag{22}$$

Based on Eq. (10)

$$\lambda'_{\alpha,N}(0) = \frac{\alpha}{1 - \frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N} \tag{23}$$

and since $\rho_\alpha(x) = x^{\frac{1}{\alpha}}$, then

$$\rho'_\alpha(1) = \frac{1}{\alpha}. \tag{24}$$

By substituting Eqs. (23) and (24) in the stability condition (Eq. (22)), we obtain the inequality

$$1 - \frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N > p. \tag{25}$$

We will show that if α and N are related so that $\frac{1}{(N+1)^\alpha} = 1-p$ and $0 < \alpha < 1$, then the stability condition in Eq. (25) is fulfilled. To this end, we rely on Eq. (15), so that

$$\begin{aligned}
& 1 - \frac{N+1}{\alpha}\binom{\alpha}{N+1}(-1)^N \\
> \; & 1 - \frac{N+1}{\alpha}\,\frac{\alpha}{(N+1)^{\alpha+1}} \\
= \; & 1 - \frac{1}{(N+1)^\alpha} \\
= \; & p
\end{aligned}$$

as required from the stability condition in Eq. (25).

(m) Since we consider a right-regular ensemble where the parameter α is defined so that $\rho(x) = x^{\frac{1}{\alpha}}$ is the right degree distribution, then

$$\alpha = \frac{1}{a_{\rm R} - 1} = \frac{1}{11}.$$

Based on item (c) (which yields to a closed form expression of the design rate R in terms of α and N), if $R = 0.65$ and $\alpha = \frac{1}{11}$, then we need to choose $N = 55$ (which yields a design rate of $R = 0.6499$). The left degree distribution of the LDPC ensemble is calculated by Eq. (10), and it is of the form $\lambda(x) = \sum_{i=2}^{56} \lambda_i x^{i-1}$.

Considering the case where the communications takes place over a BEC, the threshold of this ensemble under iterative message-passing decoding is calculated numerically by finding the minimum of the function

$$g(x) = \frac{x}{\lambda(1-\rho(1-x))} \quad x \in (0,1].$$

Numerical calculation of this threshold gives that $p^* = 0.3460$, so the (multiplicative) gap to capacity is

$$\varepsilon = 1 - \frac{R}{1-p^*} = 0.0063$$

which is a rather small value. Fig. 1 depicts the left degree distribution of the LDPC ensemble. The numerical values which are provided in this item are based on the MATLAB programs **design_BEC.m** and **choose.m**. The former MATLAB program enables to design right-regular ensembles of LDPC codes on the BEC, given the design rate R and the required right degree a_R of the ensemble.

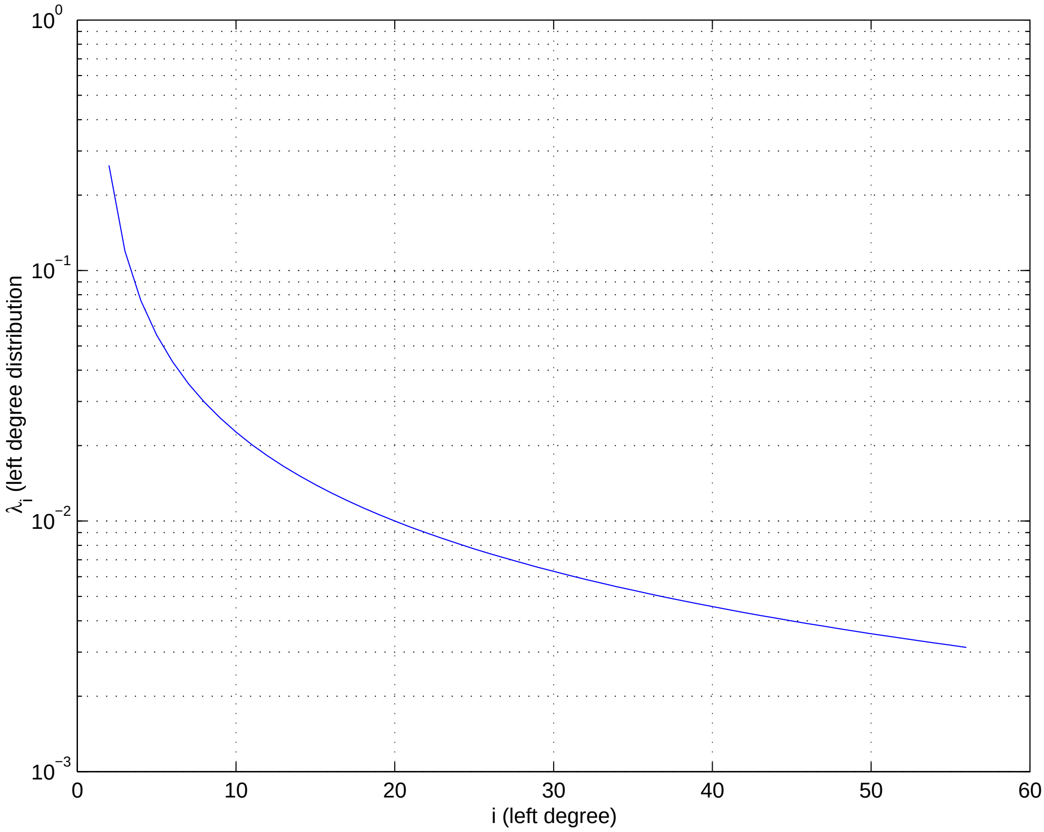

Figure 1: The left degree distribution of the designed ensemble of right-regular LDPC codes.