Information-Theoretic Lower Bounds on the Bit Error Probability of Codes on Graphs

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Abstract — **We present lower bounds on the bit error probability of binary linear block codes, and on the gap between the channel capacity and the rate of these codes for which reliable communication is achievable. The results are valid for memoryless binary-input output-symmetric (MBIOS) channels. The lower bounds provide a quantitative measure for the number of cycles of bipartite graphs which represent good error-correcting codes. The tightness of the lower bounds is especially pronounced for the binary erasure channel (BEC); to this end, we analyze a sequence of ensembles of low-density parity-check (LDPC) codes which closely approach these bounds.**

I. Introduction

If a linear block code can be represented by a cycle-free factor graph which only includes variable nodes and parity-check nodes, then maximum-likelihood soft decision decoding can be performed with low complexity [1]. Theorem 5 in [1] indicates that these graphs represent codes with very poor minimum distance. The bounds in [1] only refer to the minimum distance of cycle-free codes, and in this work we derive lower bounds on the bit error probability and the gap to capacity for codes on graphs which are represented by bipartite graphs (with or without cycles). We derive in this work information-theoretic lower bounds on the bit error probability of a binary linear code used over MBIOS channels. The bounds are expressed in terms of the *density* of an arbitrary parity-check matrix which represents the binary linear code (i.e., the number of ones in the matrix normalized per information bit), and they are valid for codes whose bipartite graphs are with or without cycles. We introduce a quantitative measure for the cycles in a bipartite graph which represents a binary linear code (where the graph only includes variable nodes and parity-check nodes). The bounds provide an interpretation for the tradeoff which exists between the bit error probability and gap to capacity of an arbitrary LDPC code (i.e., its performance limitations), and the density of an arbitrary parity-check matrix which represents the code (where the latter affects the decoding complexity per information bit and per iteration, under a message-passing iterative decoding algorithm). We present in this work quantitative results which indicate that in order to approach the channel capacity with vanishing bit error probability, LDPC codes should not have too sparse parity-check matrices, as otherwise their inherent gap to capacity becomes large. In fact, we show that the density of any parity-check matrix and the fundamental number of cycles of the graph which represents the code should increase at least like $\log \frac{1}{\varepsilon}$ where ε designates the gap to capacity under iterative message-passing decoding (see [2], [3]). The tightness of the lower bounds which are derived in this work (and whose validity is for any sequence of codes with vanishing bit error probability) is demonstrated for a general MBIOS channel, and these bounds are demonstrated to be especially tight for a BEC. The latter thing is demonstrated by a refinement of the analysis in [4] for the optimal sequence of ensembles of LDPC codes on the BEC (which is a sequence of ensembles of *right-regular* LDPC codes). Numerical results for the lower bounds on the bit erasure probability over a BEC are plotted in Fig. 1; these curves are given as a function of the normalized density of an arbitrary parity-check matrix which represents a binary linear block code (the normalized density is scaled so that it is equal to unity for cycle-free codes, and it increases with the number of cycles of the underlying graph). These curves are depicted for different values of the gap ε to capacity. The material presented here relies on [3].

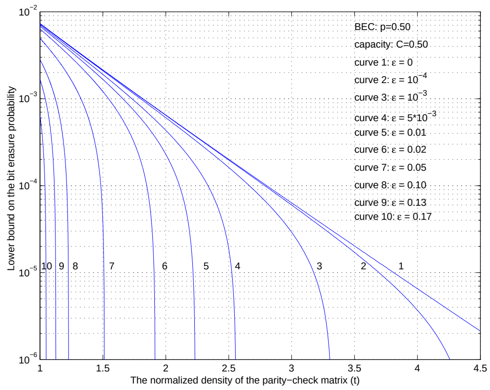

Figure 1: Lower bounds on the bit erasure probability for any binary linear code which is transmitted over a BEC. The bounds are depicted in terms of the normalized density of an arbitrary parity-check matrix which represents the code, and the curves correspond to code rates which are a fraction $1 - \varepsilon$ of the channel capacity (for different values of ε). The erasure probability of the BEC is $p = 0.500$ (which yields a capacity of one-half bits per channel use).

References

[1] T. Etzion, A. Trachtenberg and A. Vardy, "Which codes have cycle-free Tanner graphs ?," *IEEE Trans. on Information Theory*, vol. 45, no. 6, pp. 2173–2181, September 1999.

[2] A. Khandekar and R. J. McEliece, "On the complexity of reliable communications on the erasure channel", *Proocedings ISIT 2001*, p. 1, Washington, D.C., June 24–29, 2001.

[3] I. Sason and R. Urbanke, "Parity-check density versus performance of binary linear codes over memoryless symmetric channels," to appear in *IEEE Trans. on Information Theory.* [Online]. Available: http://lthiwww.epfl.ch/∼eeigal.

[4] A. Shokrollahi, "New sequences of time erasure codes approaching channel capacity," *Proceedings 13th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, Lecture Notes in Computer Science 1719, Springer Verlag, pp. 65–76, 1999.