TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering

Handout 4 — **Codes on Graphs and Iterative Decoding**

Homework 3, due **May 16, 2004.** — Dr. Igal Sason

Problem 1. Fig. 1 presents a bipartite graph of a binary linear block code.

(a) Assume the transmission takes place over a binary erasure channel (BEC), and that the received sequence is $(?, ?, ?, 1, 0, 0, 0, 0)$ (where a question mark designates an erasure). Perform iterative message-passing decoding for this case.

(b) Repeat item (b) in the case where the received sequence is equal to $(?, 1, ?, 1, ?, 1, 1, 0)$. Explain why the iterative message-passing decoder fails in the latter case. Perform maximum-likelihood (ML) decoding, and show that it is successful (find also the transmitted codeword). It exemplifies the sub-optimality of the iterative message-passing decoding algorithm, as compared to optimal ML decoding.

(c) What is the rate of this code ?

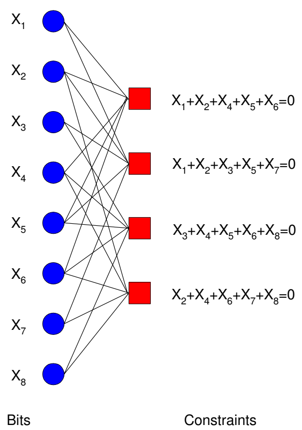

Figure 1: Tanner graph of a binary linear block code.

Problem 2. Assume that a binary $(2^m - 1, 2^m - m - 1, 3)$ Hamming code (where $m \geq 3$) is transmitted over a BEC. Show that for any erasure pattern with two erasures or less, the iterative message-passing decoding algorithm performs successfully.

Hint: The columns of an arbitrary parity-check matrix which represents a binary Hamming code of length $2^m - 1$ and dimension $2^m - m - 1$ form the set of all the $2^m - 1$ non-zero binary vectors of length m.

Problem 3. This problem refers to the gap to capacity of ensembles of LDPC codes under iterative message-passing decoding, where transmission takes place over a BEC.

(a) Prove that the threshold of an ensemble of LDPC codes with left and right degree distributions of $\lambda(\cdot)$ and $\rho(\cdot)$, respectively, is given by the minimum of the function

$$g(x) = \frac{x}{\lambda\left(1 - \rho(1 - x)\right)} \quad x \in (0, 1].$$

(b) Calculate numerically the thresholds of the ensembles of $(3, 5)$, $(6, 10)$ and $(9, 15)$ regular LDPC codes on the BEC. Compare your result with the Shannon capacity limit for the ensemble of fully random block codes of the same rate.

(c) Explain the inherent gap to capacity for the above ensembles of regular LDPC codes in item (a). Base your explanation on the derivation in class of the lower bound on the average right degree of LDPC ensembles.

Problem 4. Consider an ensemble (n, λ, ρ) of LDPC codes which is transmitted over a BEC and iteratively decoded by a message-passing algorithm. Let $p^* = p^*(\lambda, \rho)$ be the threshold of this ensemble. Clearly, we should have $R \leq 1 - p^*$, and it is natural to use the (multiplicative) gap as a measure of performance. Assume that the ensemble achieves vanishing bit erasure probability under iterative message-passing decoding, and the asymptotic rate (as $n \to \infty$) is $R = (1 - \varepsilon)(1 - p^*)$. Show that

$$\varepsilon \geq \frac{R^{a_{\mathrm{R}}-1}(1-R)}{1 + R^{a_{\mathrm{R}}-1}(1-R)}$$

where a_{R} is the average right degree of the ensemble [Hint: rely on the derivation of the lower bound on the average right degree which we obtained in class.]

Problem 5. We consider here three ensembles of (n, λ, ρ) LDPC codes with the following pairs of degree distributions.

Ensemble 1:

$$\lambda(x) = 0.292908x + 0.103672x^2 + 0.212223x^3 + 0.391197x^{11} \qquad \rho(x) = x^{14}$$

Ensemble 2:

$$\lambda(x) = 0.066393x + 0.600469x^2 + 0.333138x^9 \qquad \rho(x) = x^{14}$$

Ensemble 3:

$$\lambda(x) = 0.133275x + 0.485814x^2 + 0.380911x^9 \qquad \rho(x) = x^{14}.$$

Assume that the communications takes place over a BEC, and the codes are iteratively decoded by a message-passing algorithm.

(a) Show that these ensembles have the same design rate R, and calculate it.

(b) Show that in the limit where $n \to \infty$, the number of edges in the bipartite graphs of typical codes from these ensembles is the same, and calculate its common value for large enough values of n.

(c) Calculate the *decoding complexity per information bit* for these LDPC ensembles in the limit where $n \to \infty$.

(d) Calculate a lower bound on the gap to capacity ε for ensembles of LDPC codes with the design rate R which was calculated in item (a) and the common right degree distribution of these ensembles. *Hint*: rely on Problem 4.

(e) Calculate the thresholds and the gap to capacity of these ensembles.

(f) Rank these LDPC ensembles by taking into account their asymptotic complexity per information bit under iterative message-passing decoding, and also their asymptotic gap to capacity. Explain your answer.