On Universal Properties of Capacity-Approaching LDPC Code Ensembles

Igal Sason

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel

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Degree Distributions of LDPC Code Ensembles

Consider the case where transmission takes place over a memoryless, binary-input output-symmetric (MBIOS) channel.

- Let a designate the pdf of the log-likelihood ratio (LLR) at the channel output given that the channel input is zero. Then, the symmetry property holds (i.e., $a(l) = e^{l} a(-l)$ for $l \in \mathbb{R}$).
- Consider LDPC code ensembles whose design rate forms a fraction $1 - \varepsilon$ of the channel capacity with a target bit error probability P_{b}.

Question

What can be said about the degree distributions of the LDPC code ensembles in this setting ?

Degree Distributions of LDPC Code Ensembles (Cont.)

In this work

- Linear programming (LP) bounds on the degree distributions of LDPC code ensembles are derived.
- They provide upper bounds on the fraction of edges or nodes up to degree k where k is a parameter.
- They are general since they hold even under ML decoding (and, hence, also under any sub-optimal decoding algorithm).
- The bounds also apply to finite-length codes.
- Analytical solutions of these bounds are obtained via Lagrange duality, and these bounds are easy to calculate.

A Brief Outline of the Derivation of the LP Bounds

- A lower bound on the conditional entropy for binary linear block codes transmitted over MBIOS channels gets the form

$$\frac{H(\mathbf{X}|\mathbf{Y})}{n} \geq R - C + \frac{1-R}{2\ln 2} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p-1)}$$

where

$$g_p \triangleq \int_0^{\infty} a(l)(1+e^{-l}) \tanh^{2p}\left(\frac{l}{2}\right) dl, \quad p \in \mathbb{N}.$$

and $\Gamma(x) \triangleq \sum_k \Gamma_k x^k$ forms the degree distribution of the parity-check nodes, from the node perspective, of an arbitrary representation of the code by a **full-rank** parity-check matrix (Wiechman & Sason, IEEE Trans. on IT, Feb. 2007).

- Fano inequality.
- $g_p \geq (g_1)^p$, for every $p \in \mathbb{N}$, with equality for the BSC.
- An adaptation of these results to LDPC code ensembles, whose parity-check matrices are not necessarily full rank (i.e., the parity-check equations are linearly dependent), is needed.
- The above IT bound is proved to hold for every code from a binary LDPC code ensemble when the code rate R is replaced by the design rate (R_{d}) of the ensemble ($R \geq R_{\text{d}}$).
- The derivation of the LP bounds finally relies on the equality

$$\frac{1}{2\ln 2}\sum_{k=1}^{\infty}\frac{u^k}{k(2k-1)} = 1 - h_2\left(\frac{1-\sqrt{u}}{2}\right), \quad \forall\, u \in [0,1].$$

where h_2 designates the binary entropy function on base 2.

LP1 bound for the degree distribution of the parity-check nodes for LDPC code ensembles

$$
\begin{aligned}
&\text{maximize} \quad \sum_{i=1}^{k} \rho_i, \quad k = 1, 2, \ldots \\
&\text{subject to} \\
&\begin{cases}
\displaystyle\sum_{i=1}^{\infty} \left\{ \left[1 - h_2\left(\frac{1-g_1^{\frac{i}{2}}}{2} \right) \right] \frac{\rho_i}{i} \right\} \leq \frac{\varepsilon C + h_2(P_\mathrm{b})}{1-(1-\varepsilon)C} \sum_{i=1}^{\infty} \frac{\rho_i}{i} \\
\displaystyle\sum_{i=1}^{\infty} \rho_i = 1 \\
\rho_i \geq 0, \quad i = 1, 2, \ldots
\end{cases}
\end{aligned}
$$

where the optimization variables are $\{\rho_i\}_{i \geq 1}$. The quantity g_1 above depends on the channel statistics only.

LP2 bound for the degree distribution: Universal for all equi-capacity MBIOS channels

Replace the parameter g_1 with the channel capacity C (where, in general, $g_1 \geq C$ with equality for the BEC).

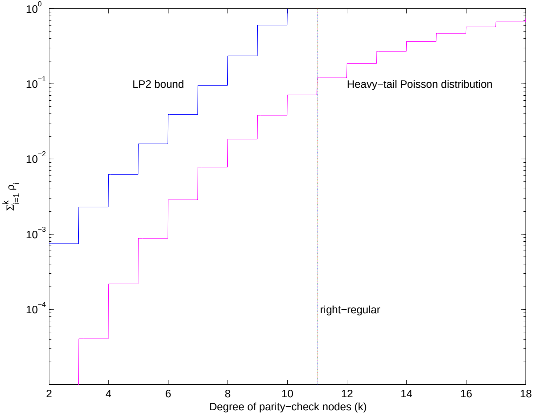

Figure: The universal LP2 bound versus the heavy-tail Poisson degree distribution, and the degree distribution of the right-regular LDPC ensemble. It refers to the fraction of edges which are attached to parity-check nodes of degree $\leq k$ for an integer $k \geq 2$. Considered here is a BEC whose capacity is $\frac{1}{2}$ bit per channel use, and where 99.9% of capacity is achieved under iterative message-passing decoding with vanishing bit erasure probability.

Asymptotic Behavior of the Degree Distributions

Corollary

If the asymptotic bit error/ erasure probability vanishes, then the following properties hold for an arbitrary finite degree i

$$L_i = O(1)\,, \qquad R_i = O(\varepsilon)\,,$$
$$\lambda_i = O\left(\frac{1}{\ln \frac{1}{\varepsilon}}\right), \quad \rho_i = O\left(\frac{\varepsilon}{\ln \frac{1}{\varepsilon}}\right).$$

where $\{L_i\}$ and $\{R_i\}$ are the degree distributions of the variable and parity-check nodes, respectively, and $\{\lambda_i\}$ and $\{\rho_i\}$ are the corresponding degree distributions from the edge perspective.

- These bounds hold under ML decoding (or any other algorithm).
- The upper bounds on the left degree distribution look at first glance looser than those for the right degree distribution (due to the additional factor ε in the latter case).
- However, it is not an artifact of the bounding technique, as it indeed reflects reality, e.g.:
  - For various capacity-achieving degree distributions on the BEC with iterative message-passing decoding, the fraction of degree-2 variable nodes tends to $\frac{1}{2}$.
  - The upper bound on the fraction of edges connected to degree-2 variable nodes (λ_2) is shown in this work to be obtained for the right-regular LDPC code ensemble of Shokrollahi which achieves capacity on the BEC under iterative decoding.

Information-Theoretic Lower Bounds on the Tradeoff Between the Graphical Complexity and Performance

Question

Consider the representation of a finite-length binary linear block code by an arbitrary bipartite graph. How simple can such a graphical representation be as a function of the channel model, target block error probability, and code rate ?

Answer $\Rightarrow$

- Information-theoretic lower bounds which measure the inherent graphical complexity of finite-length LDPC codes as a function of their achievable gap to capacity.
- Provides a measure of the sub-optimality of explicit constructions of LDPC codes by comparing to information-theoretic bounds.

Graphical Complexity versus Performance (Cont.)

- We provide in this work an information-theoretic lower bound on the graphical complexity which depends on the channel model, target block error probability, and the code rate.
- This bound relies on two previous information-theoretic results:
  - A new lower bound on the average left/ right degrees in the bipartite graph as a function of the channel, target block/ bit error probability, and the achievable gap to capacity.
  - Sphere-packing lower bounds: We rely here on the classical 1959 sphere-packing bound of Shannon, and the recently introduced ISP bound (Wiechman & Sason, IEEE Trans. on IT, May 2008).
- The graphical complexity, defined as the number of edges in the bipartite graph, is simply the product of the block length and the average left degree.

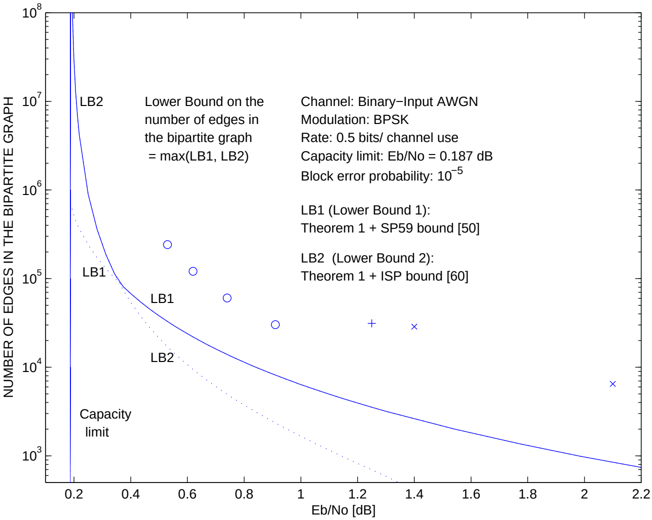

Figure: A comparison between the graphical complexity of various efficient LDPC code ensembles and an information-theoretic lower bound.

Cardinality of the Fundamental System of Cycles of Good LDPC Code Ensembles

- Binary Linear block codes which are represented by cycle-free bipartite graphs are not good even under ML decoding.
- A theoretical treatment of cycle-free codes was provided by T. Etzion, A. Trachtenberg and A. Vardy, "Which codes have cycle-free Tanner graphs ?," *IEEE Trans. on Information Theory*, vol. 45, no. 6, pp. 2173–2181, September 1999.

Question

What can be said about the cardinality of the fundamental system of cycles of LDPC code ensembles as a function of the achievable gap (in rate) to capacity ?

Theorem

Let $\{(n,\lambda,\rho)\}$ be a sequence of LDPC code ensembles transmitted over an MBIOS channel. Suppose that the design rate is a fraction $1-\varepsilon$ of the channel capacity C, and the average bit error probability of this sequence vanishes under some decoding algorithm as $n \to \infty$. Consider the average cardinality of the fundamental system of cycles, $\beta_n(\mathcal{G})$, where the graphs $\mathcal{G}$ are chosen uniformly at random from the LDPC code ensemble (n,λ,ρ). Then, the following result holds:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\beta_n(\mathcal{G})]}{n} \geq \frac{(1-C)\,\ln\left(g_1\,\left[1-2h_2^{-1}\left(\frac{1-C}{1-(1-\varepsilon)C}\right)\right]^{-2}\right)}{\ln\left(\frac{1}{g_1}\right)} - 1$$

where $g_1 \triangleq \mathbb{E}\left[\tanh^2\left(\frac{L}{2}\right)\right]$ and L forms the LLR at the channel output.

Corollary

The average cardinality of the fundamental system of cycles grows at least like $\log \frac{1}{\varepsilon}$ *where the achievable design rate forms a fraction* $1 - \varepsilon$ *of the channel capacity.*

$\Rightarrow$ The fundamental system of cycles becomes unbounded as the achievable gap to capacity vanishes (even under ML decoding).

Essence of the proof of this theorem: A combination of an improved lower bound on the average right degree (which behaves like $\log \frac{1}{\varepsilon}$), which follows from the lower bound on the conditional entropy, with some simple arguments from graph theory.

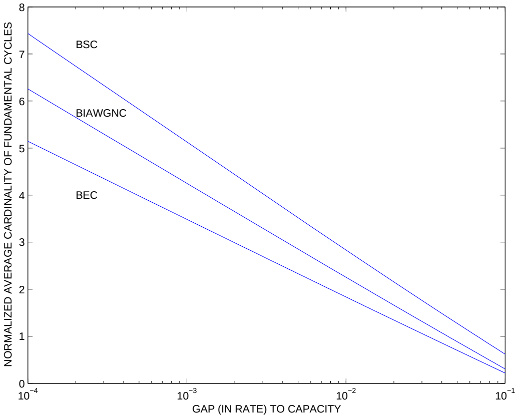

Figure: Asymptotic lower bounds on the average cardinality of the fundamental system (see Theorem 1). The bounds refer to the BSC, BIAWGNC and BEC where the design rate is $\frac{1}{2}$ bit per channel use.

Full Paper Version

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