On the Concentration of the Crest Factor for OFDM Signals

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Abstract—**This paper applies several concentration inequalities to prove concentration results for the crest factor of OFDM signals. The considered approaches are, to the best of our knowledge, new in the context of establishing concentration for OFDM signals.**

Index Terms—**Concentration of measures, crest-factor, OFDM signals.**

I. INTRODUCTION

Orthogonal-frequency-division-multiplexing (OFDM) is a modulation that converts a high-rate data stream into a number of low-rate steams that are transmitted over parallel narrow-band channels. OFDM is widely used in several international standards for digital audio and video broadcasting, and for wireless local area networks. For a textbook providing a survey on OFDM, see e.g. [7, Chapter 19].

One of the problems of OFDM is that the peak amplitude of the signal can be significantly higher than the average amplitude. This issue makes the transmission of OFDM signals sensitive to non-linear devices in the communication path such as digital to analog converters, mixers and high-power amplifiers. As a result of this drawback, it increases the symbol error rate and it also reduces the power efficiency of OFDM signals as compared to single-carrier systems. Commonly, the impact of nonlinearities is described by the distribution of the crest-factor (CF) of the transmitted signal [5], but its calculation involves time-consuming simulations even for a small number of sub-carriers. The expected value of the CF for OFDM signals is known to scale like the logarithm of the number of sub-carriers of the OFDM signal (see [5], [9, Section 4] and [13]).

In this paper, we consider two of the main approaches for proving concentration inequalities, and apply them to derive concentration results for the crest factor of OFDM signals. The first approach is based on martingales, and the other approach is Talagrand's method for proving concentration inequalities in product spaces. It is noted that some of these concentration inequalities can be derived using ideas from information theory (see, e.g., [6] and references therein).

Considering the martingale approach for proving concentration results, the Azuma-Hoeffding inequality is by now a well-known methodology that has been often used to prove concentration of measures. It is due to Hoeffding [3] who proved this inequality for a sum of independent and bounded random variables, and Azuma [1] who later extended it to bounded-difference martingales. In the context of communication and information theoretic aspects, Azuma's inequality was used during the last decade in the coding literature for establishing concentration results for codes defined on graphs and iterative decoding algorithms (see, [8] and references therein). Some other martingale-based concentration inequalities were also recently applied to the performance evaluation of random coding over non-linear communication channels [14]. McDiarmid's inequality is an improved version of Azuma's inequality in the special case where one considers the concentration of a function $f : \mathbb{R}^n \to \mathbb{R}$ of n independent RVs when the variation of $f(x_1, \ldots, x_n)$ w.r.t. each of its coordinates is bounded (and when all the other $n - 1$ components are kept fixed). Under this setting, it gives an improvement by a factor of 4 in the exponent. This inequality is applied in this paper in order to prove the concentration of the crest factor of OFDM signals around the expected value.

A second approach for proving concentration inequalities in product spaces was developed by Talagrand in his seminal paper [11]. It forms in general a powerful probabilistic tool for establishing concentration results for coordinate-wise Liphschitz functions of independent random variables (see also, e.g., [2, Section 2.4.2] and [6, Section 4]). This approach was used in [4] to prove concentration inequalities, in the large system limit, for a code division multiple access (CDMA) system. Talagrand's inequality is used in this paper to prove a concentration result (near the median) of the crest factor of OFDM signals, and it also enables to obtain an upper bound on the distance between the median and the expected value.

A stronger concentration inequality for the crest factor of OFDM signals was introduced in [5, Theorem 3] under some assumptions on the probability distribution of the considered problem (the reader is referred to the two conditions in [5, Theorem 3], followed by [5, Corollary 5]). These requirements are not needed in the following analysis, and the derivation of the concentration inequalities here is rather simple and it provides some further insight to this issue.

II. SOME CONCENTRATION INEQUALITIES

In the following, we present briefly essential background on concentration inequalities that is required for the analysis in this paper. In the next section, we will apply these probabilistic

tools for obtaining concentration inequalities for the crest factor of OFDM signals.

A. Azuma's Inequality

Azuma's inequality[1] forms a useful concentration inequality for bounded-difference martingales [1]. In the following, this inequality is introduced.

Theorem 1: **[Azuma's inequality]** Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ be a discrete-parameter real-valued martingale sequence such that for every $k \in \mathbb{N}$, the condition $|X_k - X_{k-1}| \leq d_k$ holds a.s. for some non-negative constants $\{d_k\}_{k=1}^{\infty}$. Then

$$\mathbb{P}(|X_n - X_0| \geq r) \leq 2 \exp\left(-\frac{r^2}{2\sum_{k=1}^{n} d_k^2}\right) \quad \forall r \geq 0. \quad (1)$$

The concentration inequality stated in Theorem 1 was proved in [3] for independent bounded random variables, followed by a discussion on sums of dependent random variables; this inequality was later derived in [1] for bounded-difference martingales. The reader is referred, e.g., to [6] for a proof.

B. A Refined Version of Azuma's Inequality

The following refined version of Azuma's inequality was introduced in [10] (which includes some other approaches for refining Azuma's inequality).

Theorem 2: Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ be a discrete-parameter real-valued martingale. Assume that, for some constants $d, \sigma > 0$, the following two requirements are satisfied a.s.

$$|X_k - X_{k-1}| \leq d,$$
$$\text{Var}(X_k|\mathcal{F}_{k-1}) = \mathbb{E}\left[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\right] \leq \sigma^2$$

for every $k \in \{1, \ldots, n\}$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|X_n - X_0| \geq \alpha n) \leq 2 \exp\left(-n D\left(\frac{\delta + \gamma}{1 + \gamma}\Big\|\frac{\gamma}{1 + \gamma}\right)\right) \quad (2)$$

where

$$\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d} \quad (3)$$

and

$$D(p\|q) \triangleq p \ln\left(\frac{p}{q}\right) + (1-p) \ln\left(\frac{1-p}{1-q}\right), \quad \forall p, q \in [0, 1] \quad (4)$$

is the divergence (a.k.a. relative entropy or Kullback-Leibler distance) between the two probability distributions $(p, 1-p)$ and $(q, 1-q)$. If $\delta > 1$, then the probability on the left-hand side of (2) is equal to zero.

Proof: The idea of the proof of Theorem 2 is essentially similar to the proof of [2, Corollary 2.4.7]. The full proof is provided in [10, Section III]. ∎

Proposition 1: Let $\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}$ be a discrete-parameter real-valued martingale. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|X_n - X_0| \geq \alpha\sqrt{n}) \leq 2 \exp\left(-\frac{\delta^2}{2\gamma}\right)\left(1 + O\left(n^{-\frac{1}{2}}\right)\right) \quad (5)$$

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[1]Azuma's inequality is also known as the Azuma-Hoeffding inequality. Since this inequality is referred several times in this paper, it will be named from this point as Azuma's inequality for the sake of brevity.

where γ and δ are introduced in (3).

Proof: This inequality follows from Theorem 2 (see [10, Appendix H]). ∎

C. McDiarmid's Inequality

In the following, we state McDiarmid's inequality (see [6, Theorem 3.1]).

Theorem 3: Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a family of independent random variables with X_k taking values in a set A_k for each k. Suppose that a real-valued function f, defined on $\prod_k A_k$, satisfies

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq c_k$$

whenever the vectors $\mathbf{x}$ and $\mathbf{x}'$ differ only in the k-th coordinate. Let $\mu \triangleq \mathbb{E}[f(X)]$ be the expected value of $f(X)$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|f(X) - \mu| \geq \alpha) \leq 2 \exp\left(-\frac{2\alpha^2}{\sum_k c_k^2}\right).$$

This inequality is proved with the aid of martingales. It has some nice applications which were exemplified in the context of algorithmic discrete mathematics (see [6, Section 3]).

D. Talagrand's inequality

Talagrand's inequality is an approach used for establishing concentration results on product spaces, and this technique was introduced in Talagrand's landmark paper [11].

We provide in the following two definitions that will be required for the introduction of a special form of Talagrand's inequalities.

Definition 1 (Hamming distance): Let $\mathbf{x}, \mathbf{y}$ be two n-length vectors. The Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ is the number of coordinates where $\mathbf{x}$ and $\mathbf{y}$ disagree, i.e.,

$$d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^{n} I_{\{x_i \neq y_i\}}$$

where I stands for the indicator function.

The following suggests a generalization and normalization of the previous distance metric.

Definition 2: Let $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ (i.e., a is a non-negative vector) satisfy $||a||^2 = \sum_{i=1}^{n}(a_i)^2 = 1$. Then, define

$$d_a(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^{n} a_i I_{\{x_i \neq y_i\}}.$$

Hence, $d_{\mathrm{H}}(\mathbf{x}, \mathbf{y}) = \sqrt{n}\, d_a(\mathbf{x}, \mathbf{y})$ for $a = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$.

The following is a special form of Talagrand's inequalities ([6, Chapter 4], [11], [12]).

Theorem 4 (Talagrand's inequality): Let the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of independent random variables with X_k taking values in a set A_k, and let $A \triangleq \prod_{k=1}^{n} A_k$. Let $f : A \to \mathbb{R}$ satisfy the condition that, for every $\mathbf{x} \in A$, there exists a non-negative, normalized n-length vector $a = a(x)$ such that

$$f(\mathbf{x}) \leq f(\mathbf{y}) + \sigma d_a(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in A \quad (6)$$

for some fixed value $\sigma > 0$. Then, for every $\alpha \geq 0$,

$$\mathbb{P}(|f(X) - m| \geq \alpha) \leq 4 \exp\left(-\frac{\alpha^2}{4\sigma^2}\right) \qquad (7)$$

where m is the median of $f(X)$ (i.e., $\mathbb{P}(f(X) \leq m) \geq \frac{1}{2}$ and $\mathbb{P}(f(X) \geq m) \geq \frac{1}{2}$). The same conclusion in (7) holds if the condition in (6) is replaced by

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \sigma d_a(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{y} \in A. \qquad (8)$$

Remark 1: In the special case where the condition for the function f in Theorem 4 (Talagrand's inequality) is satisfied with the additional property that the vector a on the right-hand side of (6) is *independent* of x (i.e., the value of this vector is fixed), then the concentration inequality in (7) follows from McDiarmid's inequality. To verify this observation, the reader is referred to [6, Theorem 3.6] followed by the discussion in [6, p. 211] (leading to [6, Eqs. (3.12) and (3.13)]).

III. APPLICATION: CONCENTRATION OF THE CREST-FACTOR FOR OFDM SIGNALS

A. Background

Given an n-length codeword $\{X_i\}_{i=0}^{n-1}$, a single OFDM baseband symbol is described by

$$s(t; X_0, \ldots, X_{n-1}) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} X_i \exp\left(\frac{j \, 2\pi it}{T}\right), \quad 0 \leq t \leq T. \qquad (9)$$

Lets assume that $X_0, \ldots, X_{n-1}$ are i.i.d. complex RVs with $|X_i| = 1$. Since the sub-carriers are orthonormal over $[0, T]$, then a.s. the power of the signal s over this interval is 1. The CF of the signal s, composed of n sub-carriers, is defined as

$$\mathrm{CF}_n(s) \triangleq \max_{0 \leq t \leq T} |s(t)|. \qquad (10)$$

From [9, Section 4] and [13], it follows that the CF scales with high probability like $\sqrt{\log(n)}$ for large n. In [5, Theorem 3 and Corollary 5], a concentration inequality was derived for the CF of OFDM signals. It states that for every $c \geq 2.5$

$$\mathbb{P}\left(\left|\mathrm{CF}_n(s) - \sqrt{\log(n)}\right| < \frac{c \log \log(n)}{\sqrt{\log(n)}}\right) = 1 - O\left(\frac{1}{(\log(n))^4}\right).$$

Remark 2: The analysis used to derive this rather strong concentration inequality (see [5, Appendix C]) requires some assumptions on the distribution of the X_i's (see the two conditions in [5, Theorem 3] followed by [5, Corollary 5]). These requirements are not needed in the following analysis, and the derivation of the two concentration inequalities in this paper is simple, though weaker concentration results are obtained.

In the following, Azuma's inequality and a refined version of this inequality are considered under the assumption that $\{X_j\}_{j=0}^{n-1}$ are independent complex-valued random variables with magnitude 1, attaining the M points of an M-ary PSK constellation with equal probability.

B. Establishing Concentration of the Crest-Factor via Azuma's Inequality and a Refined Version

1) Proving Concentration via Azuma's Inequality: In the following, Azuma's inequality is used to derive a concentration result. Let us define

$$Y_i = \mathbb{E}[\,\mathrm{CF}_n(s) \,|\, X_0, \ldots, X_{i-1}], \quad i = 0, \ldots, n \qquad (11)$$

Based on a standard construction of Doob's martingales, $\{Y_i, \mathcal{F}_i\}_{i=0}^n$ is a martingale where $\mathcal{F}_i$ is the σ-algebra that is generated by the first i symbols $(X_0, \ldots, X_{i-1})$ in (9). Hence, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n$ is a filtration. This martingale has also bounded jumps, and

$$|Y_i - Y_{i-1}| \leq \frac{2}{\sqrt{n}}$$

for $i \in \{1, \ldots, n\}$ since revealing the additional i-th coordinate X_i affects the CF, as is defined in (10), by at most $\frac{2}{\sqrt{n}}$ (see the first part of Appendix A). It therefore follows from Azuma's inequality that, for every $\alpha > 0$,

$$\mathbb{P}(|\mathrm{CF}_n(s) - \mathbb{E}[\mathrm{CF}_n(s)]| \geq \alpha) \leq 2 \exp\left(-\frac{\alpha^2}{8}\right) \qquad (12)$$

which demonstrates the concentration of this measure around its expected value.

2) Proof of Concentration via Proposition 1: In the following, we rely on Proposition 1 to derive an improved concentration result. For the martingale sequence $\{Y_i\}_{i=0}^n$ in (11), Appendix A gives that a.s.

$$|Y_i - Y_{i-1}| \leq \frac{2}{\sqrt{n}}, \quad \mathbb{E}\big[(Y_i - Y_{i-1})^2 | \mathcal{F}_{i-1}\big] \leq \frac{2}{n} \qquad (13)$$

for every $i \in \{1, \ldots, n\}$. Note that the conditioning on the σ-algebra $\mathcal{F}_{i-1}$ is equivalent to the conditioning on the symbols $X_0, \ldots, X_{i-2}$, and there is no conditioning for $i = 1$. Let $Z_i = \sqrt{n} Y_i$. Proposition 1 therefore implies that for an arbitrary $\alpha > 0$

$$\begin{aligned}
&\mathbb{P}(|\mathrm{CF}_n(s) - \mathbb{E}[\mathrm{CF}_n(s)]| \geq \alpha) \\
&= \mathbb{P}(|Y_n - Y_0| \geq \alpha) \\
&= \mathbb{P}(|Z_n - Z_0| \geq \alpha\sqrt{n}) \\
&\leq 2 \exp\left(-\frac{\alpha^2}{4}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\right) \qquad (14)
\end{aligned}$$

(since $\delta = \frac{\alpha}{2}$ and $\gamma = \frac{1}{2}$ in the setting of Proposition 1). Note that the exponent of the last concentration inequality is doubled as compared to the bound that was obtained in (12) via Azuma's inequality, and the term which scales like $O\left(\frac{1}{\sqrt{n}}\right)$ on the right-hand side of (14) is expressed explicitly for finite n (see [10, Appendix H]).

C. Establishing Concentration via McDiarmid's Inequality

In the following, McDiarmid's inequality is applied to prove a concentration inequality for the crest factor of OFDM

signals. To this end, let us define

$$U \triangleq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{i-1}, X_i, \ldots, X_{n-1}) \right|$$

$$V \triangleq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \right|.$$

Then, this implies that

$$\begin{aligned}
|U - V| &\leq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{i-1}, X_i, \ldots, X_{n-1}) \right. \\
&\qquad \left. - s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \right| \\
&= \max_{0 \leq t \leq T} \frac{1}{\sqrt{n}} \left| (X_{i-1} - X'_{i-1}) \exp\left(\frac{j \, 2\pi i t}{T} \right) \right| \\
&= \frac{|X_{i-1} - X'_{i-1}|}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}
\end{aligned} \tag{15}$$

where the last inequality holds since $|X_{i-1}| = |X'_{i-1}| = 1$. Hence, McDiarmid's inequality in Theorem 3 implies that, for every $\alpha \geq 0$,

$$\mathbb{P}(|\mathrm{CF}_n(s) - \mathbb{E}[\mathrm{CF}_n(s)]| \geq \alpha) \leq 2 \exp\left(-\frac{\alpha^2}{2} \right) \tag{16}$$

which demonstrates concentration around the expected value. It is noted that McDiarmid's inequality provides an improvement in the exponent by a factor of 4 as compared to Azuma's inequality. It also improves the exponent by a factor of 2 as compared to Proposition 1 in the considered case (where $\gamma = \frac{1}{2}$).

The same kind of result applies easily to QAM-modulated OFDM signals, since the RVs are bounded which therefore enables to get a similar result to (15).

D. Establishing Concentration via Talagrand's Inequality

In the following, Talagrand's inequality is applied to prove a concentration inequality for the crest factor of OFDM signals. Let us assume that $X_0, Y_0, \ldots, X_{n-1}, Y_{n-1}$ are i.i.d. bounded complex RVs, and also for simplicity

$$|X_i| = |Y_i| = 1.$$

In order to apply Talagrand's inequality to prove concentration, note that

$$\begin{aligned}
&\max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{n-1}) \right| - \max_{0 \leq t \leq T} \left| s(t; Y_0, \ldots, Y_{n-1}) \right| \\
&\leq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{n-1}) - s(t; Y_0, \ldots, Y_{n-1}) \right| \\
&\leq \frac{1}{\sqrt{n}} \left| \sum_{i=0}^{n-1} (X_i - Y_i) \exp\left(\frac{j \, 2\pi i t}{T} \right) \right| \\
&\leq \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |X_i - Y_i| \\
&\leq \frac{2}{\sqrt{n}} \sum_{i=0}^{n-1} I_{\{x_i \neq y_i\}} \\
&= 2 d_a(X, Y)
\end{aligned}$$

where

$$a \triangleq \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right) \tag{17}$$

is a non-negative unit-vector of length n (note that a in this case is independent of x). Hence, Talagrand's inequality in Theorem 4 implies that, for every $\alpha \geq 0$,

$$\mathbb{P}(|\mathrm{CF}_n(s) - m_n| \geq \alpha) \leq 4 \exp\left(-\frac{\alpha^2}{16} \right), \quad \forall \alpha > 0 \tag{18}$$

where m_n is the median of the crest factor for OFDM signals that are composed of n sub-carriers. This inequality demonstrates the concentration of this measure around its median. As a simple consequence of (18), one obtains the following result.

Corollary 1: The median and expected value of the crest factor differ by at most a constant, independently of the number of sub-carriers n.

Proof: By Talagrand's inequality in (18), it follows that

$$\begin{aligned}
&\left| \mathbb{E}[\mathrm{CF}_n(s)] - m_n \right| \\
&\leq \mathbb{E} \left| \mathrm{CF}_n(s) - m_n \right| \\
&\overset{(a)}{=} \int_0^\infty \mathbb{P}(|\mathrm{CF}_n(s) - m_n| \geq \alpha) \, d\alpha \\
&\leq \int_0^\infty 4 \exp\left(-\frac{\alpha^2}{16} \right) d\alpha \\
&= 8\sqrt{\pi}
\end{aligned}$$

where equality (a) holds since for a non-negative random variable Z

$$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z \geq t) \, dt.$$

∎

Remark 3: This result applies in general to an arbitrary function f satisfying the condition in (6), where Talagrand's inequality in (7) implies that (see, e.g., [6, Lemma 4.6])

$$\left| \mathbb{E}[f(X)] - m \right| \leq 4\sigma\sqrt{\pi}.$$

Remark 4: By comparing (18) with (16), it follows that Mc-Diarmid's inequality provides an improvement in the exponent. This is consistent with Remark 1 and the fixed value of the non-negative normalized vector in (17).

IV. SUMMARY

This paper derives four concentration inequalities for the crest-factor (CF) of OFDM signals under the assumption that the symbols are independent. The first two concentration inequalities rely on Azuma's inequality and a refined version of it, and the last two concentration inequalities are based on Talagrand's and McDiarmid's inequalities. Although these concentration results are weaker than some existing results from the literature (see [5] and [13]), they establish concentration in a rather simple way and provide some insight to the problem. The use of these bounding techniques, in the context of concentration for OFDM signals, seems to be new. The improvement of McDiarmid's inequality is by a factor of 4 in the exponent as compared to Azuma's inequality, and by a factor of 2 as compared to the refined version of Azuma's inequality in Proposition 1. Note however that Proposition 1 may be in general tighter than McDiarmid's inequality (if

$\gamma < \frac{1}{4}$ in the setting of Proposition 1). It also follows from Talagrand's method that the median and expected value of the CF differ by at most a constant, independently of the number of sub-carriers.

Some other new refined versions of Azuma's inequality were introduced in [10], followed by some applications in information theory and communications. This work is aimed to stimulate the use of some refined versions of concentration inequalities, based on the martingale approach and Talagrand's approach, in information-theoretic aspects.

APPENDIX A
PROOF OF THE PROPERTIES IN (13) FOR OFDM SIGNALS

Consider an OFDM signal from Section III-A. The sequence in (11) is a martingale due to basic properties of martingales. From (10), for every $i \in \{0, \ldots, n\}$

$$Y_i = \mathbb{E}\left[\max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{n-1}) \right| \,\Big|\, X_0, \ldots, X_{i-1} \right].$$

The conditional expectation for the RV Y_{i-1} refers to the case where only $X_0, \ldots, X_{i-2}$ are revealed. Let X'_{i-1} and X_{i-1} be independent copies, which are also independent of $X_0, \ldots, X_{i-2}, X_i, \ldots, X_{n-1}$. Then, for every $1 \leq i \leq n$,

$$Y_{i-1} = \mathbb{E}\Big[\max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \right| \\ \Big| X_0, \ldots, X_{i-2} \Big]$$
$$= \mathbb{E}\Big[\max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \right| \\ \Big| X_0, \ldots, X_{i-2}, X_{i-1} \Big].$$

Since $|\mathbb{E}(Z)| \leq E(|Z|)$, then for $i \in \{1, \ldots, n\}$

$$|Y_i - Y_{i-1}| \leq \mathbb{E}_{X'_{i-1}, X_i, \ldots, X_{n-1}} \Big[|U - V| \,\Big|\, X_0, \ldots, X_{i-1} \Big] \quad (19)$$

where

$$U \triangleq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X_{i-1}, X_i, \ldots, X_{n-1}) \right|$$
$$V \triangleq \max_{0 \leq t \leq T} \left| s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \right|.$$

From (9)

$$|U - V| \leq \max_{0 \leq t \leq T} \Big| s(t; X_0, \ldots, X_{i-1}, X_i, \ldots, X_{n-1}) \\ - s(t; X_0, \ldots, X'_{i-1}, X_i, \ldots, X_{n-1}) \Big|$$
$$= \max_{0 \leq t \leq T} \frac{1}{\sqrt{n}} \left| (X_{i-1} - X'_{i-1}) \exp\left(\frac{j \, 2\pi i t}{T} \right) \right|$$
$$= \frac{|X_{i-1} - X'_{i-1}|}{\sqrt{n}}. \quad (20)$$

By assumption, $|X_{i-1}| = |X'_{i-1}| = 1$, and therefore a.s.

$$|X_{i-1} - X'_{i-1}| \leq 2 \Longrightarrow |Y_i - Y_{i-1}| \leq \frac{2}{\sqrt{n}}.$$

In the following, an upper bound on the conditional variance

$$\mathrm{Var}(Y_i \mid \mathcal{F}_{i-1}) = \mathbb{E}\big[(Y_i - Y_{i-1})^2 \mid \mathcal{F}_{i-1} \big]$$

is obtained. Since $\big(\mathbb{E}(Z)\big)^2 \leq \mathbb{E}(Z^2)$ for a real-valued RV Z, then from (19) and (20)

$$\mathbb{E}\big[(Y_i - Y_{i-1})^2 \mid \mathcal{F}_{i-1} \big] \leq \frac{1}{n} \cdot \mathbb{E}_{X'_{i-1}}\big[|X_{i-1} - X'_{i-1}|^2 \mid \mathcal{F}_i \big]$$

where $\mathcal{F}_i$ is the σ-algebra that is generated by $X_0, \ldots, X_{i-1}$. Due to a symmetry argument of the PSK constellation, then it follows that

$$\mathbb{E}\big[(Y_i - Y_{i-1})^2 \mid \mathcal{F}_{i-1} \big]$$
$$\leq \frac{1}{n} \mathbb{E}_{X'_{i-1}}\big[|X_{i-1} - X'_{i-1}|^2 \mid \mathcal{F}_i \big]$$
$$= \frac{1}{n} \mathbb{E}\big[|X_{i-1} - X'_{i-1}|^2 \mid X_0, \ldots, X_{i-1} \big]$$
$$= \frac{1}{n} \mathbb{E}\big[|X_{i-1} - X'_{i-1}|^2 \mid X_{i-1} \big]$$
$$= \frac{1}{n} \mathbb{E}\big[|X_{i-1} - X'_{i-1}|^2 \mid X_{i-1} = e^{\frac{j\pi}{M}} \big]$$
$$= \frac{1}{nM} \sum_{l=0}^{M-1} \left| e^{\frac{j\pi}{M}} - e^{\frac{j(2l+1)\pi}{M}} \right|^2$$
$$= \frac{4}{nM} \sum_{l=1}^{M-1} \sin^2\left(\frac{\pi l}{M} \right) = \frac{2}{n}.$$

To clarify the last equality, note that if $x \in \mathbb{R}$ and $m \in \mathbb{N}$

$$\sum_{l=1}^{m} \sin^2(lx) = \frac{m+1}{2} - \frac{\sin\big((m+1)x\big) \cos(mx)}{2 \sin(x)}.$$

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