Achieving Marton’s Region for Broadcast Channels Using Polar Codes

Marco Mondelli

Joint work with S. Hamed Hassani, Igal Sason, Rüdiger Urbanke

School of Computer and Communication Sciences, EPFL, Switzerland
Department of Electrical Engineering, Technion, Israel

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Polar Coding

Capacity-achieving for binary-input memoryless symmetric channels with complexity $\Theta(n \log n)$ and error probability $\sim 2^{-\sqrt{n}}$. 
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- Non-binary and non-symmetric point to point channels.
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- Lossless and lossy source coding.
Polar Coding

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Useful also in many other scenarios:

- Non-binary and non-symmetric point to point channels.
- Lossless and lossy source coding.
- Multi-user channels: broadcast, multiple access, relay, interference, wiretap...
**Broadcast Channel: Superposition Coding**

Broadcast channel: $Y_1$ and $Y_2$ given $X$ and an auxiliary RV $V$.

Information-theoretic region:

\[ R_1 < I(X; Y_1|V) \]

\[ R_2 < I(V; Y_2) \]

\[ R_1 + R_2 < I(X; Y_1) \]

Polar region by Goela et al.

Broadcast Channel: Superposition Coding

\[
p_{Y_1, Y_2 | X}
\]

\(V\) auxiliary RV s.t. \(V - X - (Y_1, Y_2)\)

Information-theoretic region

\[
\begin{align*}
R_1 &< I(X; Y_1 | V) \\
R_2 &< I(V; Y_2) \\
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Polar region by Goela et al.*

\[
\begin{align*}
R_1 &< I(X; Y_1 | V) \\
R_2 &< I(V; Y_2) \\
\bullet \ p_{Y_1 | V} &\succ p_{Y_2 | V}
\end{align*}
\]

Broadcast Channel: Comparison of Superposition Regions

BSC(0.11) \rightarrow Y_1

BEC(0.2) \rightarrow Y_2

Information-theoretic region

Polar region by Goela et al.

Time sharing

Focus of this talk.

Achieve with polar codes the information-theoretic region.
Broadcast Channel: Comparison of Superposition Regions

\[ \text{BSC}(0.11) \]
\[ \text{BEC}(0.4) \]

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Information-theoretic region

Polar region by Goela et al.

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Broadcast Channel: Polar Codes for Marton’s Region

Main result

Achieve with polar codes Marton’s region with common message.
Achieving Marton’s Region for Broadcast Channels Using Polar Codes

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Achieve with polar codes Marton’s region with common message.

\[
X \xrightarrow{p_{Y_1, Y_2 | X}} Y_1, Y_2 \xrightarrow{V, V_1, V_2} \text{auxiliary RV s.t. } X = \phi(V, V_1, V_2)
\]

\[
\begin{align*}
R_0 &< \min\{I(V; Y_1), I(V; Y_2)\} \\
R_0 + R_1 &< I(V, V_1; Y_1) \\
R_0 + R_2 &< I(V, V_2; Y_2) \\
R_0 + R_1 + R_2 &< I(V, V_1; Y_1) + I(V_2; Y_2 | V) - I(V_1; V_2 | V) \\
R_0 + R_1 + R_2 &< I(V, V_2; Y_2) + I(V_1; Y_1 | V) - I(V_1; V_2 | V)
\end{align*}
\]

- Common information at rate \( R_0 \)
Broadcast Channel: Polar Codes for Marton’s Region

Main result

Achieve with polar codes Marton’s region with common message.

\[ P_{Y_1,Y_2|X} \]

\[ X \rightarrow Y_1 \rightarrow Y_2 \rightarrow V, V_1, V_2 \text{ auxiliary RV s.t. } X = \phi(V, V_1, V_2) \]

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\end{align*}
\]

- Common information at rate \( R_0 \)
- Best known achievable region
- Combination of superposition and binning
Primitives: Lossless Compression

**Problem statement.** Given $X \sim p_X$, compress $X^{1:n} = (X^1, \ldots, X^n)$ into a vector of size $\approx nH(X)$. 
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**Basic fact:** $U^{1:n} = X^{1:n} G_n$ is polarized, with $G_n = \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \otimes \log_2 n$. 
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- High entropy positions: $U^i$ uniform and independent of $U^{1:i-1}$,
  $\mathcal{H}_X = \{i : Z(U^i \mid U^{1:i-1}) \geq 1 - \delta\}$.

- Low entropy positions: $U^i$ deterministic function of $U^{1:i-1}$,
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Send bits in $\mathcal{L}_X^c$. 

\[ \approx nH(X) \quad \approx n(1 - H(X)) \]
Primitives: Lossless Compression with Side Information

Problem statement. Given \((X, Y) \sim p_{X,Y}\), compress \(X^{1:n}\) with side information \(Y^{1:n}\) into a vector of size \(\approx nH(X|Y)\).
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- Low entropy positions: \(U^i\) deterministic function of \((U^{1:i-1}, Y^{1:n})\),
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Send bits in \(\mathcal{L}_{X|Y}^c\).
Primitives: Transmission over Binary-Input DMCs

**Problem statement.** Given $X \sim p_X$, transmit at rate $I(X; Y)$. 

\[ X \rightarrow p_{Y|X} \rightarrow Y \]
Primitives: Transmission over Binary-Input DMCs

**Problem statement.** Given $X \sim p_X$, transmit at rate $I(X; Y)$.

Set $U^{1:n} = X^{1:n} G_n$ and interpret $Y^{1:n}$ as side information.
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Set $U^{1:n} = X^{1:n} G_n$ and interpret $Y^{1:n}$ as side information.

Information bits in $\mathcal{I} = \mathcal{H}_X \cap \mathcal{L}_{X|Y}$:

- $i \in \mathcal{H}_X \Rightarrow U^i$ uniform and independent of $U^{1:i-1}$.
- $i \in \mathcal{L}_{X|Y} \Rightarrow U^i$ decodable given $(U^{1:i-1}, Y^{1:n})$.
- $\mathcal{H}_{X|Y} \subseteq \mathcal{H}_X$, $\mathcal{L}_X \subseteq \mathcal{L}_{X|Y}$.
Polar Codes for Superposition Region

Problem statement. Given \((V, X) \sim p_V p_{X|V}\), transmit at rates

\[
\begin{align*}
R_1 &= I(X; Y_1) - I(V; Y_2) \\
R_2 &= I(V; Y_2)
\end{align*}
\]

when \(I(V; Y_1) < I(V; Y_2) < I(X; Y_1)\).
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Any other rate pair in information-theoretic region achievable with similar polar schemes.
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**Basics of superposition coding**

- \(V\) contains message of user 2 and is decoded by both users.
- \(X\) contains message of user 1 and, given \(V\), is decoded by user 1.
Positions of \( U_1^{1:n} = X^{1:n} G_n \)

- \( V = \) side information on \( X \).
- Given \( V \), transmission of \( X \) over DMC \( p_{Y_1|X} \).
Positions of $U_1^{1:n} = X^{1:n} G_n$

- $V = \text{side information on } X$.
- Given $V$, transmission of $X$ over DMC $p_{Y_1|X}$.

Information bits for user 1 in $\mathcal{I}^{(1)} = \mathcal{H}_{X|V} \cap \mathcal{L}_{X|V, Y_1}$:

- $i \in \mathcal{H}_{X|V} \Rightarrow U^i$ uniform and independent of $(U^{1:i-1}, V^{1:n})$.
- $i \in \mathcal{L}_{X|V, Y_1} \Rightarrow U^i$ decodable given $(U^{1:i-1}, V^{1:n}, Y_1^{1:n})$. 

$\approx nI(X; Y_1|V)$
Positions of $U_{2}^{1:n} = V_{1}^{1:n} G_{n}$

- Information intended for user 2 but decodable by both users.
Positions of $U_2^{1:n} = V^{1:n} G_n$

- Information intended for user 2 but decodable by both users.
- Positions decodable by user 1 in $\mathcal{I}_V^{(1)} = \mathcal{H}_V \cap \mathcal{L}_V|Y_1$.
- Positions decodable by user 2 in $\mathcal{I}_V^{(2)} = \mathcal{H}_V \cap \mathcal{L}_V|Y_2$. 

![Diagram of regions](image.png)
Positions of $U^{1:n} = V^{1:n} G_n$

- Information intended for user 2 but decodable by both users.
- Positions decodable by user 1 in $\mathcal{I}_V^{(1)} = \mathcal{H}_V \cap \mathcal{L}_{V|Y_1}$.
- Positions decodable by user 2 in $\mathcal{I}_V^{(2)} = \mathcal{H}_V \cap \mathcal{L}_{V|Y_2}$.

- $\mathcal{I}_V^{(1)} \cap \mathcal{I}_V^{(2)}$: both users decode.
- $\mathcal{D}^{(2)} = \mathcal{I}_V^{(2)} \setminus \mathcal{I}_V^{(1)}$: only user 2 decodes $(p_{Y_1|V} \succ p_{Y_2|V} \Rightarrow \mathcal{D}^{(2)} = \emptyset)$.
- $\mathcal{D}^{(1)} = \mathcal{I}_V^{(1)} \setminus \mathcal{I}_V^{(2)}$: only user 1 decodes.
Single Chaining Construction

First, suppose $|\mathcal{D}^{(1)}| = |\mathcal{D}^{(2)}|$. 

Transmit $k$ polar blocks. Repeat positions in $\mathcal{D}^{(1)}$ into $\mathcal{D}^{(2)}$ of next block. User 1 decodes "forward" and user 2 decodes "backwards". Rate loss $1 = k$. 

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- Transmit $k$ polar blocks.
- Repeat positions in $\mathcal{D}^{(1)}$ into $\mathcal{D}^{(2)}$ of next block.

- User 1 decodes “forward” and user 2 decodes “backwards”.
- Rate loss $\sim 1/k$. 

![Diagram showing the single chaining construction withFrozen and DATA regions labeled.](image-url)
Double Chaining Construction

In general, $|\mathcal{D}^{(1)}| < |\mathcal{D}^{(2)}|$ (\(I(V; Y_1) < I(V, Y_2)\)).
Double Chaining Construction

In general, \(|\mathcal{D}^{(1)}| < |\mathcal{D}^{(2)}|\) \((I(V; Y_1) < I(V, Y_2))\).

- Not enough positions in \(\mathcal{D}^{(1)}\) to repeat \(\mathcal{D}^{(2)}\).
- Repeat extra positions in \(\mathcal{I}^{(1)}\) of previous block.

\[
\begin{align*}
U_1^{1:n} & \approx nI(X; Y_1 | V) \\
& \approx n(I(V; Y_2) - I(V; V_1)) \\
U_2^{1:n} & \quad \text{FROZEN} \\
& \quad \text{DATA} \\
& \quad \mathcal{D}^{(1)} \\
& \quad \text{DATA}
\end{align*}
\]
Double Chaining Construction

In general, $|\mathcal{D}^{(1)}| < |\mathcal{D}^{(2)}|$ ($I(V; Y_1) < I(V; Y_2)$).

- Not enough positions in $\mathcal{D}^{(1)}$ to repeat $\mathcal{D}^{(2)}$.
- Repeat extra positions in $\mathcal{I}^{(1)}$ of previous block.

\[ R_1 = I(X; Y_1) - I(V; Y_2), \quad R_2 = I(V; Y_2). \]
Conclusions

**Main result**

Achievability *with polar codes* of:

1) Superposition region
2) Binning region
3) Marton’s region with common message combining 1) and 2)
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Main result

Achievability with polar codes of:

1) Superposition region
2) Binning region
3) Marton’s region with common message combining 1) and 2)

- Chaining constructions useful in various multi-user scenarios.