On Achievable Rates and Complexity of LDPC Codes for Parallel Channels: Information-Theoretic Bounds and Applications

Igal Sason   Gil Wiechman

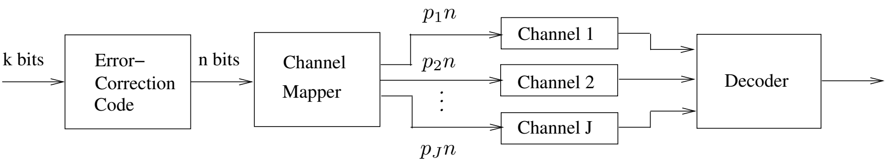
Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel
{sason@ee, igillw@tx}.technion.ac.il

2006 International Symposium on Information Theory
Seattle, Washington
July 10, 2006

Outline

Parallel Channels

- Transmission takes place over a set of J independent memoryless binary-input output-symmetric (MBIOS) channels.
- Each code bit is a-priori assigned to one of the J channels.
- A fraction p_j of the code bits is transmitted over the j'th channel.



Why Parallel Channels ?

Parallel channels are used to model various scenarios:

- Punctured LDPC codes.
- Non-uniformly error protected codes.
- Multi-level codes.
- LDPC coded modulation.
- etc.

Fundamental Questions Regarding LDPC Codes

Question

How good can LDPC codes be, even under ML Decoding ?

Answer to this question $\Rightarrow$

- Quantitative measure of the inherent loss of sub-optimal and practical iterative message-passing decoding algorithms.

Fundamental Questions Regarding LDPC Codes

Question

How good can LDPC codes be, even under ML Decoding ?

Answer to this question $\Rightarrow$

- Quantitative measure of the inherent loss of sub-optimal and practical iterative message-passing decoding algorithms.

Question

How sparse can parity-check matrices of binary linear codes be, as a function of their gap (in rate) to capacity?

Answer to this question$\Rightarrow$

- Lower bounds on the decoding complexity per iteration.

Bounds on Achievable Rates

Information-theoretic bounds for a single MBIOS channel

- Gallager (Ph.D. thesis 1961)
- Burshtein *et al.* (IEEE Trans. on IT, September 2002)
- Sason and Urbanke (IEEE Trans. on IT, July 2003)
- Wiechman and Sason (Allerton 2005)

Bounds referring to ensemble averages for a single MBIOS channel

- Meason *et al*. (ITW 2004)
- Montanari (IEEE Trans. on IT, September 2005)

Bounds on Achievable Rates

Bounds for punctured codes

- Pfister *et al.* (IEEE Trans. on IT, July 2005)

Bounds for parallel MBIOS channels

- Pishro-Nik *et al.* (IEEE Trans. on IT, July 2005)
- Liu *et al.* (IEEE Trans. on Information Theory, April 2006)
- Sason and Goldenberg (Arxiv, July 2006)

Information-Theoretic Bounds on Parity-Check Density

Goal: Achieving a fraction $1-\varepsilon$ of channel capacity

Theorem (Sason & Urbanke, IEEE Trans. on IT, July 03)

1. For any sequence of binary linear block codes achieving a fraction $1-\varepsilon$ of capacity, parity-check density grows at least like $\ln \frac{1}{\varepsilon}$.
2. Logarithmic behavior achievable under ML decoding for any MBIOS channel.
3. For the BEC, it is achievable under iterative decoding.

Un-Quantized Approach (Wiechman & Sason, Allerton 2005)

- Work directly with the soft LLR values at the output of the channel.
- Tightens the coefficient of the logarithmic growth for all MBIOS channels, except for the BEC and the BSC.

Main Results

The main Results of this work:

- Generalization of the information-theoretic bounds for parallel MBIOS channels.
- The bounds serve to evaluate the tradeoff between performance and decoding complexity per iteration for codes transmitted over parallel channels.
- Application of these bounds towards assessing the performance-complexity tradeoff of randomly and intentionally punctured LDPC codes transmitted over an MBIOS channel.

The Un-Quantized Approach

- Define an *equivalent* channel whose output is the LLR of the original communication channel.
- LLR is divided into sign and absolute value.
- Channel symmetry property

  $\Rightarrow$ new channel is a multiplicative channel, where the binary input (converted to +1,-1) multiplies an independent noise.
- Noise is distributed according to the *pdf* of the LLR of the original channel, given that the transmitted symbol is 0.
- Use output magnitude as side information on the noise, and calculate the syndrome w.r.t. the *sign* of the received sequence.

Lower Bound on Cond. Entropy for an MBIOS Channel

Let $\mathcal{C}$ be a binary linear block code of length n and design rate R_{d}.

- Communication over MBIOS channel with capacity C bits/ch. use.
- $\mathbf{x}$, $\mathbf{y}$ - transmitted codeword and received sequence, respectively.
- a - *pdf* of the LLR given that the transmitted symbol is 0.
- For an arbitrary parity-check matrix of $\mathcal{C}$, let Γ_k designate the fraction of the parity-checks involving k variables, and define $\Gamma(x) \triangleq \sum_k \Gamma_k x^k$.

Lower Bound on Cond. Entropy for an MBIOS Channel

Let $\mathcal{C}$ be a binary linear block code of length n and design rate R_{d}.

Theorem

The conditional entropy of the transmitted codeword given the received sequence satisfies

$$\frac{H(\mathbf{X}|\mathbf{Y})}{n} \geq 1 - C - (1 - R_d)\left(1 - \frac{1}{2\ln(2)} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p-1)}\right)$$

$$g_p \triangleq \int_0^{\infty} a(l)(1 + e^{-l}) \tanh^{2p}\left(\frac{l}{2}\right) dl$$

Lower Bound on Cond. Entropy for Parallel Channels

Let $\mathcal{C}$ be a binary linear block code of length n and design rate R_{d}.

Theorem

The conditional entropy of the transmitted codeword given the received sequence satisfies

$$\frac{H(\mathbf{X}|\mathbf{Y})}{n} \geq 1 - \sum_{j=1}^{J} p_j C_j - (1 - R_{\mathrm{d}}) \cdot \left(1 - \frac{1}{2n(1-R_{\mathrm{d}})\ln 2} \sum_{p=1}^{\infty} \frac{\sum_{m=1}^{n(1-R_{\mathrm{d}})} \prod_{j=1}^{J} (g_{j,p})^{\beta_{j,m}}}{p(2p-1)}\right)$$

where

$$g_{j,p} \triangleq \int_0^{\infty} a(l;j)\,(1+e^{-l}) \tanh^{2p}\left(\frac{l}{2}\right) dl, \qquad j \in \{1,\ldots,J\}, \quad p \in \mathbb{N}.$$

Notes

Problem

The values $\beta_{j,m}$ are not usually known.
Therefore the bound cannot be practically evaluated for specific codes.

Notes

Problem

The values $\beta_{j,m}$ are not usually known.
Therefore the bound cannot be practically evaluated for specific codes.

Solution

Consider the expected conditional entropy over an ensemble of codes.

Notes

Problem

The calculation of the expectation over the bound is not tractable

Notes

Problem

The calculation of the expectation over the bound is not tractable

Suggestion

Bound the expectation using Jensen's inequality.
Leads to an inherent loss in the tightness of the bounds.

Notes

Problem

The calculation of the expectation over the bound is not tractable

Observation

- We only consider sequences of ensembles where $n \to \infty$.
- We only need the limit of the expectation when $n \to \infty$.

Notes

Problem

The calculation of the expectation over the bound is not tractable

Observation

- We only consider sequences of ensembles where $n \to \infty$.
- We only need the limit of the expectation when $n \to \infty$.

- The calculation of the limit is possible
- The following bounds are valid only for sequences of *ensembles.*

Upper Bound on the Achievable Rates

Consider a sequence of LDPC ensembles, whose block lengths tend to infinity.

- Transmission over J parallel MBIOS channels.
  Capacity of j'th channel: C_j bits/ch. use.
- p_j denotes the asymptotic fraction of code bits transmitted over the j'th channel.
- q_j denotes the asymptotic fraction of edges in graph connected to code bits transmitted over the j'th channel.

Upper Bound on the Achievable Rates

Theorem

A necessary condition for this sequence to achieve vanishing bit error probability (even under ML decoding) is that the design rate R_d satisfies

$$R_d \leq 1 - \frac{1 - \sum_{j=1}^{J} p_j C_j}{1 - \frac{1}{2\ln 2} \sum_{p=1}^{\infty} \left\{ \frac{1}{p(2p-1)} \Gamma\left(\sum_{j=1}^{J} q_j \, g_{j,p} \right) \right\}}$$

where $\Gamma(x) = \sum_{i=2}^{\infty} \Gamma_i x^i$ is the right degree distribution from the node perspective.

Example: Parallel BECs

Example

For the particular case where the J parallel MBIOS channels are BECs where the erasure probability of the j^{th} channel is ε_j, the common design rate of the sequence of LDPC ensembles is upper bounded by

$$R_d \leq 1 - \frac{\sum_{j=1}^{J} p_j \varepsilon_j}{1 - \Gamma\left(1 - \sum_{j=1}^{J} q_j \, \varepsilon_j\right)} .$$

This coincides with [Pishro-Nik et al. IEEE Trans. on IT, July 2005].

Lower Bound on the Parity Check Density

- Consider a sequence of LDPC ensembles, whose block lengths tend to infinity.
- Assume this sequence achieves a fraction $1 - \varepsilon$ of the average capacity $\overline{C} \triangleq \sum_{j=1}^{J} p_j C_j$ with vanishing bit-error probability.

Theorem

The asymptotic density of their parity-check matrices satisfies

$$\liminf_{m \to \infty} \Delta_m \geq K_1 + K_2 \ln \frac{1}{\varepsilon}$$

where K_1 and K_2 are coefficients which depend on the pdfs *of the J parallel channels and on the values of p_j and q_j.*

- Same form as the bound for a single MBIOS channel.

Intentionally Punctured Codes

- Introduced by Ha and McLaughlin (IEEE Trans. on IT, November 2004)
- Code bits are separated according to the degree of the corresponding node.
- Each set is punctured at a different rate.

Intentionally Punctured Codes

Intentional Puncturing Can be modeled as transmission over a set of parallel channels.

- Each channel transmits bits whose corresponding nodes have a fixed degree.
- The channels are composed of a concatenation of a BEC (which models the puncturing) and the communication channel.
- The fraction q_j of edges transmitted over channel j depends on the left degree distribution of the ensemble.
- The bounds depend on both the left and right degree distributions.

Numerical Results

- Original ensemble design rate 1/2.
- Transmission over binary input AWGN channel.
- Puncturing patterns optimized for iterative decoding.
- Provides bound on inherent loss due to iterative decoding.

Design rate	Capacity limit	Lower bound (ML decoding)	Iterative (IT) Decoding	Fractional gap to cap. (ML vs. IT)
0.500	0.187 dB	0.270 dB	0.393 dB	$\geq$ 40.3%
0.592	0.635 dB	0.716 dB	0.857 dB	$\geq$ 36.4%
0.671	1.083 dB	1.171 dB	1.330 dB	$\geq$ 35.6%
0.774	1.814 dB	1.927 dB	2.115 dB	$\geq$ 37.2%
0.838	2.409 dB	2.547 dB	2.781 dB	$\geq$ 37.1%
0.912	3.399 dB	3.607 dB	3.992 dB	$\geq$ 35.1%

Summary

- Information-theoretic bounds on the thresholds and parity-check density of binary linear block codes for parallel channels.
- Can be applied to punctured codes, non-uniform error protection, multilevel codes and other scenarios
- Bounds on thresholds under ML decoding and exact thresholds under iterative decoding enable to assess the inherent loss due to ensemble structure and the sub-optimality of iterative decoding.
- Lower bounds on the parity-check density enable to assess the performance-complexity tradeoff under iterative decoding.
- The bounds enable the re-derivation of previously reported results or improved versions of those results.

Further Reading

This talk is based on the paper:

- I. Sason and G. Wiechman, “On achievable rates and complexity of LDPC codes for parallel channels with application to puncturing,” accepted to *IEEE Trans. on Information Theory*, June 2006 (currently under revision).
- Available at:

  `http://arxiv.org/abs/cs.IT/0508072`

Thank you for your attention !