On Achievable Rates and Complexity of LDPC Codes for Parallel Channels: Information-Theoretic Bounds and Applications

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Outline

1. Background and Motivation

2. Information-Theoretic Bounds for Parallel Channels

3. Application: Intentionally Punctured Codes

4. Summary
Parallel Channels

- Transmission takes place over a set of $J$ independent memoryless binary-input output-symmetric (MBIOS) channels.
- Each code bit is a-priori assigned to one of the $J$ channels.
- A fraction $p_j$ of the code bits is transmitted over the $j$’th channel.
Why Parallel Channels?

Parallel channels are used to model various scenarios:

- Punctured LDPC codes.
- Non-uniformly error protected codes.
- Multi-level codes.
- LDPC coded modulation.
- etc.
Fundamental Questions Regarding LDPC Codes

Question

*How good can LDPC codes be, even under ML Decoding?*

Answer to this question ⇒

- Quantitative measure of the inherent loss of sub-optimal and practical iterative message-passing decoding algorithms.
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Question

How sparse can parity-check matrices of binary linear codes be, as a function of their gap (in rate) to capacity?

Answer to this question ⇒

- Lower bounds on the decoding complexity per iteration.
Bounds on Achievable Rates

Information-theoretic bounds for a single MBIOS channel
- Gallager (Ph.D. thesis 1961)
- Burshtein et al. (IEEE Trans. on IT, September 2002)
- Wiechman and Sason (Allerton 2005)

Bounds referring to ensemble averages for a single MBIOS channel
- Meason et al. (ITW 2004)
- Montanari (IEEE Trans. on IT, September 2005)
Bounds on Achievable Rates

Bounds for punctured codes
- Pfister et al. (IEEE Trans. on IT, July 2005)

Bounds for parallel MBIOS channels
- Pishro-Nik et al. (IEEE Trans. on IT, July 2005)
- Liu et al. (IEEE Trans. on Information Theory, April 2006)
- Sason and Goldenberg (Arxiv, July 2006)
Background and Motivation

Related Work

**Information-Theoretic Bounds on Parity-Check Density**

Goal: Achieving a fraction $1 - \varepsilon$ of channel capacity

**Theorem (Sason & Urbanke, IEEE Trans. on IT, July 03)**

1. For any sequence of binary linear block codes achieving a fraction $1 - \varepsilon$ of capacity, parity-check density grows at least like $\ln \frac{1}{\varepsilon}$.
2. Logarithmic behavior achievable under ML decoding for any MBIOS channel.
3. For the BEC, it is achievable under iterative decoding.

**Un-Quantized Approach (Wiechman & Sason, Allerton 2005)**

- Work directly with the soft LLR values at the output of the channel.
- Tightens the coefficient of the logarithmic growth for all MBIOS channels, except for the BEC and the BSC.
Main Results

The main Results of this work:

- Generalization of the information-theoretic bounds for parallel MBIOS channels.

- The bounds serve to evaluate the tradeoff between performance and decoding complexity per iteration for codes transmitted over parallel channels.

- Application of these bounds towards assessing the performance-complexity tradeoff of randomly and intentionally punctured LDPC codes transmitted over an MBIOS channel.
The Un-Quantized Approach

- Define an \textit{equivalent} channel whose output is the LLR of the original communication channel.

- LLR is divided into sign and absolute value.

- Channel symmetry property
  \Rightarrow new channel is a multiplicative channel, where the binary input (converted to +1,-1) multiplies an independent noise.

- Noise is distributed according to the \textit{pdf} of the LLR of the original channel, given that the transmitted symbol is 0.

- Use output magnitude as side information on the noise, and calculate the syndrome w.r.t. the \textit{sign} of the received sequence.
Lower Bound on Cond. Entropy for an MBIOS Channel

Let $C$ be a binary linear block code of length $n$ and design rate $R_d$.

- Communication over MBIOS channel with capacity $C$ bits/ch. use.
- $x, y$ - transmitted codeword and received sequence, respectively.
- $a$ - pdf of the LLR given that the transmitted symbol is 0.
- For an arbitrary parity-check matrix of $C$, let $\Gamma_k$ designate the fraction of the parity-checks involving $k$ variables, and define $\Gamma(x) \triangleq \sum_k \Gamma_k x^k$. 
Lower Bound on Cond. Entropy for an MBIOS Channel

Let $C$ be a binary linear block code of length $n$ and design rate $R_d$.

**Theorem**

The conditional entropy of the transmitted codeword given the received sequence satisfies

$$\frac{H(X|Y)}{n} \geq 1 - C - (1 - R_d) \left( 1 - \frac{1}{2 \ln(2)} \sum_{p=1}^{\infty} \frac{\Gamma(g_p)}{p(2p - 1)} \right)$$

where

$$g_p \triangleq \int_{0}^{\infty} a(l)(1 + e^{-l}) \tanh^{2p} \left( \frac{l}{2} \right) dl$$
Lower Bound on Cond. Entropy for Parallel Channels

Let $C$ be a binary linear block code of length $n$ and design rate $R_d$.

**Theorem**

The conditional entropy of the transmitted codeword given the received sequence satisfies

$$\frac{H(X|Y)}{n} \geq 1 - \sum_{j=1}^{J} p_j C_j - (1 - R_d)$$

$$\cdot \left( 1 - \frac{1}{2n(1 - R_d) \ln 2} \sum_{p=1}^{\infty} \frac{\sum_{m=1}^{n(1 - R_d)} \prod_{j=1}^{J} (g_{j,p})^{\beta_{j,m}}}{p(2p - 1)} \right)$$

where

$$g_{j,p} \triangleq \int_{0}^{\infty} a(l; j) \left( 1 + e^{-l} \right) \tanh^{2p} \left( \frac{l}{2} \right) dl, \quad j \in \{1, \ldots, J\}, \quad p \in \mathbb{N}.$$
Notes

Problem

The values $\beta_{j,m}$ are not usually known. Therefore the bound cannot be practically evaluated for specific codes.
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Solution
Consider the expected conditional entropy over an ensemble of codes.
Notes

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The calculation of the expectation over the bound is not tractable
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Suggestion
Bound the expectation using Jensen’s inequality.
 Leads to an inherent loss in the tightness of the bounds.
Notes

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Observation
- We only consider sequences of ensembles where $n \to \infty$.
- We only need the limit of the expectation when $n \to \infty$. 
## Notes

### Problem

The calculation of the expectation over the bound is not tractable.

### Observation

- We only consider sequences of ensembles where $n \to \infty$.
- We only need the limit of the expectation when $n \to \infty$.
- The calculation of the limit is possible.
- The following bounds are valid only for sequences of ensembles.
Upper Bound on the Achievable Rates

Consider a sequence of LDPC ensembles, whose block lengths tend to infinity.

- Transmission over \( J \) parallel MBIOS channels. Capacity of \( j \)'th channel: \( C_j \) bits/ch. use.
- \( p_j \) denotes the asymptotic fraction of code bits transmitted over the \( j \)'th channel.
- \( q_j \) denotes the asymptotic fraction of edges in graph connected to code bits transmitted over the \( j \)'th channel.
Upper Bound on the Achievable Rates

Theorem

A necessary condition for this sequence to achieve vanishing bit error probability (even under ML decoding) is that the design rate $R_d$ satisfies

$$R_d \leq 1 - \frac{1}{2 \ln 2} \sum_{p=1}^{\infty} \left\{ \frac{1}{p(2p-1)} \Gamma \left( \sum_{j=1}^{J} q_j g_{j,p} \right) \right\}$$

where $\Gamma(x) = \sum_{i=2}^{\infty} \Gamma_i x^i$ is the right degree distribution from the node perspective.
Example: Parallel BECs

For the particular case where the $J$ parallel MBIOS channels are BECs where the erasure probability of the $j^{th}$ channel is $\varepsilon_j$, the common design rate of the sequence of LDPC ensembles is upper bounded by

$$R_d \leq 1 - \sum_{j=1}^{J} p_j \varepsilon_j \quad \frac{1 - \Gamma \left( 1 - \sum_{j=1}^{J} q_j \varepsilon_j \right)}{1 - \Gamma \left( 1 - \sum_{j=1}^{J} q_j \varepsilon_j \right)}.$$  

This coincides with [Pishro-Nik et al. IEEE Trans. on IT, July 2005].
Lower Bound on the Parity Check Density

- Consider a sequence of LDPC ensembles, whose block lengths tend to infinity.
- Assume this sequence achieves a fraction $1 - \varepsilon$ of the average capacity $\bar{C} = \sum_{j=1}^{J} p_j C_j$ with vanishing bit-error probability.

**Theorem**

The asymptotic density of their parity-check matrices satisfies

$$\liminf_{m \to \infty} \Delta_m \geq K_1 + K_2 \ln \frac{1}{\varepsilon}$$

where $K_1$ and $K_2$ are coefficients which depend on the pdfs of the $J$ parallel channels and on the values of $p_j$ and $q_j$.

- Same form as the bound for a single MBIOS channel.
Intentionally Punctured Codes

- Introduced by Ha and McLaughlin (IEEE Trans. on IT, November 2004)

- Code bits are separated according to the degree of the corresponding node.

- Each set is punctured at a different rate.
Intentionally Punctured Codes

Intentional Puncturing Can be modeled as transmission over a set of parallel channels.

- Each channel transmits bits whose corresponding nodes have a fixed degree.
- The channels are composed of a concatenation of a BEC (which models the puncturing) and the communication channel.
- The fraction $q_j$ of edges transmitted over channel $j$ depends on the left degree distribution of the ensemble.
- The bounds depend on both the left and right degree distributions.
Numerical Results

- Original ensemble design rate 1/2.
- Transmission over binary input AWGN channel.
- Puncturing patterns optimized for iterative decoding.
- Provides bound on inherent loss due to iterative decoding.

<table>
<thead>
<tr>
<th>Design rate</th>
<th>Capacity limit</th>
<th>Lower bound (ML decoding)</th>
<th>Iterative (IT) Decoding</th>
<th>Fractional gap to cap. (ML vs. IT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.500</td>
<td>0.187 dB</td>
<td>0.270 dB</td>
<td>0.393 dB</td>
<td>≥ 40.3%</td>
</tr>
<tr>
<td>0.592</td>
<td>0.635 dB</td>
<td>0.716 dB</td>
<td>0.857 dB</td>
<td>≥ 36.4%</td>
</tr>
<tr>
<td>0.671</td>
<td>1.083 dB</td>
<td>1.171 dB</td>
<td>1.330 dB</td>
<td>≥ 35.6%</td>
</tr>
<tr>
<td>0.774</td>
<td>1.814 dB</td>
<td>1.927 dB</td>
<td>2.115 dB</td>
<td>≥ 37.2%</td>
</tr>
<tr>
<td>0.838</td>
<td>2.409 dB</td>
<td>2.547 dB</td>
<td>2.781 dB</td>
<td>≥ 37.1%</td>
</tr>
<tr>
<td>0.912</td>
<td>3.399 dB</td>
<td>3.607 dB</td>
<td>3.992 dB</td>
<td>≥ 35.1%</td>
</tr>
</tbody>
</table>
Summary

- Information-theoretic bounds on the thresholds and parity-check density of binary linear block codes for parallel channels.
- Can be applied to punctured codes, non-uniform error protection, multilevel codes and other scenarios.
- Bounds on thresholds under ML decoding and exact thresholds under iterative decoding enable to assess the inherent loss due to ensemble structure and the sub-optimality of iterative decoding.
- Lower bounds on the parity-check density enable to assess the performance-complexity tradeoff under iterative decoding.
- The bounds enable the re-derivation of previously reported results or improved versions of those results.
Further Reading

This talk is based on the paper:


Available at:

http://arxiv.org/abs/cs.IT/0508072

Thank you for your attention!