On Rényi Entropy Power Inequalities

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Outline

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   - Definitions and Motivation
   - The Question

2 A New Rényi EPI

3 Further Tightening the Rényi EPI
   - The Optimization Problem
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Entropy Power

Definition 1 (Entropy Power)

Let $X$ be a $d$-dimensional random vector (r.v.) with differential entropy $h(X)$. The entropy power of $X$ is

$$N(X) = \exp \left( \frac{2}{d} h(X) \right).$$
Entropy Power

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$$N(X) = \exp \left( \frac{2}{d} h(X) \right) .$$

- $\frac{2}{d}$ in the exponent implies homogeneity of order 2:
  $$N(\lambda X) = \lambda^2 N(X), \quad \forall \lambda \in \mathbb{R}.$$

- If $X \sim N(0, \sigma^2 I_d)$, then $h(X) = \frac{d}{2} \log(2\pi e\sigma^2)$, and
  $$N(X) = 2\pi e\sigma^2 .$$

In some definitions the entropy power is normalized by $2\pi e$. 
The Entropy Power Inequality

Introduced by Shannon in his 1948 fundamental paper: “A mathematical theory of communication.”

The Entropy Power Inequality (EPI)

Let $\{X_k\}_{k=1}^n$ be independent r.v.’s. Then,

$$N\left(\sum_{k=1}^n X_k\right) \geq \sum_{k=1}^n N(X_k)$$

and equality holds if and only if $\{X_k\}_{k=1}^n$ are Gaussians with proportional covariances.
Applications of the EPI

Converse theorems for...

- The capacity region of the Gaussian broadcast channel - Bergmans, 1974
- Multi-terminal rate-distortion theory (the quadratic Gaussian CEO problem) - Oohama, 1998.
- The capacity region of the Gaussian broadcast MIMO channel - Weingarten, Steinberg & Shamai, 2006.
Rényi’s Entropy

Definition 2

Let $X$ be a $d$-dimensional r.v. with density $f_X$, and $\alpha \in (0, 1) \cup (1, \infty)$. The order-$\alpha$ Rényi entropy of $X$ is

$$h_\alpha(X) = \frac{\alpha}{1 - \alpha} \log \|f_X\|_\alpha = \frac{1}{1 - \alpha} \log \left( \int_{\mathbb{R}^d} f_X^\alpha(x) \, dx \right).$$
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By continuous extension in $\alpha$, we have

- $h_0(X) = \log \mu(\text{supp}(f_X))$.
- $h_1(X) = h(X) = -\int_{\mathbb{R}^d} f_X(x) \log f_X(x) \, dx$.
- $h_\infty(X) = -\log(\text{ess sup}(f_X))$.

where $\mu$ is the Lebesgue measure in $\mathbb{R}^d$. 
Properties of Rényi’s Entropy

Let $X$ be a $d$-dimensional r.v. with density.

- $h_\alpha(X)$ is continuous in $\alpha \in [0, \infty]$
- $h_\alpha(X)$ is monotonically non-increasing in $\alpha \in [0, \infty]$, 
  \[ 0 \leq \beta \leq \alpha \implies h_\beta(X) \geq h_\alpha(X). \]
- If $X = (X_1, \ldots, X_d)$ has independent elements, then 
  \[ h_\alpha(X) = \sum_{k=1}^{d} h_\alpha(X_k), \quad \forall \alpha \in [0, \infty]. \]
  (similar to Shannon’s entropy)
- Unlike Shannon’s entropy, 
  \[ h_\alpha(X) \nleq \sum_{k=1}^{d} h_\alpha(X_k). \]
Rényi’s Entropy Power

**Definition 3**

- Let $X$ be a $d$-dimensional r.v. with density.
- Let $\alpha \in [0, \infty]$.

The Rényi entropy power of $X$ is

$$N_\alpha(X) = \exp \left( \frac{2}{d} h_\alpha(X) \right).$$
Rényi’s Entropy Power

**Definition 3**

- Let $X$ be a $d$-dimensional r.v. with density.
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*The Rényi entropy power of $X$ is*

$$N_\alpha(X) = \exp \left( \frac{2}{d} h_\alpha(X) \right).$$

**Homogeneity of order 2:**

$$N_\alpha(\lambda X) = \lambda^2 N_\alpha(X), \ \forall \lambda \in \mathbb{R}, \ \alpha \in [0, \infty].$$
An Application of The Rényi Entropy - Example

\[ P^n_X \xrightarrow{X^n \in X^n} f_n \xrightarrow{m \in \{1, \ldots, 2^{nR}\}} \phi_n \xrightarrow{L = \{x^n \in X^n: f_n(x^n) = m\}} \]

- Fixed \( \rho > 0 \): rate \( R \) is called achievable if there exist encoders \( \{f_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} \mathbb{E}[|L|^{\rho}] = 1 \).

- Direct and converse results: \(^1\)

\[
R > H_{\frac{1}{1+\rho}}(X) \Rightarrow R \text{ is achievable} \\
R < H_{\frac{1}{1+\rho}}(X) \Rightarrow R \text{ is not achievable}
\]

A Rényi EPI (R-EPI)?

Formally,
1. Let \( \{X_k\}_{k=1}^n \) be \( d \)-dimensional independent r.v.’s with densities.
2. Let \( \alpha \in [0, \infty] \), \( n \in \mathbb{N} \).

Does there exist a positive constant \( c_{\alpha}^{(n,d)} \) such that

\[
N_{\alpha} \left( \sum_{k=1}^{n} X_k \right) \geq c_{\alpha}^{(n,d)} \sum_{k=1}^{n} N_{\alpha}(X_k)
\]
A Rényi EPI (R-EPI)?

- Formally,
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\[
N_{\alpha} \left( \sum_{k=1}^{n} X_k \right) \geq c_{\alpha}^{(n,d)} \sum_{k=1}^{n} N_{\alpha}(X_k) ?
\]

- For independent Gaussian random vectors with proportional covariances, \( N_{\alpha} \left( \sum_{k=1}^{n} X_k \right) = \sum_{k=1}^{n} N_{\alpha}(X_k) \), for every \( \alpha \in [0, \infty] \).

\[
\implies c_{\alpha}^{(n,d)} \leq 1, \quad \forall \alpha \in [0, \infty].
\]
Related Work - EPI

1. Shannon, 1948 - the entropy power inequality (EPI)
   ▶ Many information-theoretic proofs have been suggested (e.g., Stam - 1959, Gou-Shamai-Verdú - 2006, Rioul - 2011).

2. Zamir and Feder, 1993: a vector generalization of the EPI.

3. Baron and Madiman, 2007: Some generalizations of the EPI, and connection to the CLT.

4. EPI for discrete random variables:
   ▶ Jog and Anantharam, 2014.
   ▶ Telatar et al., 2014.

Related Work - Rényi EPI

1. Bercher and Vignat (BV), 2002: for every $\alpha \in [0, \infty]$, 
   \[ N_\alpha \left( \sum_{k=1}^{n} X_k \right) \geq \max_{1 \leq k \leq n} N_\alpha(X_k). \]


3. Bobkov and Chistyakov (BC), 2015: for every $\alpha > 1$, 
   \[ c_\alpha = \frac{1}{e} \alpha^{\frac{1}{\alpha - 1}} \] (independently of $d$ and $n$).


5. Xu, Melbourne & Madiman, ISIT 2016: reverse Rényi EPIs for $s$-concave densities ($s = 1 \Rightarrow \text{log-concavity}$).

Our work provides the tightest R-EPIs known so far, for $\alpha > 1$. 
Theorem 1

Let

- $\{X_k\}_{k=1}^n$ be $d$–dimensional independent r.v’s with densities.
- $\alpha > 1$, $\alpha' = \frac{\alpha}{\alpha-1}$.
- $n \in \mathbb{N}$.

Then, the following R-EPI holds:

$$N_\alpha \left( \sum_{k=1}^n X_k \right) \geq c_\alpha^{(n)} \sum_{k=1}^n N_\alpha(X_k),$$

with

$$c_\alpha^{(n)} = \alpha^{\frac{1}{\alpha-1}} \left( 1 - \frac{1}{n\alpha'} \right)^{n\alpha'-1}.$$
Theorem 1 Implications

Theorem 1 \(\Rightarrow\) BC bound

Theorem 1 improves the R-EPI by Bobkov and Chistyakov \((c_\alpha = \frac{1}{e} \alpha^{\frac{1}{\alpha-1}})\) for every \(\alpha > 1\) and \(n \in \mathbb{N}\); for every \(\alpha > 1\), it asymptotically coincides with the R-EPI by Bobkov and Chistyakov as \(n \to \infty\).
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**Theorem 1 ⇒ BC bound**

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**Theorem 1 ⇒ EPI**

If \( \alpha \downarrow 1 \), Theorem 1 coincides with the EPI.
Theorem 1 Implications

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**Theorem 1 ⇒ EPI**

If \( \alpha \downarrow 1 \), Theorem 1 coincides with the EPI.

**Asymptotic Tightness of the Result in Theorem 1**

If \( n = 2 \) and \( \alpha \to \infty \), \( c_\alpha^{(n)} \) tends to \( \frac{1}{2} \) which is optimal; achieved when \( X_1 \) and \( X_2 \) are uniformly distributed in the cube \([0, 1]^d\).
$c^{(n)}_{\alpha}$ as a function of $\alpha$

**Figure:** $c^{(n)}_{\alpha}$ as a function of $\alpha$ for $n = 2, 3, 10$ and $n \to \infty$
Outline of the Proof of Theorem 1

Main Tool: The Sharpened Young’s Inequality

Let $p, q, r \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ be non-negative functions. Then

$$\|f * g\|_r \leq \left( \frac{A_p A_q}{A_r} \right)^{\frac{d}{2}} \|f\|_p \|g\|_q,$$

where $A_t = t^{\frac{1}{t}} t'^{-\frac{1}{t'}}$ and $t' = \frac{t}{t-1}$. Equality holds if and only if $f$ and $g$ are Gaussians or $r = p = q = 1$. 
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where $A_t = t \frac{1}{t} t' \frac{1}{t'}$ and $t' = \frac{t}{t-1}$. Equality holds if and only if $f$ and $g$ are Gaussians or $r = p = q = 1$.

- Reversed for $p, q, r \in (0, 1]$.
- Using mathematical induction:

$$\| f_1 * \ldots * f_n \|_{\nu} \leq A \prod_{k=1}^{n} \| f_k \|_{\nu_k}, \quad A = \left( \frac{1}{A_{\nu}} \prod_{k=1}^{n} A_{\nu_k} \right)^{\frac{d}{2}}.$$
Outline of the Proof of Theorem 1

Young's sharpened inequality and the monotonicity property of the Rényi entropy yield the following observation.

Let $P^n = \{ t \in \mathbb{R}^n : t_k \geq 0, \sum_{k=1}^n t_k = 1 \}$ be the probability simplex and let $\alpha > 1$. If $\sum_{k=1}^n N_\alpha(X_k) = 1$, then

$$\log N_\alpha \left( \sum_{k=1}^n X_k \right) \geq f_0(t), \ \forall t \in P^n,$$

where

- $f_0(t) = \frac{\log \alpha}{\alpha - 1} - D(t||N_\alpha) + \alpha' \sum_{k=1}^n \left( 1 - \frac{t_k}{\alpha'} \right) \log \left( 1 - \frac{t_k}{\alpha'} \right)$.
- $N_\alpha = (N_\alpha(X_1), \ldots, N_\alpha(X_n))$.
- $D(t||N_\alpha) = \sum_{k=1}^n t_k \log \left( \frac{t_k}{N_\alpha(X_k)} \right)$. 

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Outline of the Proof of Theorem 1

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\[
\log N_\alpha \left( \sum_{k=1}^{n} X_k \right) \geq f_0(t), \quad \forall t \in P^n,
\]

\( \implies \) The R-EPI can be tightened by maximizing \( f_0(t) \).
Outline of the Proof of Theorem 1

Young’s sharpened inequality and the monotonicity property of the Rényi entropy yield the following observation.

Let $\mathcal{P}^n = \{ t \in \mathbb{R}^n : t_k \geq 0, \sum_{k=1}^n t_k = 1 \}$ be the probability simplex and let $\alpha > 1$. If $\sum_{k=1}^n N_{\alpha}(X_k) = 1$, then

$$\log N_{\alpha} \left( \sum_{k=1}^n X_k \right) \geq f_0(t), \quad \forall t \in \mathcal{P}^n,$$

$\implies$ The R-EPI can be tightened by maximizing $f_0(t)$.

- The solution of the optimization problem leads to an implicit bound in most cases.
- Instead, we take a sub-optimal choice $t_k = N_{\alpha}(X_k)$ (it can be verified to be optimal if $N_{\alpha}(X_k)$ is independent of $k$).
- Some more steps yield Theorem 1.
Sub Optimality

- BV bound for $n = 2$:

  $$N_\alpha(X_1 + X_2) \geq \max \{N_\alpha(X_1), N_\alpha(X_2)\}$$
  $$\geq \frac{1}{2} (N_\alpha(X_1) + N_\alpha(X_2)), \alpha \in [0, \infty].$$

- Theorem 1 for $n = 2$ and $\alpha \to \infty$ yields

  $$N_\infty(X_1 + X_2) \geq \frac{1}{2} (N_\infty(X_1) + N_\infty(X_2)).$$

- Since the maximal value of two numbers is larger than or equal to their average, the BV bound is tighter than our bound in Theorem 1 for $n = 2$ and large enough $\alpha$’s (unless $N_\infty(X_1) = N_\infty(X_2)$).
The Optimization Problem

Recall that \( \log \mathcal{N}_\alpha \left( \sum_{k=1}^{n} X_k \right) \geq f_0(t), \quad \forall \ t \in \mathcal{P}^n. \)

- The optimization problem is not convex

maximize \( f_0(t_1, t_2, \ldots, t_{n-1}, t_n) \)
subject to \( t_k \geq 0, \quad k \in \{1, \ldots, n\}, \)
\( \sum_{k=1}^{n} t_k = 1 \)
The Optimization Problem

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\text{maximize} & \quad f_0(t_1, t_2, \ldots, t_{n-1}, t_n) \\
\text{subject to} & \quad t_k \geq 0, \quad k \in \{1, \ldots, n\}, \\
& \quad \sum_{k=1}^n t_k = 1
\end{align*}$$

- An equivalent problem

$$\begin{align*}
\text{maximize} & \quad f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k) \\
\text{subject to} & \quad t_k \geq 0, \quad k \in \{1, \ldots, n-1\}, \\
& \quad \sum_{k=1}^{n-1} t_k \leq 1
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\end{align*}
\]

- This problem can be shown to be convex by a non trivial use of the next result from matrix theory (Bunch et al. 1978).
Rank–One Modification Theorem (Bunch et al. 1978)

Let

- $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix with the eigenvalues $d_1 \leq d_2 \leq \ldots \leq d_n$.
- $C$ be a rank-one modification of $D$ i.e., $C = D + \rho zz^T$, where $z \in \mathbb{R}^n$, $\rho \in \mathbb{R}$, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be its eigenvalues.

Then,
Rank–One Modification Theorem (Bunch et al. 1978)

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- $C$ be a rank-one modification of $D$ i.e., $C = D + \rho z z^T$, where $z \in \mathbb{R}^n$, $\rho \in \mathbb{R}$, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be its eigenvalues.

Then,

1. If $\rho > 0$, then $d_1 \leq \lambda_1 \leq d_2 \leq \lambda_2 \leq \ldots \leq d_n \leq \lambda_n$. If $\rho < 0$, then $\lambda_1 \leq d_1 \leq \lambda_2 \leq d_2 \leq \ldots \leq \lambda_n \leq d_n$.

2. If $d_j \neq d_i$ and $z_i, \rho \neq 0$, then the inequalities are strict, and for every $i \in \{1, \ldots, n\}$, $\lambda_i$ is a zero of $W(x) = 1 + \rho \sum_{j=1}^{n} \frac{z_i^2}{d_j - x}$.
Applying The Rank–One Modification Theorem

1. The Hessian matrix of $f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k)$:

   $$\nabla^2 f_0 = D + \rho 11^T$$

2. The Rank–One Modification Theorem is used to prove that $\nabla^2 f_0$ is negative semi-definite, hence $f_0$ is concave.
Applying The Rank–One Modification Theorem

1. The Hessian matrix of $f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k)$:

$$\nabla^2 f_0 = D + \rho \mathbb{1} \mathbb{1}^T$$

2. The Rank–One Modification Theorem is used to prove that $\nabla^2 f_0$ is negative semi-definite, hence $f_0$ is concave.

3. The optimization problem

$$\begin{align*}
\text{maximize} & \quad f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k) \\
\text{subject to} & \quad t_k \geq 0, \quad k \in \{1, \ldots, n-1\}, \\
& \quad \sum_{k=1}^{n-1} t_k \leq 1
\end{align*}$$

is convex.

4. The solution can be found by solving the KKT conditions.
The KKT Conditions

- The optimization problem

\[
\text{maximize } \quad f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k)
\]

\[
\text{subject to } \quad t_k \geq 0, \quad k \in \{1, \ldots, n-1\},
\]

\[
\sum_{k=1}^{n-1} t_k \leq 1
\]

- Assume w.l.o.g that \( N_\alpha(X_k) \leq N_\alpha(X_n), \quad k \in \{1, \ldots, n-1\} \).

- Set \( c_k = \frac{N_\alpha(X_k)}{N_\alpha(X_n)}, \quad k \in \{1, \ldots, n-1\} \).
The KKT Conditions

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\text{maximize} & \quad f_0(t_1, t_2, \ldots, t_{n-1}, 1 - \sum_{k=1}^{n-1} t_k) \\
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After some simplifications, the KKT conditions are:

1. \(t_k(\alpha' - t_k) = c_k t_n(\alpha' - t_n), \quad k \in \{1, \ldots, n-1\}\)
2. \(\sum_{k=1}^{n} t_k = 1\)
3. \(t_k \geq 0, \quad k \in \{1, \ldots, n\}\)
Theorem 2

Let $X_1, \ldots, X_n$ be $d$-dimensional independent r.v’s with densities and assume, w.l.o.g, that $N_\alpha(X_k) \leq N_\alpha(X_n), \quad k \in \{1, \ldots, n - 1\}$. 
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- Let $X_1, \ldots, X_n$ be $d$-dimensional independent r.v’s with densities and assume, w.l.o.g, that $N_\alpha(X_k) \leq N_\alpha(X_n)$, $k \in \{1, \ldots, n - 1\}$.
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- Let $c_k = \frac{N_\alpha(X_k)}{N_\alpha(X_n)}, \ k \in \{1, \ldots, n - 1\}$.
- Let $t_n \in [0, 1]$ be the unique solution of $t_n + \sum_{k=1}^{n-1} \psi_k(t_n) = 1$ with

$$
\psi_k(x) = \frac{\alpha' - \sqrt{\alpha'^2 - 4c_k x(\alpha'-x)}}{2}, \ x \in [0, 1].
$$
Theorem 2

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\psi_k(x) = \frac{\alpha' - \sqrt{\alpha'^2 - 4c_k x(\alpha' - x)}}{2}, \quad x \in [0, 1].
$$

Define $t_k = \psi_k(t_n)$, $k \in \{1, \ldots, n - 1\}$.

Then, the following R-EPI holds:

$$
N_\alpha \left( \sum_{k=1}^{n} X_k \right) \geq e^{f_0(t_1, \ldots, t_n)} \sum_{k=1}^{n} N_\alpha(X_k),
$$
Theorem 2 Implications

Theorem 2 ⇒ Theorem 1

Theorem 2 improves the R-EPI in Theorem 1 unless $N_{\alpha}(X_k)$ is independent of $k$; in the latter case, the two R-EPIs coincide.
Theorem 2 Implications

Theorem 2 ⇒ Theorem 1

- Theorem 2 improves the R-EPI in Theorem 1 unless $N_\alpha(X_k)$ is independent of $k$; in the latter case, the two R-EPIs coincide.
- However, Theorem 1 gives a closed-form bound.
## Theorem 2 Implications

<table>
<thead>
<tr>
<th>Theorem 2 ⇒ Theorem 1</th>
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<tbody>
<tr>
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Theorem 2 Implications

**Theorem 2 ⇒ Theorem 1**
- Theorem 2 improves the R-EPI in Theorem 1 unless $N_\alpha(X_k)$ is independent of $k$; in the latter case, the two R-EPIs coincide.
- However, Theorem 1 gives a closed-form bound.
- Recall that Theorem 1 ⇒ BC bound & the EPI.

**Theorem 2 ⇒ BV Bound**
- Improves the BV bound ($N_\alpha(\sum_{k=1}^{n} X_k) \geq \max_{1 \leq k \leq n} N_\alpha(X_k)$).
- Both bounds asymptotically coincide as $\alpha \to \infty$ if and only if
  $$\sum_{k=1}^{n-1} N_\infty(X_k) \leq N_\infty(X_n)$$
Further Tightening the Rényi EPI

A Tighter Rényi EPI

Closed-Form Expression of Theorem 2 for \( n = 2 \)

**Corollary 1**

Let

- \( X_1 \) and \( X_2 \) be \( d \)-dimensional independent r.v’s with densities.
- \( \alpha > 1, \alpha' = \frac{\alpha}{\alpha - 1} \).
- \( \beta_\alpha = \frac{N_\alpha(X_1)}{N_\alpha(X_2)} \) (Recall that w.l.o.g \( N_\alpha(X_1) \leq N_\alpha(X_2) \)).
- \( t_\alpha = \begin{cases} \frac{\alpha'(\beta_\alpha+1)-2\beta_\alpha-\sqrt{(\alpha'(\beta_\alpha+1))^2-8\alpha'\beta_\alpha+4\beta_\alpha}}{2(1-\beta_\alpha)} & \text{if } \beta_\alpha < 1 \\ \frac{1}{2} & \text{if } \beta_\alpha = 1 \end{cases} \)
Closed-Form Expression of Theorem 2 for $n = 2$

**Corollary 1**

The following R-EPI holds:

$$N_\alpha(X_1 + X_2) \geq c_\alpha \left( N_\alpha(X_1) + N_\alpha(X_2) \right),$$

where

$$c_\alpha = \alpha^{1-\alpha} \exp \left\{ -d(t_\alpha \| \frac{\beta_\alpha}{\beta_\alpha + 1}) \right\} \left( 1 - \frac{t_\alpha}{\alpha'} \right)^{\alpha'-t_\alpha} \left( 1 - \frac{1 - t_\alpha}{\alpha'} \right)^{\alpha'-1+t_\alpha}$$

and $d(x\|y)$ is the binary relative entropy

$$d(x\|y) = x \log \left( \frac{x}{y} \right) + (1 - x) \log \left( \frac{1-x}{1-y} \right), \quad 0 \leq x, y \leq 1.$$
Closed-Form Expression of Theorem 2 for $n = 2$

For $n = 2$ (two summands), our tightest bound in Theorem 2 is asymptotically tight when $\alpha \to \infty$ and is achieved by two independent $d$-dimensional random vectors uniformly distributed in the cubes $[0, \sqrt{N_1}]^d$ and $[0, \sqrt{N_2}]^d$. 
Comparing the R-EPIs \((n = 3)\)

![Graph comparing R-EPIs for n = 3](image)

**Figure:** A comparison of the R-EPIs from Bobkov&Chistyakov (BC), Bercher&Vignat (BV), Theorem 1 and Theorem 2 for \(n = 3\)
Summary - Analytical Tools

- Theorem 1:
  1. The sharpened Young’s inequality
  2. Monotonicity of the Rényi entropy power in its order
Summary - Analytical Tools

- **Theorem 1:**
  1. The sharpened Young’s inequality
  2. Monotonicity of the Rényi entropy power in its order

- **Theorem 2 - a further improvement:**
  1. The rank-one modification theorem - proving convexity
  2. Convex optimization and solution of the KKT conditions
Publications


Further Research: R-EPI For $\alpha \in [0, 1)$

- Are our bounding techniques extendible to $\alpha < 1$?
Further Research: R-EPI For $\alpha \in [0, 1)$

- Are our bounding techniques extendible to $\alpha < 1$?
- Unfortunately, not. In this case, Young’s inequality and the monotonicity property of the Rényi entropy power yield inequalities in opposite directions.
Further Research: R-EPI For $\alpha \in [0, 1)$

- For $\alpha = 0$, one can use the Brunn-Minkowski (BM) inequality:

$$
\mu^\frac{1}{d}(A + B) \geq \mu^\frac{1}{d}(A) + \mu^\frac{1}{d}(B).
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It is conjectured that $c_\alpha = 1$ for all $\alpha \in (0, 1)$. This needs to be proved.
R-EPI For $\alpha \in [0, 1)$

**Proposition 1 (R-EPI for $\alpha \in [0, 1)$)**

Let $\{X_k\}_{k=1}^n$ be independent uniformly distributed random vectors and let $\alpha \in [0, 1)$. Then the following R-EPI holds,

$$N_\alpha \left( \sum_{k=1}^n X_k \right) \geq \sum_{k=1}^n N_\alpha (X_k)$$
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**Proof.**

1. Monotonicity of the Rényi entropy power in its order:

$$N_\alpha (\sum_{k=1}^n X_k) \geq N_1 (\sum_{k=1}^n X_k)$$

2. EPI: $N_1 (\sum_{k=1}^n X_k) \geq \sum_{k=1}^n N_1 (X_k)$.

3. For uniformly distributed random vectors, $N_1 (X_k) = N_\alpha (X_k)$
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- The uniform and Gaussian cases satisfy the conjecture.
Further Research: More Topics

1. Rényi EPIs for discrete random vectors
2. Possible generalizations of the Rényi EPI in a way which generalizes the result by Feder and Zamir (IEEE Trans. on IT, 1993)
3. Possible generalizations with Rényi measures of the extended EPIs by Barron and Madiman (IEEE Trans. on IT, 2007)
4. Possible strengthening of the Rényi EPI by restriction to some families of distributions, e.g.,
   - extension of EPIs by Toscani (2015) for log-concave distributions;
   - extension of EPIs by Courtade (ISIT 2016).
Backup
Example: Data Filtering (FIR)

- Let $Y_k = 2X_k - X_{k-1} - X_{k-2}$ be an output of a FIR filter, where \( \{X_k\} \) are i.i.d. random variables.

- Using the homogeneity of $N_\alpha(\cdot)$, we can consider the difference $h_2(Y) - h_2(X)$:

$$N_2(Y_k) \geq c_2 \left( 4N_2(X_k) + N_2(X_{k-1}) + N_2(X_{k-2}) \right)$$

$$= c_2 6 N_2(X_k)$$
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3. Bobkov and Chistyakov: $h_2(Y) - h_2(X) \geq 0.7425$.
4. Bercher and Vignat: $h_2(Y) - h_2(X) \geq 0.6931$. 
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4. Bercher and Vignat: $h_2(Y) - h_2(X) \geq 0.6931$.
5. If $X_k$ is a Gaussian: $h_2(Y) - h_2(X) = 0.8959$. 
An Application of The Rényi Entropy - Example


- A task is drawn from a finite set $\mathcal{X}$ with probability $P$.
- The task should be described with a fixed number of bits.
- **No task should be neglected. Not even the atypical ones** (classic source coding cannot be used).
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- The task should be described with a fixed number of bits.
- **No task should be neglected. Not even the atypical ones** (classic source coding cannot be used).
- The encoder partitions $\mathcal{X}$ into $M$ subsets, $f : \mathcal{X} \to \{1, \ldots, M\}$, such that for every $x \in \mathcal{X}$, $f^{-1}(f(x))$ is the subset that contains $x$. 
An Application of The Rényi Entropy - Example

Encoding Tasks Theorem

Let \( \{X_i\}_{i=1}^\infty \) be a source over \( \mathcal{X} \). Let \( \rho > 0 \).

1. **Direct:** If \( R > \limsup_{n \to \infty} \frac{1}{n} H_{1+\rho}(X^n) \), then there exist encoders \( f_n : \mathcal{X}^n \to \{1, \ldots, 2^{nR}\} \) such that

\[
\lim_{n \to \infty} E \left[ \| f_n^{-1}(f_n(X^n)) \|^\rho \right] = 1.
\]

2. **Converse:** If \( R < \liminf_{n \to \infty} \frac{1}{n} H_{1+\rho}(X^n) \), then for any choice of encoders \( f_n : \mathcal{X}^n \to \{1, \ldots, 2^{nR}\} \),

\[
\lim_{n \to \infty} E \left[ \| f_n^{-1}(f_n(X^n)) \|^\rho \right] = \infty.
\]
Proof of Theorem 1 - Outline

1. Assume w.l.o.g that $\sum_{k=1}^{n} N_\alpha(X_k) = 1$ (homogeneity of the Rényi entropy power)

2. $\log N_\alpha(\sum_{k=1}^{n} X_k) \geq f(t)$
   \begin{align*}
   &= \frac{\log \alpha}{\alpha - 1} - D(t\|\underline{N}_\alpha) + \alpha' \sum_{k=1}^{n} \left(1 - \frac{t_k}{\alpha'} \right) \log \left(1 - \frac{t_k}{\alpha'} \right)
   \end{align*}

3. Choose $t_k = N_\alpha(X_k)$ such that $D(t\|\underline{N}_\alpha) = 0$

4. From the convexity of $f(x) = (1 - x) \log(1 - x)$, $x \in (0, 1)$,
   \begin{align*}
   \left(1 - \frac{t_k}{\alpha'} \right) \log \left(1 - \frac{t_k}{\alpha'} \right) \geq \log \left(1 - \frac{1}{n\alpha'} \right) + \frac{\log e}{n\alpha'} \\
   &- \frac{t_k}{\alpha'} \left[\log e + \log \left(1 - \frac{1}{n\alpha'} \right)\right]
   \end{align*}

5. Combining 2., 3. and 4., yields the desired result (since $\sum_{k=1}^{n} t_k = 1$)
Discussion - Tightness

- The R-EPI in Theorem 2 provides the tightest R-EPI known to date for $\alpha \in (1, \infty)$. 
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- Nevertheless, one of the inequalities involved in its derivation is loose:
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  - Monotonicity of the Rényi entropy power in its order: Equality only for uniformly distributed random vectors.
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- Nevertheless, one of the inequalities involved in its derivation is loose:
  - The sharpened Young’s inequality: equality only for Gaussians.
  - Monotonicity of the Rényi entropy power in its order: Equality only for uniformly distributed random vectors.
- For $\alpha = \infty$ and $n = 2$, the sharpened Young’s inequality reduces to:

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}.$$  

- Equality holds if $f$ and $g$ are scaled versions of a uniform distribution on the same convex set.
- This is consistent with the fact that the R-EPIs in Theorems 1 and 2 are asymptotically tight for $n = 2$ by letting $\alpha \to \infty$. 