On the Error Exponents of Improved Tangential-Sphere Bounds

Moshe Twitto

Department of Electrical Engineering
Technion-Israel Institute of Technology

Joint work with Igal Sason
Outline

1. Background

2. The Tangential-Sphere Bound and Improved Versions
   - Tangential-Sphere Bound (TSB)
   - Improvements on the TSB
   - Error Exponents of Improved TSB
   - Numerical Results for Error Exponents
Background

- Tight analytical bounds emerge as a useful tool for assessing performance of coded communication systems.

- Union bounds are useless at rates above the cutoff rate of the channel $\Rightarrow$ Improved upper bounds which are not subject to the cutoff rate limitations are needed.

- The discovery of turbo-like codes has increased the motivation for the derivation of tight bounds on the decoding error probability.
General Concept for the Derivation of Improved Upper Bounds

The general concept of the improved bounding technique, as introduced by Fano (1960), is based on the inequality

\[ \Pr(\text{word error} \mid c_0) \leq \Pr(\text{word error}, y \in R \mid c_0) + \Pr(y \notin R \mid c_0) \]

- **c_0**—The transmitted codeword (linear code).
- **y**—The received vector at the output of the channel.
- **R**—An arbitrary geometrical region.

The idea is to apply the union bound to the first term in the RHS of the above inequality.

Special case: **R** is the whole \( n \)-dimensional space \( \Rightarrow \) The union bound.
Tangential-Sphere Bound (TSB)

- Introduced by Poltyrev in 1994.
- Consider the transmission of a binary linear code over an AWGN channel, using an equi-energy modulation.
- For the TSB, the region $\mathcal{R}$ is a circular, $N$ dimensional cone, with a half angle $\theta$, and a radius $r$. Denote it by $C_N(\theta)$. 
The Tangential-Sphere Bound and Improved Versions

Tangential-Sphere Bound (TSB)

Tangential-Sphere Bound (Cont.)

Pr(word error | $c_0$) $\leq$ Pr(word error, $y \in \mathcal{R} | c_0$) + Pr($y \notin \mathcal{R} | c_0$) (1)

For the TSB:

- The union bound is applied on the first term in the RHS of (1), which gives:

$$\Pr(E|c_0) \leq \sum_{i=1}^{M} \Pr(E_0 \rightarrow_i, y \in C_N(\theta) | c_0) + \Pr(y \notin C_N(\theta) | c_0)$$

$E_0 \rightarrow_i$—The event of deciding on $c_i$ rather than $c_0$. 
Geometrical interpretation of the joint event $E_{0,i} \cap y \in \mathcal{R}$
The Tangential-Sphere Bound (Cont.)

The optimization is carried over \( r \) (\( r \) and \( \theta \) are related).

Two special cases

1. \( r \to \infty \): Particularizes to the union bound.
2. \( r = 0 \): Equals to 1.

- Shown to be the optimal volume among all the volumes \( \mathcal{R} \) which posses some symmetry properties (Yousefi and Khandani).
- One of the tightest upper bounds on the ML-decoding error probability of block codes.
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Tangential-Sphere Bound (Cont.)

The final version of the bound is

\[ P_e \leq \int_{-\infty}^{\infty} \frac{dz_1}{\sqrt{2\pi}\sigma} e^{-\frac{z_1^2}{2\sigma^2}} \left\{ \sum_{k: \frac{\delta_k}{2} \leq \alpha_k} \left\{ A_k \int_{\beta_k(z_1)}^{r z_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z_2^2}{2\sigma^2}} \tilde{\gamma} \left( \frac{N-2}{2}, \frac{r^2 z_1^2 - z_2^2}{2\sigma^2} \right) dz_2 \right\} \right\} 
+ 1 - \tilde{\gamma} \left( \frac{N-1}{2}, \frac{r z_1^2}{2\sigma^2} \right) \]

where

\[ \tilde{\gamma}(a, x) \triangleq \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt, \quad a, x > 0 \]

and \( A_k \) denotes the distance spectrum of the code.
The optimal radius is obtain by the following optimization equation:

\[
\sum_{k: \frac{\delta_k}{2} < \alpha_k} A_k \int_0^{\theta_k} \sin^{N-3} \phi \, d\phi = \frac{\sqrt{\pi}}{\Gamma\left(\frac{N-2}{2}\right)} \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}
\]

\[
\theta_k = \cos^{-1} \left( \frac{\delta_k}{2r} \frac{1}{\sqrt{1 - \frac{\delta^2_k}{4N\bar{E}\bar{s}}}} \right).
\]

Notes:

1. The optimal radius is independent of the SNR.
2. There exists a unique solution for the above equation (Sason and Shamai).
Improvements on the TSB

Reminder:
The TSB is based on the inequality

\[
\Pr(\text{word error} \mid \mathbf{c}) \leq \Pr(\text{word error}, \mathbf{y} \in \mathcal{R} \mid \mathbf{c}) + \Pr(\mathbf{y} \notin \mathcal{R} \mid \mathbf{c})
\] (2)

where the union bound is applied on the first term in the RHS of (2).

Improvement:

Yousefi and Khandani suggest to improve the TSB by applying Hunter’s bound rather than the union bound.
Improvements on the TSB

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Improvements on the TSB

Hunter’s Bound

Let \( \{E_i\}, i = 1, \ldots, M \) designate a set of \( M \) events, and \( E_i^c \) designates the complementary event of \( E_i \). Then

\[
\Pr \left( \bigcup_{i=1}^{M} E_i \right) = \Pr(E_1) + \Pr(E_2 \cap E_1^c) + \ldots + \Pr(E_M \cap E_{M-1}^c \ldots \cap E_1^c)
\]

\[
\leq \Pr(E_{\pi_1}) + \sum_{i=2}^{M} \Pr(E_{\pi_i} \cap E_{\lambda_i}^c).
\]

where \( \{\pi_1, \ldots, \pi_M\} \) is an arbitrary permutation of the set \( \{1, \ldots, M\} \), and \( \{\lambda_2, \ldots, \lambda_M\} \) designates an arbitrary sequence of integers where \( \lambda_i \in \{\pi_1, \ldots, \pi_{i-1}\} \).
Improvements on the TSB (Cont.)

Applying Hunter’s bound yields

\[
\Pr \left( \bigcup_{i=1}^{M-1} E_{0 \rightarrow i}, \mathbf{y} \in C_N(\theta) \mid z_1 \right) \leq \min_{\Pi, \Lambda} \left\{ \Pr(E_{0 \rightarrow \pi_1}, \mathbf{y} \in C_N(\theta) \mid z_1) \right. \\
+ \left. \sum_{i=2}^{M-1} \Pr(E_{0 \rightarrow \pi_i}, E_{0 \rightarrow \lambda_i}^c, \mathbf{y} \in C_N(\theta) \mid z_1) \right\}
\]

where \( E_{0 \rightarrow j} \) designates the pairwise error event where the decoder decides on codeword \( \mathbf{c}_j \) rather than the transmitted codeword \( \mathbf{c}_0 \).
Geometrical interpretation of the joint event $E_{0\rightarrow i} \cap E_{0\rightarrow j}^c \cap y \in C_N(\theta)$
Problems:
- The problem of finding the optimal ordering of the events is prohibitively complex.
- The bound depends on the global geometrical properties of the code.

Important fact
The probabilities $\Pr\left( E_{0\rightarrow i}, E_{0\rightarrow j}^c \right)$ are monotonic decreasing functions of the correlation coefficients between $c_i$ and $c_j$. 
Yousefi et al. derived two versions of improved tangential-sphere bounds. These new bounds (ITSB and AHP bounds) were exemplified to outperform the TSB for short linear block codes.

In the following, we compare the error exponents associated with the TSB and its improved versions.
Error Exponents of Improved TSB

Theorem

The upper bounds ITSB, AHP and the TSB have the same error exponent, which is

\[
E(c) = \min_{0 \leq \delta \leq 1} \left\{ \frac{1}{2} \ln \left( 1 - \gamma + \gamma e^{-2r(\delta)} \right) + \frac{\gamma \Delta^2 c}{1 + \gamma \Delta^2} \right\}
\]

where

\[
\gamma = \gamma(\delta) \triangleq \frac{1 - \delta}{\delta} \left[ \sqrt{\frac{c}{c_0(\delta)}} + (1 + c)^2 - 1 - (1 + c) \right]
\]

and

\[
c_0(\delta) \triangleq \left( 1 - e^{-2r(\delta)} \right) \frac{1 - \delta}{2\delta}, \quad r(\delta) = \frac{\ln(A_l)}{N}, \quad \Delta = \sqrt{\frac{\delta}{1 - \delta}}, \quad c \triangleq \frac{E_s}{N_0}.
\]
Proof’s Outline

- Lemma: The ITSB is at least as tight as the TSB for specific codes. ⇒ It is at least as tight as the TSB for ensembles of codes.
- Lemma: Asymptotically, the AHP is at least as tight as the TSB.
- Lemma: Both the ITSB and the AHP are lower bounded by a certain function $\psi(C)$.
- We use the Chernoff bounding technique to show that the exponential versions of $\psi(C)$ and the TSB are identical.
- ⇒ The error exponents of the TSB, ITSB and AHP bounds are all identical.
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- **Lemma**: The ITSB is at least as tight as the TSB for *specific* codes. \(\Rightarrow\) It is at least as tight as the TSB for ensembles of codes.

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Error Exponents of Some Bounds

RCE—Gallager’s random coding exponent.

TSB—The error exponent of the tangential-sphere bound.

UB—The error exponent of the union bound.
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Conclusion

- The TSB and its improved versions do not achieve capacity for the ensemble of random linear block codes.
- Tight analytical bounds for structured codes are required, especially for high-rate codes where the weakness of the TSB is more pronounced.
Further Reading
