thesis algorithm. Finally, we have demonstrated a new iterative CWD technique which does not rely on *a priori* knowledge of the signal of interest. It is especially useful in applications where interference terms hinder the use of AWD synthesis techniques.

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On the Asymptotic Statistical Behavior of Empirical Cepstral Coefficients

Neri Merhav and Chin-Hui Lee

***Abstract*—The asymptotic covariance matrix of the empirical cepstrum is analyzed. We show that for Gaussian processes, cepstral coefficients derived from smoothed periodograms are asymptotically uncorrelated and their variances multiplied by the sample size *T* tend to unity. For an autoregressive process and its autoregressive cepstrum estimate, somewhat weaker results hold.**

I. INTRODUCTON

Cepstral analysis is useful in the preprocessing of many speech recognition and speaker verification systems (see, e.g., [1]-[6]). This is based on strong experimental evidence that among many types of feature vectors, the cepstrum provides the best performance in speech recognition [6] and speaker verification [2] applications.

It is of interest, in light of this fact, to investigate the asymptotic statistical properties of the empirical cepstral vector. We examine both analytically (Section II) and experimentally (Section III) the covariance matrix of this vector when extracted from a stationary random process in two cases. First, an underlying stationary Gaussian process is assumed and we confine interest to the cepstrum derived from the smoothed periodogram [7]. The cepstral components are shown to be asymptotically uncorrelated and their variances, when multipled by sample size T, tend to unity as $T \to \infty$. In the second case, an autoregressive (AR) process (not necessarily Gaussian) is assumed and we focus on the cepstrum derived from the empirical AR power spectrum density (PSD), which is a parametric estimator of the PSD. Here the covariance matrix, when multiplied by T, tends to the identity matrix in the weak norm sense (Hilbert-Schmidt), which is a weaker form of convergence than in the former case. Thus, in both cases the asymptotic covariance matrix is, in a sense, equivalent to the identity matrix independently of the underlying PSD.

This "orthonormality" property of the cepstral vector regardless of the PSD, does not exist in many other feature vectors commonly used in speech processing, e.g., the AR parameter vector, the vector of reflection coefficients, and the DFT coefficients. It is interesting to note, however, that the log-spectral energies (which are related to the cepstrum via a Fourier transform) do have the above mentioned covariance orthonormality property under some conditions [10]. This will be discussed more deeply in Section II.

One implication of these results is that, essentially, only the cepstral means carry useful information regarding the PSD, while the cepstral variances are relatively insensitive to the PSD. This observation has been also supported experimentally by earlier studies [5], [11]-[13]. Another implication of the results is that they provide some theoretical motivation for the use of diagonal covariance matrices in cepstral hidden Markov model of speech signals.

II. MAIN RESULTS

Consider a stationary process $\{y_t\}_{t \geq 1}$ with an autocorrelation sequence $R(\tau) = E(y_t y_{t+\tau})$ and power spectrum density (PSD), $S(\omega) = \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-j\omega\tau}$ satisfying the following regularity conditions.

A1) There exist $S_{\min} > 0$ and $S_{\max} < \infty$ such that $S_{\min} \leq S(\omega) \leq S_{\max}$ for all $-\pi \leq \omega < \pi$.

A2) $\{R(\omega)\}$ is absolutely summable, i.e., $\sum_{\tau=-\infty}^{\infty} |R(\tau)| < \infty$.

A3) The sequence $\{R_I(\tau)\}$, defined as the inverse Fourier transform of $1/S(\omega)$, is absolutely summable.

The cepstrum $\{c_\tau\}$ is defined as

$$c_t = \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \cdot e^{j\omega\tau} \log S(\omega), \qquad \tau = 0, 1, 2, \cdots. \tag{1}$$

Suppose now that we wish to estimate c_τ from a finite observation sequence $y_1, y_2, \cdots, y_T$ by using an estimate $\hat{S}(\omega)$ of the PSD and then substitute $\hat{S}(\omega)$ instead of $S(\omega)$ in (1). Consider the smoothed periodogram [7] for estimating the PSD, i.e.,

$$\hat{S}_L(\omega) = \sum_{\tau=-L}^{L} \hat{R}(\tau) e^{j\omega\tau}$$

where L is a fixed positive integer and $\hat{R}(\tau)$ is the empirical autocorrelation given by $\hat{R}(\tau) = T^{-1} \sum_{t=1}^{T-\tau} y_t y_{t+\tau}$. An alternative to the smoothed periodogram is a parametric estimator derived from an

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TABLE I
CEPSTRAL COVARIANCES FOR A SMOOTHED PERIODOGRAM

T	L	ρ_{11}	ρ_{22}	ρ_{33}	ρ_{12}	ρ_{23}	ρ_{13}
128	50	0.8265	0.8272	0.8239	−0.1163	−0.0892	0.0523
256	100	0.8993	0.9314	0.9102	−0.0785	−0.0786	−0.0524
512	150	0.9330	1.0227	0.9321	−0.0769	−0.0679	0.0038
1024	150	0.9905	1.0070	0.9628	−0.0587	−0.0324	0.0031

AR model, i.e.,

$$\hat{S}_q(\omega) = \hat{\sigma}^2|1 + \sum_{k=1}^{q} \hat{a}_k e^{-j\omega k}|^{-2}$$

where $\hat{\sigma}^2$ and $(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_q)$ are estimates of the gain and the AR coefficients. Define $\hat{c}_\tau^L$ and $\tilde{c}_\tau^q$ as the inverse Fourier transforms of $\log \hat{S}_L(\omega)$ and $\log \tilde{S}_q(\omega)$, respectively. The following lemmas (proved in the Appendix) describe the asymptotic covariance matrices of estimated cepstral coefficients in the two versions.

Lemma 1: Let $\{y_t\}$ be a stationary Gaussian process with PSD $S(\omega)$ satisfying A1–A3. Then, for every two fixed positive integers k and l, $\lim_{L\to\infty}\lim_{T\to\infty}T \cdot \operatorname{cov}(\hat{c}_k^L, \hat{c}_l^L) = \delta_{kl}$, where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise.

Lemma 2: Let $\{y_t\}$ be a pth order AR process, that is, $y_t = -\Sigma_{i=1}^p a_i y_{t-i} + w_t$, with all poles inside the unit circle, where $\{w_t\}$ are independently identically distributed (i.i.d) random variables with $E\{w_t\} = 0$ and $E\{w_t^k\} < \infty$ for all $k \geq 1$. Then, as $q \to \infty$, the $q \times q$ dimensional matrix whose klth entry is $\rho_{kl}^q = \lim_{T\to\infty}T \cdot \operatorname{cov}(\tilde{c}_k^q, \tilde{c}_l^q)$, tends to the identity matrix in the Hilbert–Schmidt sense, i.e., $\lim_{q\to\infty}q^{-1}\Sigma_{k,l=1}^q|\rho_{kl}^q - \delta_{kl}|^2 = 0$.

Discussion: In [10, corollary 5.6.3] a result related to Lemma 1 has been established. Specifically, consider a periodgram $\hat{S}(\omega)$ smoothed by a window whose bandwidth B_T vanishes with T but at a rate slower than $1/T$. Then, under certain assumptions $B_T T \cdot \operatorname{var}\{\ln \hat{S}(\omega)\}$ tends to a constant as $T \to \infty$. Furthermore, from [10, corollary 5.62.] it can be implied that $B_T T \cdot \operatorname{cov}\{\ln \hat{S}(\omega), \ln \hat{S}(\nu)\}$ vanishes with T whenever $\omega \neq \nu$ and $\omega \neq 2\pi - \nu$. It is tempting to think that Lemma 1 can be deduced from this covariance orthonormality property of the log periodogram because the cepstrum is obtained from the inverse Fourier transform which is a unitary transform. However, this is not quite the case unless one shows that the above cross covariances in the frequency domain decay much faster. The reason is that in practice (1) is computed by the IDFT and hence the covariance between two cepstral components is given by a weighted sum of $O(T^2)$ covariances among the DFT components $\ln \hat{S}(2\pi k/T)$ whose overall relative contribution does not necessarily vanish with T.

III. EXPERIMENTAL RESULTS

We examined experimentally the validity of Lemmas 1 and 2 for finite length data records. In our first set of experiments, Lemma 1 has been examined. In each experiment, we have generated an ensemble of 500 examples of the random process $y_t = w_t - 0.8w_{t-1} + 0.16w_{t-2}$, $t = 1, 2, \ldots, T$, where $\{w_t\}$ are zero mean, unit variance, independent Gaussian random variables. For each example i, $1 \leq 500$, we computed the empirical cepstrum vector $\hat{c}^L(i)$, where the Fourier integral of (1) was approximated by the IDFT. Finally, we computed the empirical covariance matrix over the 500 examples defined as

$$\rho = T \cdot \left[\frac{1}{500}\sum_{i=1}^{500} \hat{c}^L(i) \cdot \hat{c}^L(i)^{\#} - \left[\frac{1}{500}\sum_{i=1}^{500} \hat{c}^L(i)\right] \cdot \left[\frac{1}{500}\sum_{i=1}^{500} \hat{c}^L(i)\right]^{\#}\right] \tag{2}$$

where $^{\#}$ denotes vector transposition. Table I presents the 3×3 upper left block $\{\rho_{ij}\}_{i,j=1}^3$ of ρ for various values of T and L. As can be seen the diagonal terms ρ_{ii} are quite close to unity and the off diagonal terms are reasonably small even for relatively short data records. Generally speaking, the results improve for large T and L as expected. A similar behavior has been observed for the higher order cepstral components as well as for different shapings of the underlying PSD.

TABLE II
DISTANCE FROM THE IDENTITY MATRIX FOR AN AR(q) SPECTRUM ESTIMATE

T	$q = 1$	$q = 2$	$q = 4$	$q = 6$	$q = 10$
100	0.3768	0.2087	0.1044	0.0909	0.0759
200	0.3217	0.1805	0.1160	0.0982	0.0593
400	0.3420	0.2104	0.1124	0.0685	0.0651
800	0.4060	0.1731	0.1106	0.0857	0.0605

The second set of experiments was associated with Lemma 2. In each experiment we generated 500 T-point examples of the AR process $y_t = 0.8y_{t-1} + w_t$, with w_t as above. For each example, $\tilde{c}_\tau^q$ has been computed and the empirical covariance matrix ρ over the 500 examples has been calculated similarly to (2). Table II presents the calculated value of $\Delta = q^{-1}\Sigma_{i,j=1}^q(\rho_{ij} - \delta_{ij})^2$ for several values of q and T. As can be seen Δ decreases as q grows for every fixed T. Similar results were obtained for different underlying AR processes as well as for a uniformly distributed driving white noise.

APPENDIX

Proof of Lemma 1: Define

$$c_\tau^L = (2\pi)^{-1}\int_{-\pi}^{\pi} d\,\omega e^{j\tau\omega} \log\left[\sum_{t=-L}^{L} R(t)e^{-j\omega t}\right].$$

Since $\hat{c}_\tau^L$ satisfies a similar relation with $\{R(t)\}$ replaced by $\{\hat{R}(t)\}$, then by assumptions A1 and A2,

$$\delta\hat{c}_\tau^L \triangleq \hat{c}_\tau^L - c_\tau^L = \int_{-\pi}^{\pi}\frac{d\omega}{2\pi}e^{j\tau\omega}\log\frac{\sum_{t=-L}^{L}\hat{R}(t)e^{jt\omega}}{\sum_{t=-L}^{L}R(t)e^{-jt\omega}}$$

$$= \int_{-\pi}^{\pi}\frac{d\omega}{2\pi}e^{j\tau\omega}\left[\frac{\sum_{t=-L}^{L}\Delta\hat{R}(t)e^{-jt\omega}}{\sum_{t=-L}^{L}R(t)e^{-jt\omega}}\right] + o_p(T^{-1/2})$$

$$\triangleq \Delta\hat{c}_\tau^L + o_p(T^{-1/2}) \tag{A.1}$$

where $\Delta\hat{R}(t) \triangleq \hat{R}(t) - R(t)$ and $o_p(T^{-1/2})$ denotes a term of stochastic order less than $T^{-1/2}$. Under conditions A1 and A2, it is easy to show that $E(\hat{c}_\tau^L) - c_\tau^L = O(T^{-1})$ and hence $\operatorname{cov}\{\hat{c}_k^L, \hat{c}_l^L\} = E(\Delta\hat{c}_k^L \cdot \Delta\hat{c}_l^L) + o(T^{-1})$. Thus, the asymptotic behavior of $T \cdot \operatorname{cov}(\hat{c}_k^L, \hat{c}_l^L)$ is identical to that of $T \cdot E(\Delta\hat{c}_k^L \cdot \Delta\hat{c}_l^L)$ and we concentrate on the latter. Let

$$R_I^L(\tau) \triangleq (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega \cdot e^{j\tau\omega} / S_L(\omega)$$

where $S_L(\omega) \triangleq \Sigma_{\tau=-L}^{L} R(\tau) e^{-j\tau\omega}$. Note that $\Delta\hat{c}_k^L$, as defined in (A.1), is the inverse Fourier transform of the product of $1/S_L(\omega)$ and $\Sigma_{\tau=-L}^{L} \Delta\hat{R}(\tau) e^{-j\tau\omega}$. Hence, it is equal to the convolution between the corresponding inverse Fourier transforms $R_I^L(t)$ and $\{\Delta\hat{R}(t)\}_{t=-L}^{L}$, i.e.,

$$\Delta\hat{c}_k^L = \sum_{t=-L}^{L} \Delta\hat{R}(t) \cdot R_I^L(k-t) = \sum_{t=-L}^{L} \Delta\hat{R}(t) R_I^L(t+k)$$

where the second equality follows from the symmetry of both convolved sequences. Thus,

$$E(\Delta\hat{c}_k^L \Delta\hat{c}_l^L) = \sum_{t,s=-L}^{L} E[\Delta\hat{R}(t)\Delta\hat{R}(s)] R_I^L(t+k) R_I^L(s+l).$$

To compute $E[\Delta\hat{R}(t)\Delta\hat{R}(s)]$ we use the fact that for a zero-mean Gaussian quadruple $E(XYZW) = E(XY)E(ZW) + E(XZ)E(YW) + E(XW)E(YZ)$. Therefore,

$$\begin{aligned} E[\Delta\hat{R}(t)\Delta\hat{R}(s)] &= \frac{1}{T^2} \sum_{r=1}^{T-t} \sum_{l=1}^{T-s} [E(y_r y_{r+t} y_l y_{l+s}) - R(t)R(s)] \\ &= \frac{1}{T^2} \sum_{r=1}^{T} \sum_{l=1}^{T} [R(r-l)R(r-l-s+t) \\ &\quad + R(r-l-s)R(r-l+t)] \\ &\quad - \frac{1}{T^2} \sum_{U_{t,s}} [R(r-l)R(r-l-s+t) \\ &\quad + R(r-l-s)R(r-l+t)] \end{aligned} \qquad \text{(A.2)}$$

where $U_{t,s} = \{(r, l): T - t < r \le T \text{ or } T - s < l \le T\}$. As for the first term on the rightmost side of (A.2), we have

$$\begin{aligned} \frac{1}{T^2} \sum_{r=1}^{T} \sum_{l=1}^{T} &[R(r-l)R(r-l-s+t) \\ &+ R(r-l-s)R(r-l+t)] \\ = \frac{1}{T} &\sum_{k=-(T-1)}^{T-1} \left(1 - \frac{|k|}{T}\right) \cdot [R(k)R(k-s+t) \\ &+ R(k-s)R(k+t)]. \end{aligned} \qquad \text{(A.3)}$$

Let

$$R_2(\tau) \triangleq \sum_{t=-\infty}^{\infty} R(t)R(t+\tau) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\omega e^{j\tau\omega} S^2(\omega).$$

It is easy to see that (A.3), when multiplied by T, tends to $R_2(t - s) + R_2(t + s)$ as $T \to \infty$. The second term on the rightmost side of (A.2) can be shown to be $O(T^{-2})$ under assumption A2 and hence even when multiplied by T, tends to zero as $T \to \infty$. Combining these facts,

$$\begin{aligned} \rho_{k,l}^L \triangleq \lim_{T\to\infty} T \cdot \operatorname{cov}(\hat{c}_k^L, \hat{c}_l^L) &= \sum_{t,s=-L}^{L} [R_2(t-s) \\ &+ R_2(t+s)] R_I^L(t+k) R_I^L(s+l). \end{aligned} \qquad \text{(A.4)}$$

Taking the second limit as $L \to \infty$ and using A3, it is easy to see that

$$\rho_{kl} \triangleq \rho_{kl}^L = \sum_{t,s=-\infty}^{\infty} [R_2(t-s) + R_2(t+s)] R_I(s+l) R_I(t+k) \qquad \text{(A.5)}$$

where $R_I(\cdot)$ is as in A3. Since $R_2(t) = R(t)*R(t)$ and $R(t)*R_I(t) = \delta(t)$, where * denotes convolution and $\delta(\cdot)$ denotes the Kronecker delta function, the first term in (A.5) becomes δ_{kl} and the second is zero for every $l \ge 1$ and $k \ge 1$.

Proof of Lemma 2: Consider the AR(p) process satisfying $\Sigma_{k=0}^{p} a_k y_{t-k} = w_t$, where $\{w_t\}$ is zero-mean, unit variance white noise, $a_0 = 1$, and $a = (a_1, \cdots, a_k)$ is the vector of AR parameters. For $q \ge p$, consider the AR(q) cepstrum

$$c_\tau^q = -(2\pi)^{-1} \int_{-\pi}^{\pi} d\omega e^{j\omega\tau} \cdot \log \left| \sum_{k=0}^{q} a_k e^{-j\omega k} \right|^2$$

where $a_i = 0, p + 1 \le i \le q$, if $q > p$, and its estimate

$$\hat{c}_\tau^q = -(2\pi)^{-1} \int_{-\pi}^{\pi} d\omega e^{j\omega\tau} \cdot \log \left| \sum_{k=0}^{q} \hat{a}_k e^{-j\omega k} \right|^2$$

where $\hat{a}_0 = 1$ and $\hat{a} = (\hat{a}, \cdots, \hat{a}_q)$ is the vector of estimates of the AR parameters calculated from the Yule–Walker equations [14] for order q. Let $\Delta\hat{a} = \hat{a} - a$ and $\Delta\hat{c}^q = \hat{c}^q - c^q$, where $c_q = (c_1^q, \ldots, c_q^q)$ and $\hat{c}^q = (\hat{c}_1^q, \ldots, \hat{c}_q^q)$. It is known [8], [9] that under the conditions of Lemma 2, the asymptotic covariance matrix of $\Delta\hat{a}$ is given by $\lim_{T\to\infty} T \cdot E(\Delta\hat{a}\Delta\hat{a}^\#) = \boldsymbol{R}^{-1}$, where $\Delta\hat{a}$ is a column vector, $^\#$ denotes transposition, and $\boldsymbol{R}$ is the $q \times q$ covariance matrix of the process, with the ijth entry being $R(i - j)$. The asymptotic covariance matrix of $\Delta\hat{c}^q$ is given by $\boldsymbol{HR}^{-1}\boldsymbol{H}^\#$, where $\boldsymbol{H}$ is the Jacobian matrix whose klth element is $\partial c_k^q / \partial a_l$. Now,

$$\begin{aligned} \frac{\partial c_k^q}{\partial a_l} &= -\frac{\partial}{\partial a_l} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} e^{j\omega k} \\ &\quad \cdot \log \left[\sum_{i=0}^{p} a_i e^{-j\omega i} \sum_{r=0}^{p} a_r e^{j\omega r} \right] \\ &= -\int_{-\pi}^{\pi} \frac{d\omega}{2\pi} e^{j\omega k} \cdot S(\omega) \left[e^{-j\omega l} \sum_{i=0}^{p} a_i e^{j\omega i} + e^{j\omega l} \right. \\ &\quad \left. \cdot \sum_{i=0}^{p} a_i e^{-j\omega i} \right] \\ &= -\sum_{i=0}^{p} a_i [R(k+i-l) + R(k+l-i)] \end{aligned} \qquad \text{(A.6)}$$

where we have used the fact that $S(\omega) = |\Sigma_{k=0}^{p} a_k e^{-j\omega k}|^{-2}$. Now, since

$$\sum_{k=0}^{p} a_k R(k+s) = E(w_t y_{t+s}) = h_s$$

where $\{h_t\}_{t=0}^{\infty}$ is the impulse response of the filter $H(z) = 1/\Sigma_{k=0}^{p} a_k z^{-k}$ and since $R(\cdot)$ is symmetric, we have $\partial c_k^p / \partial a_l = h_{k-l}$ and hence $\boldsymbol{H}$ is a Toeplitz matrix. Since the process is stationary, $H(z)$ must be stable, and hence $\{h_t\}$ is absolutely summable. Since $\boldsymbol{R}$ is a covariance matrix of an absolutely summable autocorrelation sequence satisfying A1, and the spectrum of its inverse cancels the spectrum induced by $\{h_t\}$, we conclude by [15, theorems 2.1, 5.1, 5.2] that the weak norm of $(\boldsymbol{HR}^{-1}\boldsymbol{H}^\# - \boldsymbol{I}_q)$, $\boldsymbol{I}_q$ being the $q \times q$ identity matrix, vanishes as $q \to \infty$, i.e.,

$$\lim_{q\to\infty} q^{-1} \sum_{k,l=1}^{q} |\rho_{kl}^q - \delta_{kl}|^2 = 0$$

where ρ_{kl}^q is the klth entry of $\boldsymbol{HR}^{-1}\boldsymbol{H}^\#$.

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Synthesis of Spectral Densities Using Finite Automata

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***Abstract*—A method for designing a finite automaton whose output exhibits a given rational power spectral density belonging to a particular class, is presented. The method exploits the properties of circulant matrices, which allow us to build a stochastic matrix with a set of assigned eigenvalues.**

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I. INTRODUCTION

The interest for the synthesis of rational spectral densities by means of finite sequential machines (SM's) is twofold: first, this synthesis may be applied to design encoders often used for spectral shaping purposes in digital data transmission or recording; second, it may be used for the numerical simulation of processes with assigned spectral density, as an alternative method to the classical approach of passing a white noise signal through a suitable digital filter.

Let us recall that any rational spectral density may be represented in the form [1]

$$R(z) = C^2 \frac{\prod_{k=0}^{\mu-1} (z - \xi_k)(z^{-1} - \xi_k^*)}{\prod_{k=0}^{\nu-1} (z - \varrho_k)(z^{-1} - \varrho_k^*)} \qquad (1)$$

where * denotes conjugate and the zeros ξ_k and poles ϱ_k satisfy the conditions: $|\xi_k| \le 1$, $|\varrho_k| < 1$. Moreover, if the process is real, to every zero and pole there corresponds its complex conjugate.

The synthesis through a linear filter requires a factorization $R(z) = \sigma^2 h(z^{-1})h^*(z^*)$; then $h(z^{-1})$ provides the transfer function of the filter and σ^2 gives the variance of the filter input. The general problem of synthesizing an assigned rational spectral density by means of an SM fed by a suitable input formed by independent and identically distributed (i.i.d.) symbols was dealt with by Mullis and Roberts [2], who proved that such a synthesis is possible for any rational spectral density. Unfortunately, the proof is not constructive and does not convey any suggestion about the choice of the SM and of the input probabilities. A complete solution to the problem has been given by Mullis and Steiglitz [3] for a particular case, namely, for those spectral densities which are obtained by summing up elementary spectral densities having only a single pole inside the unit disk. In their solution, each elementary spectral density requires a separate SM and the machine inputs must be independent.

In the following we consider the same class of spectral densities as in [3], but use a different approach. Our solution, in which a fundamental role is played by circulant matrices as well, leads to a simpler structure involving a single sequential machine.

II. REVIEW OF THE SPECTRAL ANALYSIS OF THE OUTPUT OF AN SM

As it is known, an SM (in particular, reference will be made to a Moore machine) may be specified [4] as a quintuple $\mathcal{M} = \{\mathcal{B}, \mathcal{A}, \mathcal{S}, g, h\}$ where $\mathcal{B}$ is the input set, $\mathcal{A}$ is the output set, $\mathcal{S} = \{\sigma_1, \sigma_2, \cdots, \sigma_I\}$ is the state set, $g: \mathcal{S} \times \mathcal{B} \to \mathcal{S}$ is the state transition function, and $h: \mathcal{S} \to \mathcal{A}$ is the output function; explicitly,

$$s_{t+1} = g(s_t, b_t), \qquad a_t = h(s_t) \qquad (2)$$

where s_t, b_t, and a_t denote the state, input, and output processes.

If the input process b_t is composed of i.i.d. symbols, the state process s_t is a homogeneous Markov chain [5] whose transition probability matrix $\mathbf{\Pi}$ can be determined from the probability mass function of the input and the state transition function of the automaton. By assuming that the Markov chain is ergodic, the state probability vector $\mathbf{p} = [p(1), p(2), \cdots, p(I)]$ collecting the probabilities $p(i) = Pr[s_t = \sigma_i]$ is obtained from

$$\mathbf{p} = \mathbf{p}\mathbf{\Pi}, \qquad \sum_{i=1}^{I} p(i) = 1. \qquad (3)$$

We recall that the transition probability matrix $\mathbf{\Pi}$ of an ergodic

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