On Zero–Delay Lossy Source Coding with Side Information at the Decoder

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***Abstract*—We consider real-time, variable-rate lossy source coding in the presence of side information (SI) at the decoder. We show that time-sharing at most two scalar encoder-decoder pairs achieves optimal performance. We further demonstrate that in this setting, increasing the number of quantization levels can *reduce* the minimum average rate. Finally, two structure theorems, pertaining to source- or SI look-ahead are given.**

I. Introduction

We consider the following source coding problem. Symbols produced by a discrete memoryless source are to be encoded, transmitted noiselessly and reproduced by a decoder which has access to SI correlated to the source. Operation is in real time, that is, the encoding of each symbol and its reproduction by the decoder must be performed without any delay. The average distortion between the source and the reproduced symbols is constrained to be smaller than some predefined constant. Since no delay is allowed, the encoder must, at each stage, use an instantaneous code which is decoded without error at the receiving end.

When no distortion is allowed, this problem falls within the scope of zero-error source coding with SI, which was initially introduced by Witsenhausen in [1]. Witsenhausen considered fixed-length coding and characterized the side-information structure as a confusability graph defined on the source alphabet. With this characterization, fixed-length SI codes were equivalent to colorings of the associated graph. Alon and Orlitsky [2] considered variable-rate codes for the zero-error problem. Two classes of codes were considered and lower and upper bounds were derived for both the scalar and infinite block length regimes. The work of Alon and Orlitsky was further extended by Koulgi *et. al* [3] who showed that the asymptotic zero-error rate of transmission is the complementary graph entropy of an associated graph. It was also showed in [3] that the design of optimal code is NP-hard and a sub-optimal, polynomial time algorithm was proposed. The combination of zero-error codes and maximum per-letter distortion was considered in [4]. When the source alphabet is finite and distortion is allowed, scalar quantizer design boils down to finding the best partition of the source alphabet into disjoint subsets. The number of such subsets will be governed by the constraints which are imposed on the system (distortion, rate, encoder's output entropy etc.). In [5], Muresan and Effros proposed an algorithm for finding good partitions in various settings which include the variable rate scalar Wyner-Ziv [6] setting. However, the optimality of the partitions relied on the convexity of the subsets. Namely the subsets in each partition must be intervals in the source alphabet. It was noted by the authors that this requirement is too strong in the scalar Wyner-Ziv setting and there are many cases where the optimal partition contains subsets which are not convex. We demonstrate such a scenario in the last section of this work. Bounds on the performance of scalar, fixed-rate source codes with decoder SI were recently given in [7].

Real–time codes form a subclass of the class of causal codes, as defined by Neuhoff and Gilbert [8]. In [8], entropy coding is used on the whole sequence of reproduction symbols, introducing arbitrarily long delays. In the real time case, entropy coding has to be instantaneous, symbol–by–symbol (possibly taking into account past transmitted symbols). It was shown in [8] that for a discrete memoryless source (DMS), the optimal causal encoder consists of time-sharing between no more than two scalar encoders. Weissman and Merhav [9] extended [8] to the case where SI is also available at the decoder, encoder or both. The discussion in [9] was restricted, however, only to settings where the encoder and decoder could agree on the reconstruction symbol (i.e., the SI was used for compression, but not in the reproduction at the decoder). Non-causal coding of a source when the decoder has causal access to SI (with possibly a finite look-ahead) was considered by Weissman and El Gamal [10].

The results of [8] for causal coding can be adapted to real-time coding by replacing the arbitrary long delay entropy coding with zero-delay Huffman coding, thus showing that time-sharing at most two scalar quantizers, followed by Huffman coding, is optimal. When the SI is available to both the encoder and decoder, the results of [9] can be adapted to real-time in a similar manner, where at most two scalar quantizers followed by Huffman coding are used for every possible SI symbol. The setting where the decoder can use the SI both to decode the compressed message and to reproduce the source was left open in [9].

This paper has several contributions. Primarily, we prove a theorem that states that when SI is available to the decoder

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only, it is optimal to time-share at most two scalar encoders and decoders. The encoders transmit their messages using zero-error instantaneous codes, as defined in [2]. We also show that there is no performance gain if the decoder has non-causal access to the SI and in fact, only the current SI symbol is useful. Moreover, in the real-time setting, if we a-priori restrict attention to scalar decoders (that use only the current encoder message and SI symbol), there is no performance gain if the encoder has access to the whole sequence of source symbols in advance.

The rest of this paper is organized as follows. In Section II we give the formal setting and notation used throughout the paper. In Section III, we state and discuss the main contributions of this paper. We prove Theorem 1 in Section IV. We end this paper with some examples in Section V.

II. Preliminaries

We begin with notation conventions. Capital letters represent scalar random variables (RV's), specific realizations of them are denoted by the corresponding lower case letters, and their alphabet – by calligraphic letters. For a positive integer i, x^i will denote the vector $(x_1, \ldots, x_i)$. The source alphabet, $\mathcal{X}$, as well as all other alphabets in the sequel, is finite. The probability distribution over $\mathcal{X}$, will be denoted by $P_X(\cdot)$. When there is no room for ambiguity, we will use $P(x)$ instead of $P_X(x)$. $\mathbb{1}\{A\}$ will denote the indicator of the event A.

We investigate the following real–time problem. An encoder observes X_t and transmits a compressed version, W_t, to a decoder which observes Y_t. The decoder produces $\hat{X}_t \in \hat{\mathcal{X}}$, a reproduction of X_t, where $\hat{\mathcal{X}}$ is the reproduction alphabet. Given a constant D and a distortion measure $d : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}$, it is required that $\limsup_{n\to\infty} \frac{1}{n}\boldsymbol{E}\sum_{t=1}^{n} d(X_t, \hat{X}_t) \leq D$. Operation is in real–time. This means that the transmitted data, W_t, can be a function only of the encoder's observations no later than time t, namely, X^t. Similarly, the decoder's estimate, $\hat{X}_t$ is a function of (W^t, Y^t). Let L_n denote the total number of bits sent after observing n source symbols. The rate of the encoder is defined by $R \stackrel{\triangle}{=} \limsup_{n\to\infty} \frac{1}{n}\boldsymbol{E}L_n$. Our goal is to find the tradeoffs between R and D.

Since no delay is allowed, W_t must be encoded by an instantaneous code. Note that in general $P(w_t, y_t) \neq P(w_t)P(y_t)$ (since (X_t, Y_t) are not independent) and therefore, we will need to consider instantaneous coding of W_t in the presence of correlated SI at the decoder. We restrict the coding of W_t to be error-free. The remainder of this section will be devoted to definitions that are needed for zero error transmission in the presence of SI.

For a joint distribution $P(x, y)$, we say that $x, x' \in \mathcal{X}$ are *confusable* if there is a $y \in \mathcal{Y}$ such that $P(x, y) > 0$ and $P(x', y) > 0$. A characteristic graph G is defined on the vertex set of $\mathcal{X}$ and $x, x' \in \mathcal{X}$ are connected by an edge if they are confusable. The pair (G, P), denotes a probabilistic graph consisting of G together with the distribution P over its vertices (here P denotes the marginal on $\mathcal{X}$). We say that two vertices (x, x') are adjacent if there is an edge that connects them in G. The chromatic number of G, $\chi(G)$, is defined to be the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

We will focus only on (x, y) pairs with $P(x, y) > 0$ and thus restrict attention only to *restricted inputs* (RI) protocols, as defined in [2]. A protocol for transmitting X when the decoder knows Y, henceforth referred to as an RI protocol, is defined to be a mapping $\phi : \mathcal{X} \to \{0, 1\}^*$ such that if x and x' are confusable then $\phi(x)$ is neither equal to, nor a prefix of $\phi(x')$. An encoder that uses an RI protocol will be referred to as a SI-aware encoder. The length in bits of $\phi(x)$ will be denoted by $|\phi(x)|$. Note that for restricted inputs, the prefix condition should be kept only over edges of G. Namely, for every $y \in \mathcal{Y}$, the prefix condition should be kept over the subset $\{x : P(x, y) > 0\}$. The fact that the same $x \in \mathcal{X}$ can be contained in multiple such subsets, but can have only a single bit representation, complicates the search for the optimal RI protocol. Let $\bar{l}_Y(\phi) \stackrel{\triangle}{=} \sum_{x\in\mathcal{X}} p(x)|\phi(x)|$, where the subscript emphasizes that Y is known to the decoder. Let

$$\overline{L}_Y(X) = \min\left\{\overline{l_y}(\phi) : \ \phi \text{ is an RI protocol}\right\}. \qquad (1)$$

Upper and lower bounds on $\overline{L}_Y(X)$ in terms of the entropy of the optimal coloring are given in [2]. Finding a single-letter expression for $\overline{L}_Y(X)$ is an open problem. We will use $\overline{L}_Y(X)$ as a figure of merit and our results will be single-letter expressions, in terms $\overline{L}_Y(X)$. In Fig. 1, we give an example of bipartite graphs, formed by two joint distributions $P(x, y)$ where an edge connects (x, y) if $P(x, y) > 0$ along with the characteristic graphs and the optimal RI protocols for a uniform $P_X(\cdot)$.

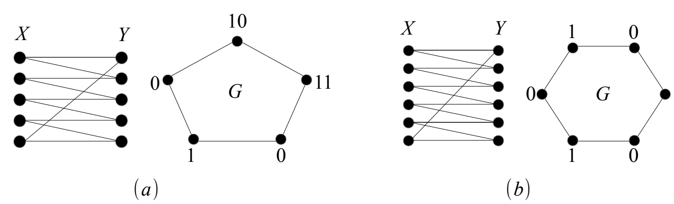

Fig. 1: Example of bipartite graphs of $P(x, y)$ along with their associated characteristic graphs G and a RI protocol for 5 (a) and 6 (b) letter alphabets with "typewriter" SI.

In Fig 1a, we used 4 different bit representations for the source symbols. These bit representations are not prefix free, but are easily seen to be uniquely decodable with the SI. The optimal bit representations imply a 4-coloring scheme for G, although $\chi(G) = 3$. In Fig. 1b, however, $\chi(G) = 2$ and indeed the optimal RI protocol uses a 2-coloring scheme. In Section V, we will return to this example, as well as another example where increasing the quantizer output alphabet reduces the rate.

When the graph G is complete, the prefix condition should be kept for all $x \in \mathcal{X}$ thus reducing the RI protocol to regular prefix coding. In this case, $\overline{L}_Y(X)$ is equal to the average Huffman codeword length of X.

In the proof the converses of our theorems, we use a "genie" that reveals common information to both encoder and decoder, thus we define a conditional RI protocol. Let the triplet (X, Y, Z) be distributed with some joint distribution $P(x, y, z)$. The information that is known to both parties will be denoted by Z, while X,Y continue to play the roles of source output and the SI respectively. For any $z \in \mathcal{Z}$, let $\Phi(z)$ denote the set of conditional RI protocols for z. Namely, the set of all RI protocols for (x, y) such that $p(x, y, z) > 0$. For any $\phi \in \Phi(z)$, let $\bar{l}_Y(\phi|z) \stackrel{\triangle}{=} \sum_{x \in \mathcal{X}} P(x|z)|\phi(x)|$ be the average length when $Z = z$. Similarly, let

$$\overline{L}_Y(X|Z = z) = \min\left\{\bar{l}_Y(\phi|z) : \phi \in \Phi(z)\right\}. \tag{2}$$

Finally, let $\overline{L}_Y(X|Z) = \boldsymbol{E}\overline{L}_Y(X|Z = z)$ where the expectation is with respect to $P_Z(\cdot)$ and we used the same abuse of notation which is commonly used with the notation of conditional entropy. It follows that $\overline{L}_Y(X|Z) \le \overline{L}_Y(X)$ since the set of RI protocols which are valid without the common knowledge of Z is contained in the set of conditional RI protocols which are valid when Z is known at both ends. In the special case where $Z = Y$, i.e., the SI is known to both parties, the RI protocol for each y reduces to designing a Huffman code according to $P_{X|Y}(\cdot|y)$ for every $y \in \mathcal{Y}$.

III. Main Results

In this section, we state and discuss the main results of this work. The pair (R, D) is said to be achievable if there exists a rate- R encoder with causal encoding functions $W_t = f_t(X^t), t = 1, 2, \ldots$, and a decoder with causal reproduction functions, $\hat{X}_t = g_t(W^t, Y^t)$, such that the average distortion is smaller than D. Let $\mathcal{R}_{RT}(D)$ denote the infimum over all rates that are achievable with a given D, where the subscript stands for real-time. Let

$$R_{RT}(D) = \min_{h,f} \overline{L}_Y(f(X)) \tag{3}$$

where the minimization is over all deterministic functions $h : \mathcal{Z} \times \mathcal{Y} \to \hat{\mathcal{X}}$ and $f : \mathcal{X} \to \mathcal{Z}$ such that $\boldsymbol{E}d(X, h(Y, Z)) \le D$ (obviously, $|\mathcal{Z}| \le |\mathcal{X}|$). Finally, denote the lower convex hull of $R_{RT}(D)$ by $\underline{R}_{RT}(D)$. In (3), each possible f in the search domain, along with its optimal h, will incur some given average distortion. Since there is only a finite number of such functions f, the $R - D$ plain contains a finite number points. The lower convex hull of these points will give us $\underline{R}_{RT}(\cdot)$, which is therefore piecewise-linear.

The first result of this paper is the following theorem:

Theorem 1. $\mathcal{R}_{RT}(D) = \underline{R}_{RT}(D)$.

Theorem 1 implies that optimal performance is attained by time-sharing at most two scalar SI-aware quantizers along with scalar decoders. The role of the function f is to partition the source alphabet into subsets. Note that there is no sense in creating overlapping subsets since it will only increase the uncertainty at the decoder (increase the distortion) while adding edges to the characteristic graph of Z with Y (thus increasing the rate). Also, there is no loss of generality in the restriction to deterministic encoders (f) since $\overline{L}_Y(Z)$ is a concave functional of $\{P(z|x)\}$ while the distortion is linear $\{P(z|x)\}$. Therefore optimizing over the whole convex set of stochastic encoders (represented by distributions $\{P(z|x)\}$) is equivalent to optimizing only over the extreme points of this set which are the deterministic encoders.

Let $\mathcal{R}^{\boldsymbol{y}}_{RT}(D)$ denote the infimum over all rates that are achievable with a given D, with the same encoders as before and decoders that can use the whole SI sequence, i.e., $\hat{X}_t = g_t(W^t, Y^n)$. We have the following theorem:

Theorem 2. $\mathcal{R}^{\boldsymbol{y}}_{RT}(D) = \underline{R}_{RT}(D)$.

The theorem states that allowing the decoder to observe the future SI symbols will not result in a performance gain. This is in contrast to the setting of non-causal access to the source and causal access to the SI at the decoder, treated in [10], where it was shown that SI look-ahead can improve performance. The performance gain is achieved through better compression of the transmitted message and not due to better estimation of the source with the SI look-ahead. In the real-time setting, the (real-time) compression of the message cannot be improved by using the SI look-ahead and therefore there is no performance gain.

Let $\mathcal{R}^{\boldsymbol{x}}_{RT}(D)$ denote the infimum over all rates that are achievable with a given D, when non-causal encoders are allowed, i.e., $W_t = f_t(X^n)$, but the decoders are restricted to be scalar, i.e., $\hat{X}_t = g_t(W_t, Y_t)$. We have the following theorem:

Theorem 3. $\mathcal{R}^{\boldsymbol{x}}_{RT}(D) = \underline{R}_{RT}(D)$.

The last theorem states that in the real-time regime, if the reproduction functions are scalar, then scalar SI aware encoders are optimal. This is of course in contrast to the classic arbitrary delay regime.

We prove theorem 1 in the following section. The proofs of the other theorems, which follow the line of proof of Theorem 1, are omitted due to the space limitations.

IV. Proof of Theorem 1

1) Converse part: We will prove a stronger converse than needed, by revealing to the encoder at each stage all the past SI symbols and revealing all past source symbols to the decoder. We also do not rule out stochastic encoders in the converse and therefore do not assume that w^t is a function of x^t. Note that with the "genie aided" feedback and feed–forward, $(W^{t-1}, X^{t-1}, Y^{t-1})$ is known to both parties at the beginning of each stage. Therefore, the minimal average number of transmitted bits at each stage is given by $\overline{L}_{Y_t}(W_t|W^{t-1}, X^{t-1}, Y^{t-1})$. For any sequence of encoding functions which are functions of (W^{t-1}, X^t, Y^{t-1}) and any sequence of reproduction functions which are functions of (W^t, X^{t-1}, Y^t) satisfying the distortion constraint we have:

$$\begin{aligned} nR &\ge \sum_{t=1}^{n} \overline{L}_{Y_t}(W_t|W^{t-1}, X^{t-1}, Y^{t-1}) \\ &= \sum_{t=1}^{n} \int \overline{L}_{Y_t}(W_t|w^{t-1}x^{t-1}, y^{t-1}) d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{n} \int \overline{L}_{Y_t}(f_t(X_t, x^{t-1}, w^{t-1})|w^{t-1}, x^{t-1}, y^{t-1}) \times \\
&\quad d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \\
&= \sum_{t=1}^{n} \int \overline{L}_{Y_t}(f_t(X_t, x^{t-1}, w^{t-1})) d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \quad (4)
\end{aligned}
$$

where $\mu(\cdot)$ denotes the joint probability mass function of its arguments and the last equation is true since X_t is independent of $(w^{t-1}, x^{t-1}, y^{t-1})$. Now, $f_t(X_t, x^{t-1}, w^{t-1})$ can be seen as a specific choice of $f(X_t)$ in the definition of $R_{RT}(D)$. This, along with the fact that we know that $Y^{t-1} = y^{t-1}$ and $W^{t-1} = w^{t-1}$, makes the decoding function $\hat{X}_t = g_t(f_t(X_t, x^{t-1}, w^{t-1}), w^{t-1}, y^{t-1}, Y_t)$ a specific choice of $h(\cdot, \cdot)$ in the definition of $R_{RT}(D)$. We therefore have

$$
\begin{aligned}
&nR \geq \sum_{t=1}^{n} \int \overline{L}_{Y_t}(f_t(X_t, x^{t-1}, w^{t-1})) d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \\
&\geq \sum_{t=1}^{n} \int R_{RT}(\boldsymbol{E}[d(X_t, g_t(f_t(X_t, x^{t-1}, w^{t-1}), w^{t-1}, \\
&\quad y^{t-1}, Y_t))|w^{t-1}, x^{t-1}, y^{t-1}]) d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \quad (5) \\
&\geq \sum_{t=1}^{n} \int \underline{R}_{RT}(\boldsymbol{E}[d(X_t, g_t(f_t(X_t, x^{t-1}, w^{t-1}), w^{t-1}, \\
&\quad y^{t-1}, Y_t))|w^{t-1}, x^{t-1}, y^{t-1}]) d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \quad (6) \\
&\geq \sum_{t=1}^{n} \underline{R}_{RT} \left(\int \boldsymbol{E}[d(X_t, g_t(f_t(X_t, x^{t-1}, w^{t-1}), w^{t-1}, \right. \\
&\quad \left. y^{t-1}, Y_t))|w^{t-1}, x^{t-1}, y^{t-1}] d\mu(w^{t-1}, x^{t-1}, y^{t-1}) \right) \quad (7) \\
&= \sum_{t=1}^{n} \underline{R}_{RT} \left(\boldsymbol{E} \left[d(X_t, g_t(f_t(X^t, W^{t-1}), W^{t-1}, Y^t)) \right] \right) \\
&= \sum_{t=1}^{n} \underline{R}_{RT} \left(\boldsymbol{E} \left[d(X_t, \hat{X}_t) \right] \right) \\
&\geq n \underline{R}_{RT} \left(\frac{1}{n} \sum_{t=1}^{n} \boldsymbol{E} \left[d(X_t, \hat{X}_t) \right] \right) \quad (8) \\
&\geq n \underline{R}_{RT}(D), \quad (9)
\end{aligned}
$$

where (5) follows from the definition of $R_{RT}(D)$ and the discussion following (4), (6) follows from the definition of $\underline{R}_{RT}(D)$, (7) and (8) follow from the convexity of $\underline{R}_{RT}(D)$. Finally, (9) follows from the monotonicity of $\underline{R}_{RT}(D)$.

2) Direct part: The direct part of the theorem is obtained by time–sharing two scalar SI-aware quantizers. By definition of $\underline{R}_{RT}(D)$, we have that there exist (D_1, D_2, λ) such that $D = \lambda D_1 + (1-\lambda) D_2$ and $(f_1, h_1), (f_2, h_2)$ that are the achievers of $R_{RT}(D_1)$ and $R_{RT}(D_2)$, respectively, such that $\lambda R_{RT}(D_1) + (1-\lambda) R_{RT}(D_2) = \underline{R}_{RT}(D)$. Let ϕ_1,ϕ_1 be the optimal protocols for $Z_{1,t} = f_1(X_t), Z_{2,t} = f_2(X_t)$ respectively. Also, let $k_n \leq n$ be a non-decreasing sequence of integers such that $\lim_{n\to\infty} \frac{k_n}{n} = \lambda$. For every n, we use (f_1, h_1) for the first k_n stages and (f_2, h_2) for the rest of the n–block. The resulting $Z_{i,t}, i = 1, 2$, are coded with the optimal protocols ϕ_1 or ϕ_2. The average distortion of this scheme is given by

$$
\begin{aligned}
&\frac{1}{n} \sum_{t=1}^{n} \boldsymbol{E} d(X_t, g(Y^t, Z^t)) \\
&\quad = \frac{k_n}{n} \boldsymbol{E} d(X_t, h_1(Y_t, Z_{1,t})) + \frac{n-k_n}{n} \boldsymbol{E} d(X_t, h_2(Y_t, Z_{2,t})) \\
&\quad \leq \frac{k_n}{n} D_1 + \frac{n-k_n}{n} D_2 \quad (10)
\end{aligned}
$$

and therefore, $\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \boldsymbol{E} d(X_t, g(Y^t, Z^t)) \leq D$. The rate of the code is given by

$$
\begin{aligned}
\frac{1}{n} \boldsymbol{E} L_n &= \frac{1}{n} \sum_{t=1}^{k_n} \boldsymbol{E}|\phi_1(Z_{1,t})| + \frac{1}{n} \sum_{t=k_n+1}^{n} \boldsymbol{E}|\phi_2(Z_{2,t})| \\
&= \frac{k_n}{n} \overline{L}(f_1(X)) + \frac{n-k_n}{n} \overline{L}(f_2(X)) \quad (11)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R &= \limsup_{n\to\infty} \frac{1}{n} \boldsymbol{E} L_n \\
&= \lambda \overline{L}_Y(f_1(X)) + (1-\lambda) \overline{L}_Y(f_2(X)) \\
&= \underline{R}_{RT}(D). \quad (12)
\end{aligned}
$$

V. Examples

A. Lossless transmission

It is interesting to relate the above results to the lossless case. Since $D = 0$ cannot be achieved by time-sharing positive distortions, we get that

$$
\overline{L}_Y(X) = \min_{h, f: h(y, f(x)) = x} \overline{L}_Y(f(X)). \quad (13)
$$

Let $Z = f(X)$. Any f which is a coloring of the characteristic graph G and h, which is the mapping from color and y back to x, are valid candidates in the optimization problem of (13). If f is not a valid coloring, meaning that two connected x_1, x_2 will result in the same z then there is no h which can result in zero error. In essence, we are looking for the coloring for which the restricted inputs protocol will produce the smallest rate. Note that when searching for the best coloring, our performance will be affected only by the characteristic graph G_z which will be built with the "source" (Z, Y). If f is a minimal coloring, i.e, $z \in \{1, 2, \ldots, \chi(G_z)\}$, then G_z is complete. To see this, note that for a minimal coloring, if z_1 and z_2 are not connected, then these colors can be combined and this reduces the number of colors, contradicting the fact that $f(x)$ is a minimal coloring. Remember that a complete graph reduces the RI protocol to Huffman coding. This means that the SI is not helping us to code the colors. Therefore, looking for colorings which will induce a non-complete G_z (i.e., non-optimal coloring) will allow us to use the SI not only to reduce the alphabet of the encoder output, but also for the coding of it output (namely, relax the prefix condition on the codewords when the graph is complete). In the example of Figure 1a, we used a 4-coloring scheme (we had 4 different bit representations for the vertices of G) and not the optimal 3-coloring. Indeed, G_z for the 4-coloring is not complete. For a uniform source, we get an average rate of 1.4 with the 4 coloring and if we had used a 3-coloring we would get an average rate of 1.6.

B. Uniform source, fully connected SI model

Let the reconstruction alphabet be the same as the source alphabet. We use the Hamming distortion measure ($d(x, \hat{x}) = 0$ if $x = \hat{x}$ and $d(x, \hat{x}) = 1$ otherwise). The encoder partitions the source alphabet into disjoint subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$, $k \leq |\mathcal{X}|$. When the encoder observes a new source symbol, x, it sends the index of the subset containing x, using an RI protocol, as defined in Section II. With the Hamming distortion measure, the average distortion is equal to the probability of error. Therefore, the optimal decoder is the maximum likelihood decoder, namely:

$$\hat{x} = \underset{x}{\operatorname{argmax}} P(y, z|x) = \underset{x \in \mathcal{A}_z}{\operatorname{argmax}} P(y|x).$$

where z is the subset index, sent by the encoder.

Let $|\mathcal{X}| = |\mathcal{Y}| = M$ and let for a small constant p, $P(X = a) = \frac{1}{M}$, $P(y = \alpha|x = a) = 1 - p$ if $\alpha = a$ and $P(y = \alpha|x = a) = \frac{p}{M-1}$ for any $\alpha \neq a$. With this choice of the joint distribution, since the bipartite graph of $\{P(x, y)\}$ is fully connected, the bipartite graph of $\{P(y, z)\}$ will also be fully connected, regardless of the choice of Z. Therefore, the RI protocol used to describe the index of the subsets is reduced to a Huffman code for Z.

It is shown in [7] that for this distortion measure and source, only the number of partitions, and not their content (i.e., the actual alphabet letters in each subset), affects the average distortion. The distortion as a function of the number of partitions, K, is given by $\frac{p}{M-1}(|\mathcal{X}| - K)$. It turns out that in this case, $\underline{R}_{RT}(D) = L(X) - \frac{L(X)}{p} D$, which is obtained by time-sharing the two trivial quantizers: the one that does not send information ($R = 0, D = p$) and the lossless quantizer ($R = L(X), D = 0$).

C. Uniform source, given SI model

We continue with the Hamming distortion measure and a uniform source with $|\mathcal{X}| = 5$. The channel from X to Y is given in Fig. 2 along with the characteristic graph of X. In

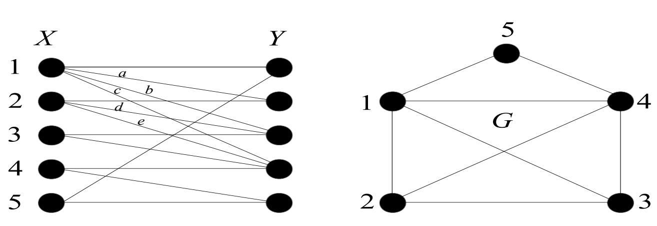

Fig. 2: $P(y|x)$ and the resulting characteristic graph.

this example we set for $\alpha \in \mathcal{X}$: $P(y = \alpha|x = \alpha) = 1 - p$ and $a = \frac{1}{2}p, b = \frac{5}{12}p, c = \frac{1}{12}p, d = \frac{3}{4}p, e = \frac{1}{4}p$. Note that the chromatic number of G is 4. This means that any partitioning of the alphabet of X into less than 4 subsets will incur a lossy reconstruction. Unlike the previous example where the SI could be used only in the reconstruction, but not to reduce the length of the transmission (since G was fully connected), here it will be used for both. Note that as in the example of Figure 1, the optimal rate for lossless transmission is actually obtained by using more subsets than the chromatic number of X. The optimal two subset partition ($|\mathcal{Z}| = 2$) is $\{1, 4\}, \{2, 3, 5\}$, yielding an average distortion of $\frac{13}{60}p$. The rate for this (and any binary) partition is 1. The optimal 3-subset partition is $\{1, 4\}, \{2, 5\}, \{3\}$ yielding an average distortion of $\frac{1}{60}p$. The average rate for this partitioning is $8/5$. However, in this case it is beneficial to split $\{2, 5\}$ and obtain a lower rate of $7/5$ (using the SI to alleviate the prefix requirement). Although we use more subsets, the rate is reduced. In Figure 3, we compare $\underline{R}_{RT}(D)$ to the performance of a system that uses Huffman codes instead of an RI protocol, i.e., a system that uses the SI only for reproduction but not for compression.

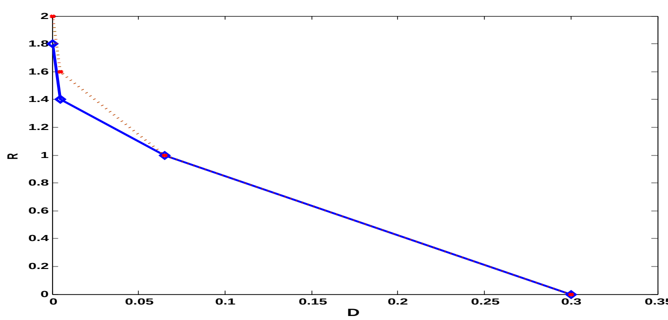

Fig. 3: $\underline{R}_{RT}(D)$ (solid) compared to a system that uses the SI only for reproduction (dotted) with $p = 0.3$.

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