Data Processing Inequalities Based on a Certain Structured Class of Information Measures With Application to Estimation Theory

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Background

Classical joint source–channel data processing inequality (DPI) for $U \rightarrow X \rightarrow Y \rightarrow V$:

$$R(D) \leq I(U;V) \leq I(X;Y) \leq C \quad \Rightarrow \quad D \geq R^{-1}(C).$$

Ziv and Zakai (1973) generalized to:

$$R_Q(D) \leq I_Q(U;V) \leq I_Q(X;Y) \leq C_Q \quad \Rightarrow \quad D \geq R_Q^{-1}(C_Q),$$

where

$$I_Q(A;B) = \mathbb{E}\left\{ \log Q\left( \frac{P(A)P(B)}{P(A,B)} \right) \right\}$$

for a general convex function $Q$ (see also Csiszár’s $f$–divergence, 1972). Further generalization (Zakai & Ziv, 1975) to multivariate convex functions

$$I_Q(A;B) = \mathbb{E}\left\{ \log Q\left( \frac{\mu_1(A,B)}{P(A,B)}, \ldots, \frac{\mu_k(A,B)}{P(A,B)} \right) \right\}.$$
Gurantz (1974) examined

\[ G(Y\mid x, x_1, \ldots, x_k) = \int \mathcal{d}y \cdot P_{Y\mid X}(y\mid x) \times 
\]

\[ Q_1 \left( \frac{P_{Y\mid X}(y\mid x_1)}{P_{Y\mid X}(y\mid x)} \right) \cdot Q_2 \left( \frac{P_{Y\mid X}(y\mid x_2)}{P_{Y\mid X}(y\mid x_1)} \right) \cdot Q_3 \left( \cdots Q_k \left( \frac{P_{Y\mid X}(y\mid x_k)}{P_{Y\mid X}(y\mid x_{k-1})} \right) \cdots \right), \]

and showed that for \( X \rightarrow Y \rightarrow Z \),

\[ G(Y, x, x_1, \ldots, x_k) \geq G(Z, x, x_1, \ldots, x_k). \]

This yields \( R_G(U; V) \leq C_G \) w.r.t. \( I_G(A; B) = \mathbb{E}\{G(B\mid A, A_1, \ldots, A_k)\} \), where \( \mathbb{E}\{\cdot\} \) is w.r.t. \( P_{AB}(a, b) \times P_A(a_1) \times \cdots \times P_A(a_k) \).

While \( I_G \) can be shown to be a special case of the ZZ75 information measure, it has an interesting structure that calls for further study.
Choice of the Convex Functions

Consider the functions

\[ Q_1(t) = -t^{a_1} \quad 0 \leq a_1 \leq 1 \]
\[ Q_i(t) = t^{a_i} \quad 0 \leq a_i \leq 1, \quad 2 \leq i \leq k \]

leading to

\[ G(Y|x_0, x_1, \ldots, x_k) = -\int_Y dy P_{Y|X}(y|x_0) \times \]
\[ \left( \frac{P_{Y|X}(y|x_1)}{P_{Y|X}(y|x_0)} \right)^{a_1} \left( \frac{P_{Y|X}(y|x_2)}{P_{Y|X}(y|x_1)} \right)^{a_2} \cdots \left( \frac{P_{Y|X}(y|x_k)}{P_{Y|X}(y|x_{k-1})} \right)^{a_k} \]

\[ = -\int_Y dy \prod_{i=0}^k P_{Y|X}^{b_i}(y|x_i) \]

where \( b_i \geq 0 \) for all \( i \) and \( \sum_{i=0}^k b_i = 1 \).
Choice of the Convex Functions (Cont’d)

Choosing \( b_i = 1/(k + 1) \) for all \( i \) yields

\[
I_G(X; Y) = -\int_Y dy \left[ \int_X dx P_X(x) P_{Y|X}^{1/(k+1)}(y|x) \right]^{k+1} = -\exp\{ -E_0(\rho, P_X) \} \bigg|_{\rho=k}.
\]

Comments:

- Gallager’s function \( E_0 \) indeed satisfies a DPI (Kaplan & Shamai 1993).
- Choice of integer \( \rho (\rho = k) \) is relatively easy:
  - Square brackets \( \rightarrow \) multidimensional integral \( \rightarrow \) swapping with \( \int dy \).
  - Generalizing from the Bhattacharyya distance \( (k = 1) \) to a general \( k \).

Questions:

- Zakai & Ziv (1975) examined the choice \( k = 1 \) in signal parameter estimation. Is \( k = 1 \) the best choice or can it be improved?
- How does the best bound of this type compare to other bounds from estimation theory?
Consider the model

\[ y(t) = x(t, u) + n(t), \quad 0 \leq t < T, \]

where \( x(t, u) \) is an arbitrary waveform, parameterized by \( u \), with

\[ \int_0^T dt \cdot x^2(t, u) = E \]

and \( n(t) \) is AWGN with spectral density \( N_0/2 \).

It is assumed that \( u \) is realization of \( U \sim \text{Unif}[\frac{1}{2}, \frac{1}{2}] \).

We are interested in lower bounds on

\[ \bar{\epsilon}^2 = \mathbb{E}(\hat{U} - U)^2 \]

in the high–SNR regime \( E/N_0 \gg 1 \).

We focus on universal lower bounds (fundamental limits), that are independent of the waveform. No bandwidth constraints are imposed.
**Calculation of $R_G(D)$**

The high–res behavior of $R_G(D)$ is as follows:

$$R_G(D) \sim \begin{cases} 
-4c\sqrt{D} & k = 1 \\
-4 \left( \frac{k}{k-2} \right)^k \cdot D & k > 2 
\end{cases}$$

where

$$c = \int_{-\infty}^{+\infty} \frac{dt}{(1 + t^2)^2}.$$ 

For $k = 2$, we have

$$\log[-R_G(D)] \sim \log D$$

in the sense that

$$\lim_{D \to 0} \frac{\log[-R_G(D)]}{\log D} = 1.$$
Calculation of $I_G(U; Y)$

For the AWGN channel

$$P(y|u) \propto \exp \left\{ -\frac{1}{N_0} \int_0^T [y(t) - x(t, u)]^2 dt \right\},$$

we have

$$I_G(U; Y) \leq - \exp \left\{ -\frac{E}{N_0} \cdot \frac{k}{(k+1)} \cdot (1 - \varrho) \right\},$$

where

$$\varrho = \frac{1}{E} \mathbb{E} \left\{ \int_0^T dt \cdot x(t, U)x(t, U') \right\} = \frac{1}{E} \int_0^T dt \cdot [\bar{x}(t)]^2,$$

and

$$\bar{x}(t) = \mathbb{E}\{x(t, U)\} = \int_{-1/2}^{+1/2} du \cdot x(t, u).$$

Note that

$$E(1 - \varrho) = \int_0^T dt \cdot \text{Var}\{x(t, U)\}.$$
DPI Estimation Error Bounds

Applying the DPI, $R_G(D) \leq I_G(U; Y)$, we get

$$\overline{\epsilon^2} \geq \begin{cases} 
\frac{1}{16c^2} \exp\{-(1 - \varrho)E/N_0\} & k = 1 \text{ (Zakai & Ziv '75)} \\
\frac{1}{4} \left(1 - \frac{2}{k}\right)^k \exp\left\{-(1 - \varrho)\frac{k}{k+1} \cdot \frac{E}{N_0}\right\} & k > 2 
\end{cases}$$

and for $k = 2$

$$\liminf_{E/N_0 \to \infty} \frac{\log \overline{\epsilon^2}}{E/N_0} \geq -\frac{2}{3} \cdot (1 - \varrho).$$

Discussion:

- $k = 2$ is the best choice of $k$ for high SNR.
- The bounds are minimized by signals with $\varrho = 0$.
- Upon setting $\varrho = 0$, the bounds are independent of the modulation.
- For the bounds to be tight, $\rho(U, U') = \int_0^T dt \cdot x(t, U)x(t, U')/E$ should be nearly zero with high probability – rapidly vanishing correlation.
- It is possible to achieve $\overline{\epsilon^2} \sim e^{-E/(3N_0)}$, e.g., by PPM. The gap is 3dB.
Comparison to Other Bounds

The Weiss–Weinstein bound (WWB) for a given modulation is

\[ WWB = \sup_{h \neq 0} \frac{h^2 \exp\{-[1 - r(h)]E/(2N_0)\}}{2(1 - \exp\{-[1 - r(2h)]E/(2N_0)\})}, \]

where

\[ r(h) = \rho(u, u + h) = \frac{1}{E} \int_0^T x(t, u)x(t, u + h)dt. \]

To derive a universal lower bound, this should be minimized over all feasible correlation functions \( r(\cdot) – \) not a trivial minimax problem. One can lower bound by solving the maximin problem, yielding

\[ WWB = \frac{e^{-E/N_0}}{2(1 - e^{-E/N_0})}. \]

But this is inferior to our earlier bounds for \( k > 1 \).
A simple consideration of $M$–ary signal detection yields

$$\bar{e}^2 \geq \frac{1}{8M^2} \cdot Q \left( \sqrt{\frac{E}{N_0} \cdot \frac{M}{M-2}} \right),$$

where $M = 4, 6, 8, \ldots$. For high SNR, this is exponentially equivalent to

$$\exp \left\{ -\frac{E}{2N_0} \cdot \frac{M}{M-2} \right\},$$

which, for large enough $M$, is arbitrarily close to $e^{-E/(2N_0)}$. This is better than our best bound $e^{-2E/(3N_0)}$.

Q: In what situations is the DPI bound superior to other bounds?
Channels with Uncertainty – AWGN with Fading

Suppose that there is an unknown nuisance parameter $A$ (e.g., fading), independent of $U$ and

$$P_{Y|U}(y|u) = \int_{-\infty}^{+\infty} da \cdot P_A(a) P_{Y|U,A}(y|u,a).$$

Think of $I_G(U;Y)$ as a functional of $P_{Y|U}$, denoted $\mathcal{I}(P_{Y|U}(\cdot|u))$, then it is a convex functional, namely,

$$\mathcal{I}(P_{Y|U}(\cdot|u)) = \mathcal{I}\left(\int_{-\infty}^{+\infty} da P_A(a) P_{Y|U,A}(\cdot|u,a)\right)$$

$$\leq \int_{-\infty}^{+\infty} da P_A(a) \mathcal{I}(P_{Y|U,A}(\cdot|u,a))$$

unknown $A$

known $A$
Consider the channel

\[ y(t) = a \cdot x(t, u) + n(t), \quad 0 \leq t < T, \]

where \( a \) and \( u \) are realizations of \( A \) and \( U \), respectively. Assume that \( A \sim \mathcal{N}(0, \sigma^2) \) is independent of \( U \).

\[
P_{Y|U}(y|u) \propto \int_{-\infty}^{+\infty} \text{d}a \cdot \frac{e^{a^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} \cdot \exp \left\{ -\frac{1}{N_0} \int_0^T \left[ y(t) - a \cdot x(t, u) \right]^2 \text{d}t \right\}
\]

\[
\propto \exp \left\{ \theta \left[ \int_0^T y(t)x(t, u) \text{d}t \right]^2 \right\}
\]

where

\[
\theta \triangleq \frac{2\sigma^2}{N_0^2 (1 + 2\sigma^2 E/N_0)}.
\]
Upon calculating $I_G(U;Y)$ for the AWGN channel with fading (under the rapidly vanishing correlation assumption), we obtain the high–SNR bounds

$$\bar{\epsilon}^2 \geq \frac{g_k}{\sigma} \cdot \sqrt{\frac{N_0}{E}}$$

with

$$g_k = \frac{1}{4\sqrt{2}} \left(1 - \frac{2}{k}\right)^k \left(1 + \frac{1}{k}\right)^{(k+1)/2}, \quad k = 1, 2, \ldots$$

The tightest bound is obtained with $k \to \infty$. Let

$$g_\infty = \lim_{k \to \infty} g_k = \frac{1}{4\sqrt{2}e^{3/2}} = 0.03944.$$ 

Thus, our asymptotic lower bound for high SNR is

$$\lim_{E/N_0 \to \infty} \inf \sqrt{\frac{E}{N_0}} \cdot \bar{\epsilon}^2 \geq \frac{0.03944}{\sigma}.$$
Comparison with Other Bounds

The Weiss–Weinstein bound:

$$\text{WWB} \propto \frac{N_0}{\sigma^2 E}.$$

The $M$–ary signal detection bound:

$$\liminf_{E/N_0 \to \infty} \sqrt{\frac{E}{N_0}} \cdot \bar{e}^2 \geq \frac{0.001758}{\sigma}.$$  

The Chazan–Zakai–Ziv bound:

$$\liminf_{E/N_0 \to \infty} \sqrt{\frac{E}{N_0}} \cdot \bar{e}^2 \geq \frac{0.00716}{\sigma},$$

a factor of 5.5 (7.4dB) smaller than the DPI bound.
Conclusion and Future Work

- We examined a family of information measures with a certain structure (Gurantz, 1974).

- For a specific choice of the convex functions – equivalent to \( E_0(\rho, P_X)|_{\rho=k} \) – an extension of the Bhattacharyya distance.

- Best choice of \( k \): \( k = 2 \) for AWGN; \( k \to \infty \) – for AWGN with fading.

- Bounds compete favorably with existing bounds, especially in situations of uncertainty. Explanation: convexity of \( \mathcal{I}_G(P_Y|U) \).

- Future work: Trying to close the gap between upper bound and universal lower bound of \( \lim_{E/N_0 \to \infty} N_0 \log \frac{e^2}{E} \).