On the Cost of Universality of Block Codes for Individual Sequences

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Abstract — **Consider a rate R block code of size k that achieves minimum distortion for a *given* individual input sequence. Such a code obviously depends on the input sequence. We seek a *universal* (sequence-independent) rate R' block code of size l ($l > k$), which guarantees a distortion not higher than that of the above optimal sequence-dependent block code. We ask the question of the cost of universality: What is the best achievable rate R' as a function of the block length l of the universal code such that the above task could be fulfilled? We provide achievable bounds that characterize this optimal rate. Dual results on the cost of universality are shown for universal channel coding and distortionless joint source-channel coding.**

Let $x_1^n = x_1 \dots x_n$ be an input sequence of length n where each x_i takes values in an alphabet of size $A < \infty$, and let y_1^n denote the reconstructed sequence, where each y_i takes values in a finite reproduction alphabet.

Consider first block-to-block (BB) codes. A BB code with rate R bits per symbol and block size k is a mapping from the set of all A^k input source strings of length k, into a subset of the output strings whose cardinality is 2^{Rk}. For a given x_1^n and a given distortion measure $d(x_1^n, y_1^n)$ let us define

$$D_k(R|x_1^n) = \min_{\{\text{all BB codes of rate } R, \text{ size } k\}} d(x_1^n, y_1^n). \quad (1)$$

The distortion-rate function above describes the performance in coding a given sequence x_1^n by the best sequence-dependent block code of size k. While in [1] it was shown that this distortion-rate function can be asymptotically attained by universal codes, our main contribution here is in providing the exact expression of the best attainable redundancy of the universal codes that attain $D_k(R|x_1^n)$, as a function of the block length. Specifically, our first result is the following:

Theorem 1 *There exists a sequence-independent block code of rate R' and size l that attains $D_k(R|x_1^n)$ for every x_1^n, if and only if*

$$R' > R_0 \triangleq \begin{cases} \log A, & l < k2^{Rk} \\ R + \frac{k2^{Rk}}{l}(\log A - R) + o\left(\frac{k2^{Rk}}{l}\right) & \textit{otherwise.} \end{cases} \quad (2)$$

The difference $r = R_0 - R$, which is the best attainable redundancy, can be interpreted as the "cost of universality" w.r.t the class of rate-distortion BB codes. Another look at eq. (2) is that in order to achieve a required redundancy r (for the interesting case where $R' < \log A$), the block size of the universal code must be about

$$l = \frac{k2^{Rk}(\log A - R)}{r}. \quad (3)$$

We next consider block-to-variable (BV) codes and obtain similar results. First, in this case the optimal performance of any BV code of block size k is given by

$$\tilde{D}_k(R|x_1^n) = \min_{\{\text{all BV codes of rate } R, \text{ size } k\}} d(x_1^n, y_1^n) \quad (4)$$

Now following well known results (see e.g. [2]) regarding the optimal redundancy of universal lossless BV codes it can be shown that $\tilde{D}_k(R|x_1^n)$ can be obtained by a universal BV code of block size l and rate R' if and only if

$$R' > R_1 \triangleq R + \frac{A^k - 1}{2l}\log l + \text{negligible terms.} \quad (5)$$

The main term in (5), resulting from the definition (4), is smaller than the main term in (2), that in turn is calculated from the definition (1) at the same distortion level; i.e., the performance of BV codes is better than BB codes. However, the redundancy term for BV codes in (5) decays as fast as $\log l/l$ and hence it is larger than the redundancy of BB codes in (2) that decays as fast as $1/l$.

In complete duality to BB rate-distortion coding, one can evaluate the cost of universality for BB channel coding for additive noise channels. In this case each noise symbol comes from an alphabet of size B and the noise sequence is treated as an individual sequence, where each k-block is taken from a subset W of B^k whose cardinality is 2^{Gk} where $G < \log B$. We define the capacity of this channel by $\log B - G \triangleq C$. We show that reliable transmission by a rate R' universal channel block code of size l (where universality here means that the code is independent of W) is possible if and only if

$$R' < C_0 \triangleq C - \frac{k2^{Gk}}{l}(\log B - G). \quad (6)$$

Finally, results on the cost of universality for joint source-channel coding can be stated in the case of lossless transmission. Specifically, we present a "separation theorem" stating that the extra cost in universal lossless joint source-channel coding is the sum of the costs associated with source coding and with channel coding separately. The problem of evaluating the cost of universlity for joint source-channel coding with distortion, remains open.

References

[1] J. Ziv, "Distortion-Rate Theory for Individual Sequences," *IEEE Trans. Inform. Theory*, Vol. IT-26, pp. 137-143, 1980.

[2] J. Rissanen, "Universal Coding, Information, Prediction and Estimation," *IEEE Trans. Information Theory*, Vol. IT-30, pp. 629–636, 1984.

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