Channel Upgradation for Non-Binary Input Alphabets and MACs

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Problem

$$W : \mathcal{X} \to \mathcal{Y} \qquad\qquad |\mathcal{Y}| \gg S$$

$$\Big\Downarrow \textbf{upgrade}(W, S)$$

$$W' : \mathcal{X} \to \mathcal{Y}' \qquad\qquad |\mathcal{Y}'| \leq S$$

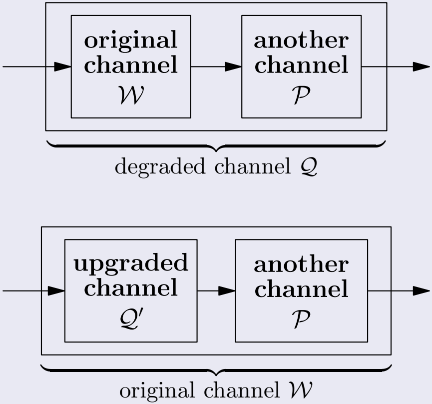
Goal: make $I(W') - I(W)$ as small as possible, where $I(\cdot)$ is symmetric capacity.

- $\mathcal{X}$ need not be binary
- W need not be symmetric
- W can be a t-user MAC $W : \mathcal{X}^t \to \mathcal{Y}$

Concurrent work

A. Ghayoori and T. A. Gulliver, Upgraded Approximation of Non-Binary Alphabets for Polar Code Construction, on arXiv

Upgradation Vs. Degradation



- $W(y|u)$ is the probability of receiving $y \in \mathcal{Y}$, given that $u \in \mathcal{X}$ was sent.
- The input is assumed to have symmetric distribution over $\mathcal{X}$.

Motivation: polar codes

- Assure Eve's channels are *very* bad (wiretap setting)
- Upper bound on rate (error correcting code)

Theorem

Let $W : \mathcal{X} \to \mathcal{Y}$ be a given channel, with $|\mathcal{X}| = q$.
Let $\mu \geq \max\{5, q(q-1)\}$ be a given fidelity parameter.
We can construct $W' : \mathcal{X} \to \mathcal{Y}'$ such that

- W' is upgraded with respect to W
- $I(W') - I(W) \leq \frac{q-1}{\mu}(2 + q \cdot \ln q)$
- $|\mathcal{Y}'| \leq S(q, \mu) \triangleq (2\mu)^q + q$

- impractical for big q (i.e. large *input* alphabet.)

Key ideas

- Use bins:
  - "close" output symbols of W go into the same bin
  - "merge" all symbols in bin to one symbol
- Use "boost" symbols (sparingly):
  - New output alphabet will have $q = |\mathcal{X}|$ special symbols, $\{\kappa_u\}_{u \in \mathcal{X}}$
  - Symbol κ_u received on W' only if u sent
  - Small probability of receiving κ_u

W

	y_1	y_2	$\cdots$
u=0	$W(y_1\|0)$	$W(y_2\|0)$	
u=1	$W(y_1\|1)$	$W(y_2\|1)$	
u=2	$W(y_1\|2)$	$W(y_2\|2)$	

Q

	y_1	y_2	$\cdots$	κ_0	κ_1	κ_2
u=0	$\alpha_0(y_1)\cdot W(y_1\|0)$	$\alpha_0(y_2)\cdot W(y_2\|0)$		ϵ_0	0	0
u=1	$\alpha_1(y_1)\cdot W(y_1\|1)$	$\alpha_1(y_2)\cdot W(y_2\|1)$		0	ϵ_1	0
u=2	$\alpha_2(y_1)\cdot W(y_1\|2)$	$\alpha_2(y_2)\cdot W(y_2\|2)$		0	0	ϵ_2

$$\epsilon_u = \sum_{y\in\mathcal{Y}} (1-\alpha_u(y))W(y|u) ,$$

$$0 \le \alpha_u(y) \le 1 , \quad \alpha_u(y) \approx 1$$

Example

W

Consider a bin $\{y_1, y_2, y_3, y_4\}$,
where

	y_1	y_2	y_3	y_4
$u = 0$	0.2625	0.0042	0.0500	$2.1 \cdot 10^{-12}$
$u = 1$	0.75	0.0118	0.1475	$5.8 \cdot 10^{-12}$
$u = 2$	0.2375	0.004	0.0525	$2.1 \cdot 10^{-12}$

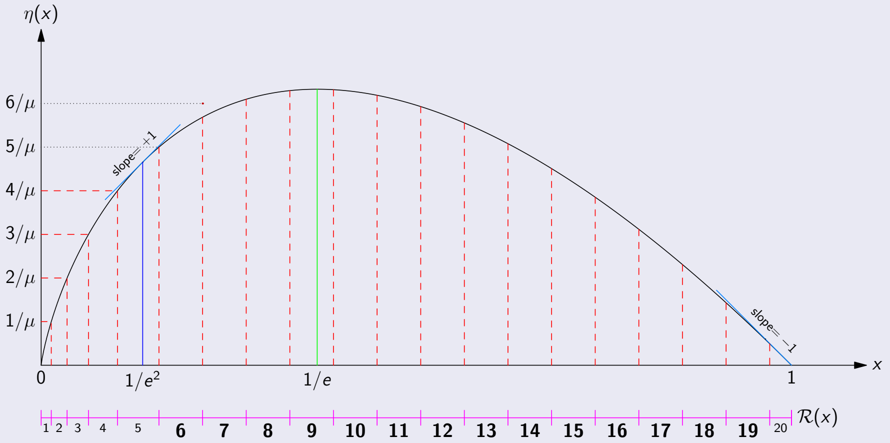
Q

We choose $\alpha_u(y)$ such that the columns are multiples of one another:

	y_1	y_2	y_3	y_4	κ_0	κ_1	κ_2
$u = 0$	0.2459	0.0038	0.0483	$1.901 \cdot 10^{-12}$	0.0185	0	0
$u = 1$	0.75	0.0118	0.1475	$5.8 \cdot 10^{-12}$	0	0	0
$u = 2$	0.2336	0.0036	0.0459	$1.806 \cdot 10^{-12}$	0	0	0.0107

$Q(y_1|u) = 63.55 \cdot Q(y_2|u) = 5.084 \cdot Q(y_3|u) = 0.1293 \cdot 10^{12} \cdot Q(y_4|u)$

Functions $\eta(x)$ and $\mathcal{R}(x)$ for $\mu = 17.2$



$$\eta(x) = -x \cdot \ln x$$

- For $y \in \mathcal{Y}$ and $u \in \mathcal{X}$, define the APP

$$\varphi_W(u|y) = \frac{W(y|u)}{\sum_{v \in \mathcal{X}} W(y|v)} .$$

- Two symbols $y_1, y_2 \in \mathcal{Y}$ are in the same bin, if

$$i(u) = \mathcal{R}(\varphi_W(u|y_1)) = \mathcal{R}(\varphi_W(u|y_2)) , \quad \text{for all } u \in \mathcal{X} .$$

In our example earlier

	y_1	y_2	y_3	y_4	
$W(y\|0)$	0.2625	0.0042	0.0500	$2.1 \cdot 10^{-12}$	
$W(y\|1)$	0.75	0.0118	0.1475	$5.8 \cdot 10^{-12}$	
$W(y\|2)$	0.2375	0.004	0.0525	$2.1 \cdot 10^{-12}$	
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21	$i(0) = 6$
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58	$i(1) = 13$
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21	$i(2) = 6$

New APP

- Denote all the output letters in bin z as $\mathcal{B}(z)$
- Define the leading input and output of the bin as

$$(u^*, y^*) = \arg \max_{\substack{u \in \mathcal{X}, \\ y \in \mathcal{B}(z)}} \varphi_W(u|y)$$

- Define a new APP measure by

$$\psi(u|z) = \min_{y \in \mathcal{B}(z)} \varphi_W(u|y) \quad \text{for all } u \neq u^*$$

and

$$\psi(u^*|z) = 1 - \sum_{u \neq u^*} \psi(u|z)$$

$$\alpha_u(y) \triangleq \frac{\psi(u|z)}{\varphi_W(u|y)} \cdot \frac{\varphi_W(u^*|y)}{\psi(u^*|z)}$$

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
where

	y_1	y_2	y_3	y_4
$W(y\|0)$	0.2625	0.0042	0.0500	$2.1 \cdot 10^{-12}$
$W(y\|1)$	0.75	0.0118	0.1475	$5.8 \cdot 10^{-12}$
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	y_1	y_2	y_3	y_4
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
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	y_1	y_2	y_3	y_4
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21

- The leading input is $u^* = \arg\max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y)\right] =$

Example (Cont.)

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$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21
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$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21

- The leading input is $u^* = \arg\max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y)\right] = \mathbf{1}$.

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
where

	y_1	y_2	y_3	y_4
$W(y\|0)$	0.2625	0.0042	0.0500	$2.1 \cdot 10^{-12}$
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$W(y\|2)$	0.2375	0.004	0.0525	$2.1 \cdot 10^{-12}$

	y_1	y_2	y_3	y_4	
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21	$\psi(0\|z) = 0.20$
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58	
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21	$\psi(2\|z) = 0.19$

- The leading input is $u^* = \arg\max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y) \right] = \mathbf{1}$.

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
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$W(y\|2)$	0.2375	0.004	0.0525	$2.1 \cdot 10^{-12}$

	y_1	y_2	y_3	y_4	
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21	$\psi(0\|z) = 0.20$
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58	$\psi(1\|z) = 1 - (0.20 + 0.19) = 0.61$
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21	$\psi(2\|z) = 0.19$

- The leading input is $u^* = \arg\max_{u\in\{0,1,2\}} \left[\max_{y\in\mathcal{B}(z)} \varphi_W(u|y)\right] = \mathbf{1}$.

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
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	y_1	y_2	y_3	y_4
$W(y\|0)$	0.2625	0.0042	0.0500	$2.1 \cdot 10^{-12}$
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	y_1	y_2	y_3	y_4
$\varphi_W(0\|y)$	0.21	0.21	0.20	0.21
$\varphi_W(1\|y)$	0.60	0.59	0.59	0.58
$\varphi_W(2\|y)$	0.19	0.20	0.21	0.21

$\psi(0|z) = 0.20$

$\psi(1|z) = 1 - (0.20 + 0.19) = 0.61$

$\psi(2|z) = 0.19$

- The leading input is $u^* = \arg\max_{u \in \{0,1,2\}} \left[\max_{y \in \mathcal{B}(z)} \varphi_W(u|y)\right] = \mathbf{1}$.
- *E.g.* $\alpha_0(y_1) = \frac{0.20}{0.21} \cdot \frac{0.60}{0.61} \approx 0.936$.
  Thus, we subtract $(\approx 0.064 \cdot W(y_1|0))$ and pass over to our boost symbol κ_0.

Example (Cont.)

Consider a bin $\mathcal{B}(z) = \{y_1, y_2, y_3, y_4\}$,
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$\psi(0|z) = 0.20$

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$\psi(2|z) = 0.19$

	y_1	y_2	y_3	y_4
$\alpha_0(y)$	$\frac{0.20}{0.21} \cdot \frac{0.60}{0.61}$	$\frac{0.20}{0.21} \cdot \frac{0.59}{0.61}$	$\frac{0.20}{0.20} \cdot \frac{0.59}{0.61}$	$\frac{0.20}{0.21} \cdot \frac{0.58}{0.61}$
$\alpha_1(y)$	1	1	1	1
$\alpha_2(y)$	$\frac{0.19}{0.19} \cdot \frac{0.60}{0.61}$	$\frac{0.19}{0.20} \cdot \frac{0.59}{0.61}$	$\frac{0.19}{0.21} \cdot \frac{0.59}{0.61}$	$\frac{0.19}{0.21} \cdot \frac{0.58}{0.61}$

Example (Cont.)

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$\psi(0|z) = 0.20$

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$\psi(2|z) = 0.19$

	y_1	y_2	y_3	y_4	
$Q(y\|0)$	0.2459	0.0038	0.0483	$1.901 \cdot 10^{-12}$	$\varepsilon_0 \approx 0.0185$
$Q(y\|1)$	0.75	0.0118	0.1475	$5.8 \cdot 10^{-12}$	$\varepsilon_1 = 0$
$Q(y\|2)$	0.2336	0.0036	0.0459	$1.806 \cdot 10^{-12}$	$\varepsilon_2 \approx 0.0107$

Proof Outline (very broadly)

Note that

Mutual Information of U and Y

$$I(W) = I(U;Y) = \ln q - \sum_{y \in \mathcal{Y}} p_W(y) \cdot H(U|Y=y))$$

Now, define

Surrogate for Mutual Information

$$\begin{aligned} J(U;Z) &= \ln q - \sum_{z \in \mathcal{Z}} p(z) \cdot H_\psi(U|Z=z)) \\ &= \ln q - \sum_{z \in \mathcal{Z}} p(z) \sum_{u \in \mathcal{X}} \eta\left[\psi(u|z)\right] \ , \end{aligned}$$

where $p(z) \triangleq \sum_{y \in \mathcal{B}(z)} p_W(y)$.

Next, prove two bounds and sum up

- First, some interesting properties regarding our quantization, the APP measure $\psi(\cdot|z)$ and the leading input imply that

$$|\eta\left(\varphi_W(u|y)\right) - \eta\left(\psi(u|z)\right)| \leq \begin{cases} 1/\mu & \text{if } u \neq u^* \ , \\ \frac{q-1}{\mu} & \text{if } u = u^* \ . \end{cases}$$

$$\Rightarrow J(U;Z) - I(W) = \sum p(y)\left[\eta(\varphi(u|y)) - \eta(\psi(u|z))\right] \leq 2 \cdot \frac{q-1}{\mu} \ ,$$

- Second,

$$\frac{1}{q}\sum_{u \in \mathcal{X}} \varepsilon_u \leq \frac{q(q-1)}{\mu} \ .$$

$$I(W') - J(U;Z) =$$
$$= \sum_z (p(z) - p_{W'}(z)) H_\psi(U|Z=z) - \sum_\kappa p_{W'}(\kappa) H(U|Z' = \kappa)$$
$$\leq \ln q \cdot \frac{1}{q} \cdot \sum_{u \in \mathcal{X}} \varepsilon_u \leq \ln q \cdot \frac{q(q-1)}{\mu} \ .$$

Thank You!