List-Decoding of Polar Codes

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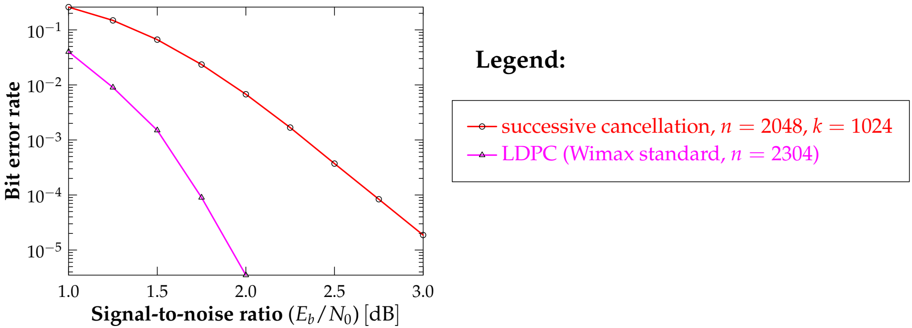
Problem and goal

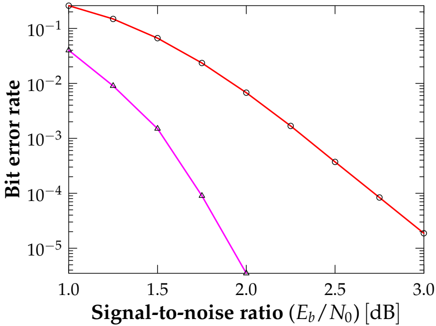
- Channel polarization is slow. For short to moderate code lengths, polar codes have disappointing performance.



- In this talk, we present a generalization of the SC decoder which greatly improves performance at short code lengths.

Avenues for improvement

From here onward, consider a polar code of length $n = 2048$ and rate $R = 0.5$, optimized for a BPSK-AWGN channel with $E_b/N_0 = 2.0$ dB.



Legend:

- Why is our polar code under-performing?
    - Is the SC decoder under-performing?
    - Are the polar codes themselves weak at this length?

A critical look at successive cancellation

Successive Cancellation Decoding

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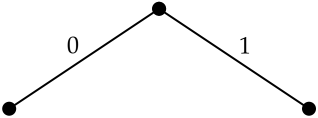
for $i = 0, 1, \ldots, n-1$ **do**
- **if** $\widehat{u}_i$ *is frozen* **then** set $\widehat{u}_i$ accordingly;
- **else**
  - **if** $W_i(\mathbf{y}_0^{n-1}, \widehat{u}_0^{i-1}|0) > W_i(\mathbf{y}_0^{n-1}, \widehat{u}_0^{i-1}|1)$ **then**
    - set $\widehat{u}_i \leftarrow 0$;
  - **else**
    - set $\widehat{u}_i \leftarrow 1$;

Potential weaknesses (interplay):

- Once an unfrozen bit is set, there is "no going back". A bit that was set at step i can not be changed at step $j > i$.
- Knowledge of the value of future frozen bits is not taken into account.

List decoding of polar codes

Key idea: Each time a decision on $\widehat{u}_i$ is needed, split the current decoding path into two paths: **try both $\widehat{u}_i = 0$ and $\widehat{u}_i = 1$.**



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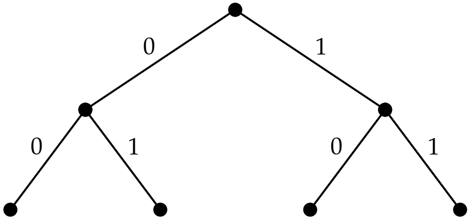

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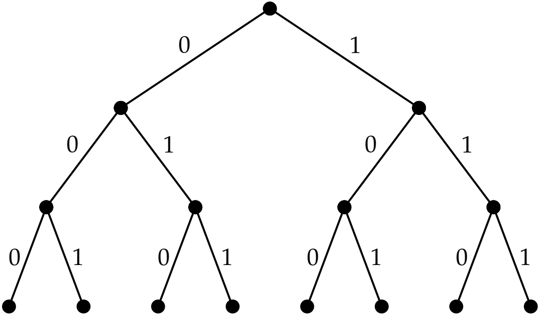

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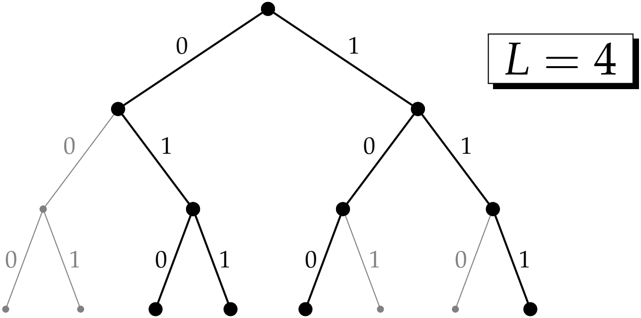

When the number of paths grows beyond a prescribed threshold L, discard the worst (least probable) paths, and keep only the L best paths.

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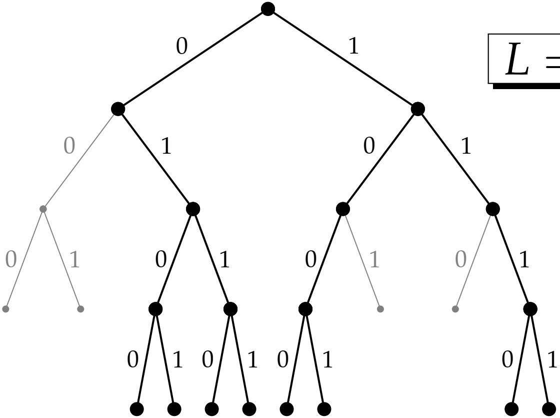

$$\boxed{L = 4}$$

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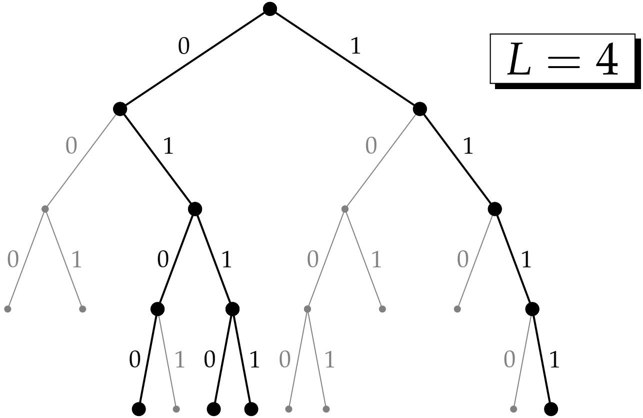

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$\boxed{\text{At the end, select the single most likely path.}}$

List-decoding: complexity issues

The idea of branching while decoding is not new. In fact a very similar idea was applied for Reed-Muller codes.

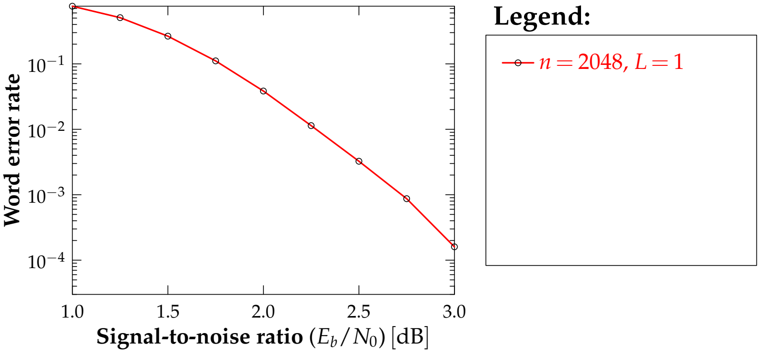
I. Dumer, K. Shabunov, Soft-decision decoding of Reed-Muller codes: recursive lists, *IEEE Trans. on Information Theory*, **52**, pp. 1260–1266, 2006.

Our contribution

- We consider list decoding of polar codes.
- However, in a naive implementation, the time would be $O(L \cdot n^2)$.
- We show that this can be done in $O(L \cdot n \log n)$ time and $O(L \cdot n)$ space.

We will return to the complexity issue later. For now, let's see how decoding performance is affected.

Approaching ML performance



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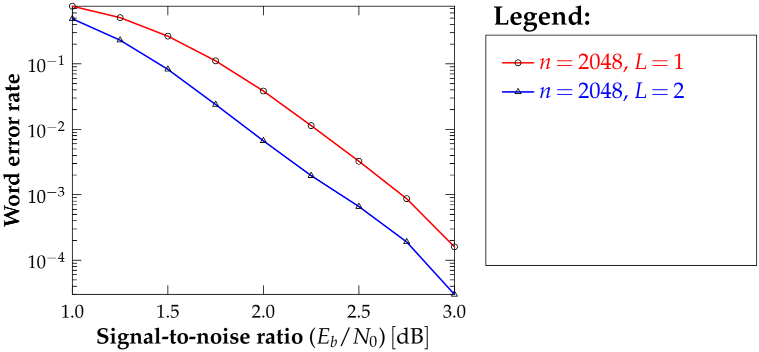

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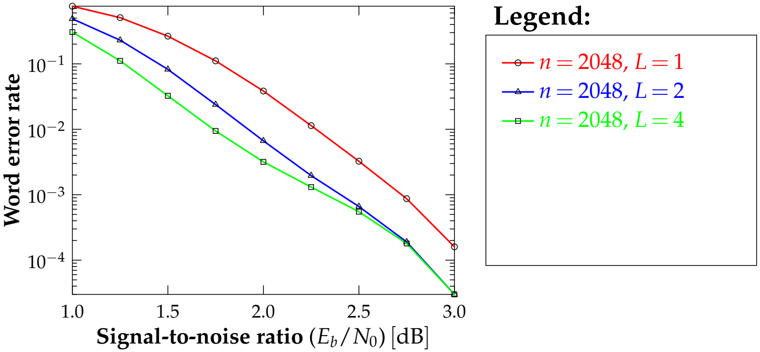

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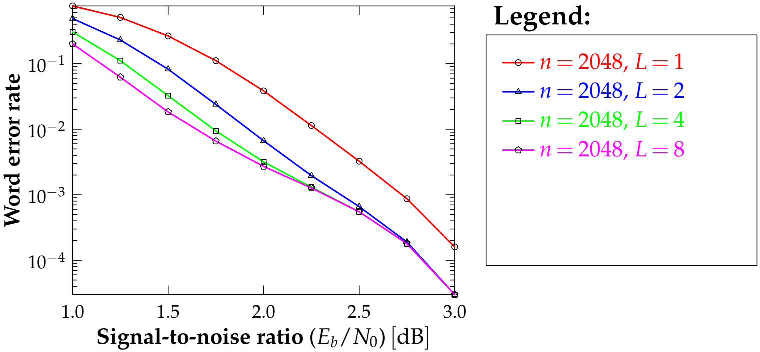

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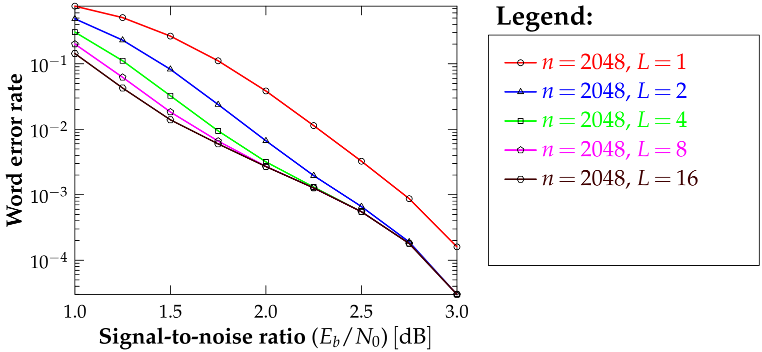

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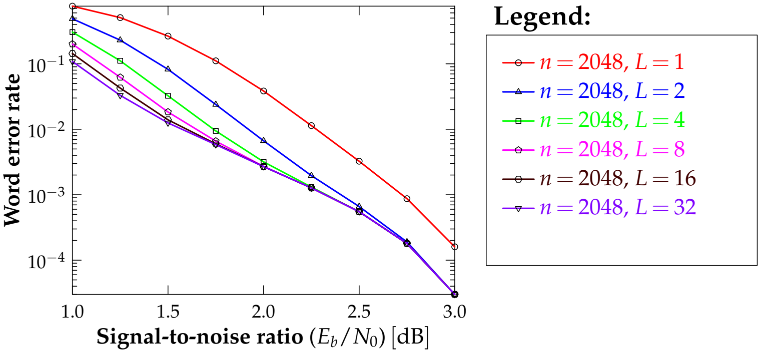

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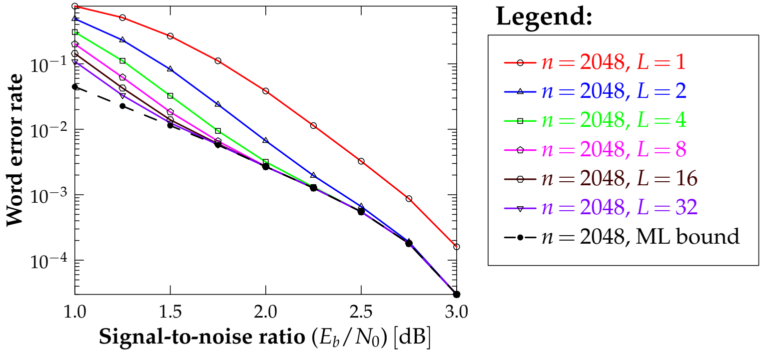

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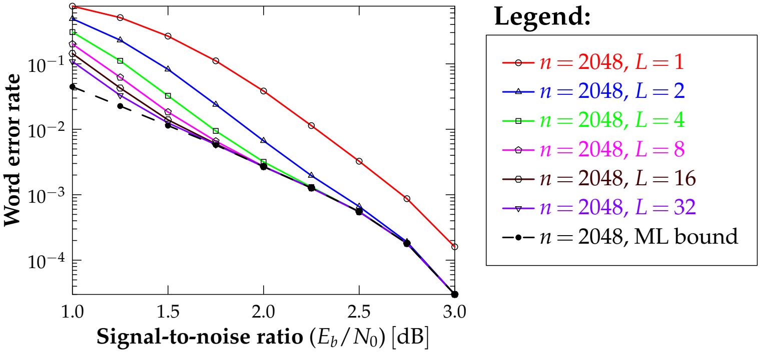

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- Good: our decoder is essentially optimal.
- Bad: Still not competitive with LDPC…

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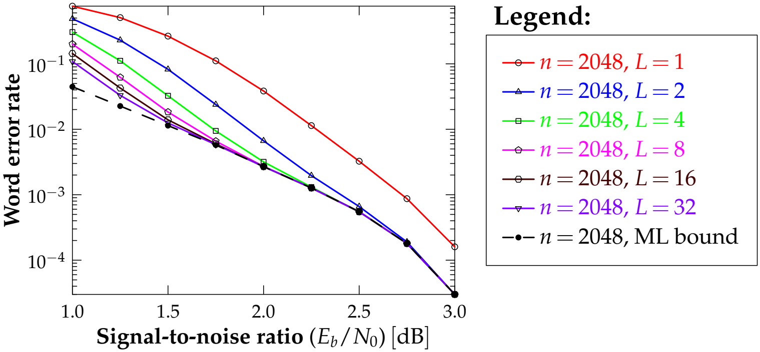

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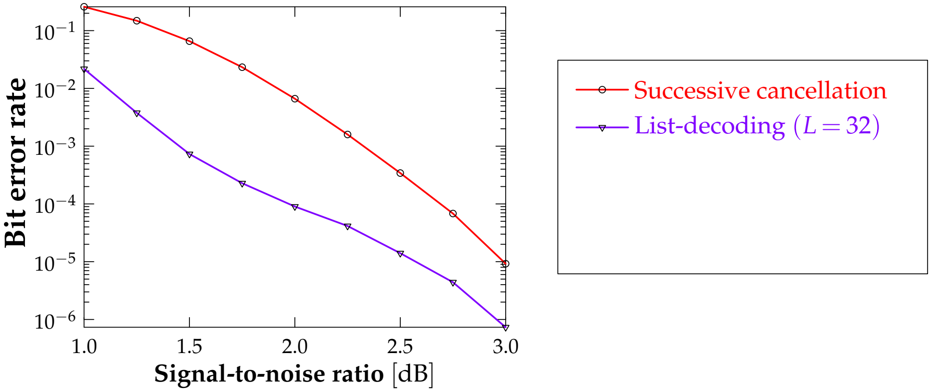
- Good: our decoder is essentially optimal.
- Bad: Still not competitive with LDPC…
- Conclusions: Must somehow "fix" the polar code.

A simple concatenation scheme

- Recall that the last step of decoding was "pick the most likely codeword from the list".
- An error: the transmitted codeword is not the most likely codeword in the list.
- However, very often, the transmitted codeword is still a member of the list.
- We need a "genie" to single-out the transmitted codeword.
- Idea: Let there be $k + r$ unfrozen bits. Of these,
  - Use the first k bits to encode information.
  - Use the last r unfrozen bits to encode the CRC value of the first k bits.
  - Pick the most probable codeword on the list with correct CRC.

Approaching LDPC performance

Simulation results for a polar code of length $n = 2048$ and rate $R = 0.5$, optimized for a BPSK-AWGN channel with $E_b/N_0 = 2.0\,$dB.



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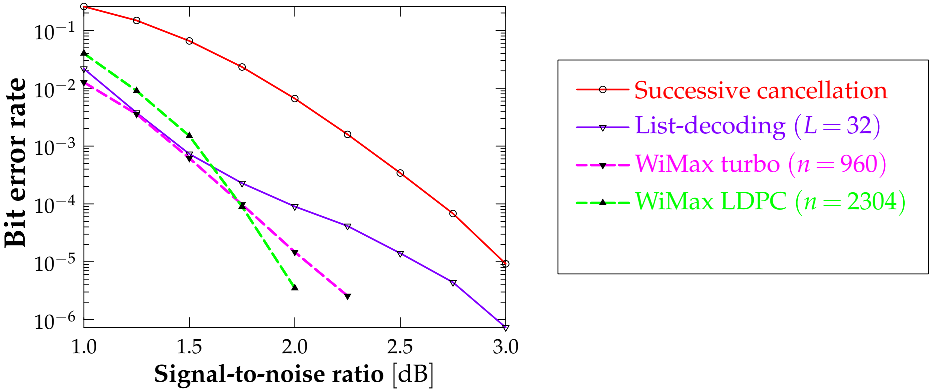

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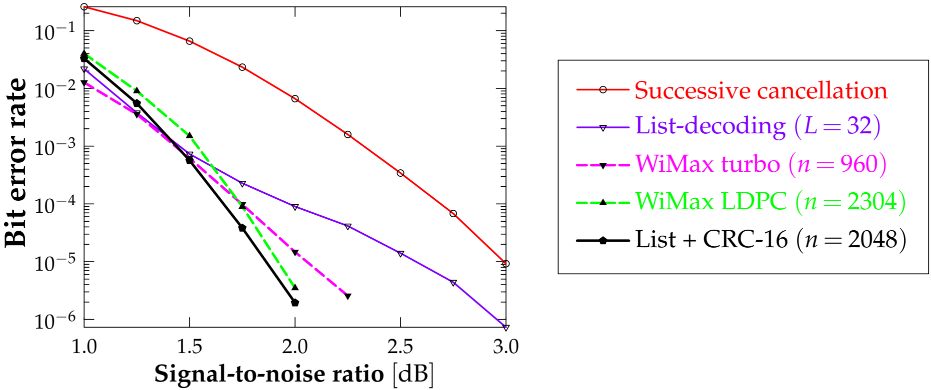


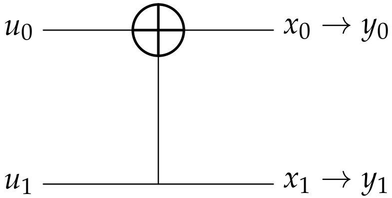
Polar codes (+CRC) under list decoding are competitive with the best LDPC codes at lengths as short as $n = 2048$.

Quadratic complexity of list decoding

Naive implementation recap

- In a naive implementation, the decoding paths are independent. They don't share information.
- Each decoding path has a set of variables associated with it. For example, at stage i, each decoding path must remember the values of the bits $\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_{i-1}$.
- It turns out (as we shall see) that each decoding path has $\Theta(n)$ memory associated with it.
- When a path is split in two, one decoding path is left with the original variables while the other must be handed a copy of them.
- Each copy operation takes $O(n)$ time.
- Thus, the overall time complexity is $O(L \cdot n^2)$.

A closer look at successive cancellation



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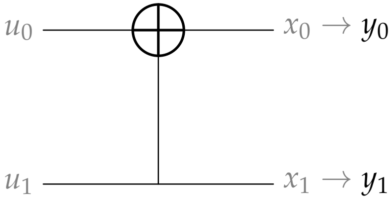

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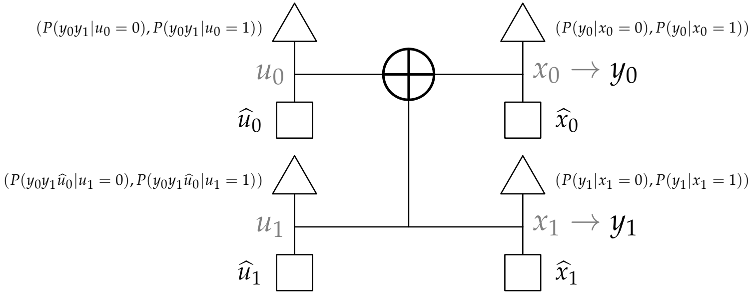

probability pair variable

boolean variable (bit)

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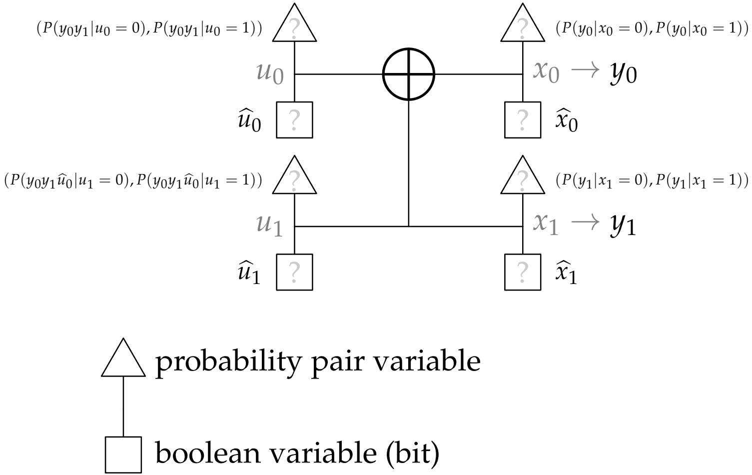

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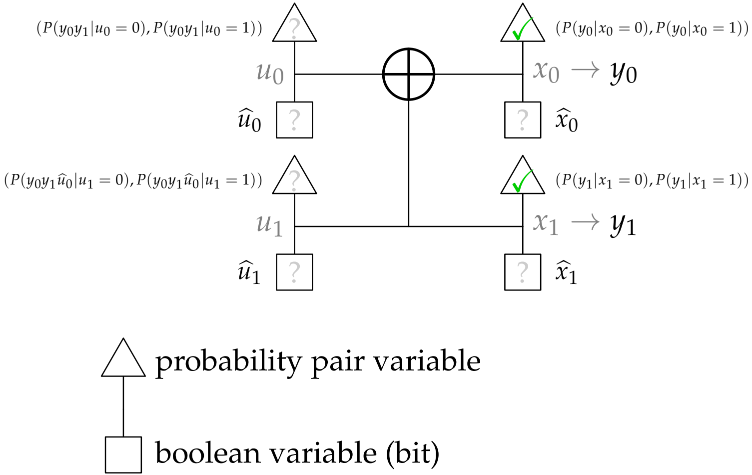

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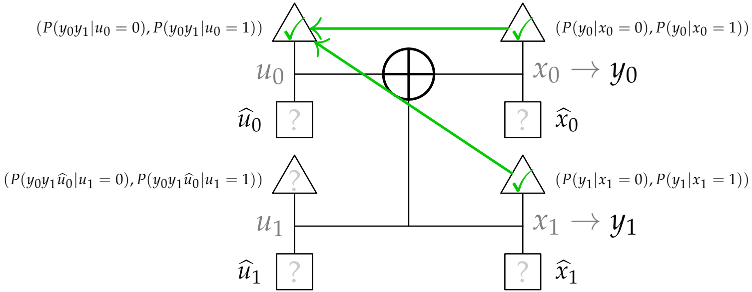

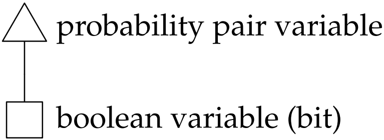

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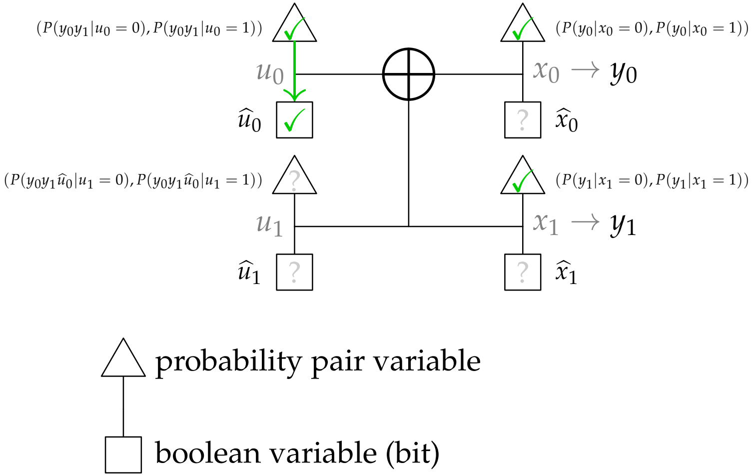

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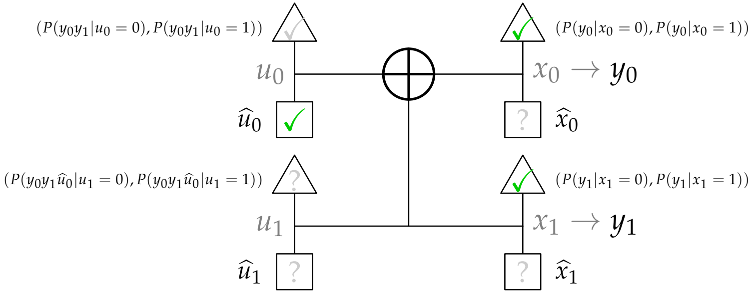

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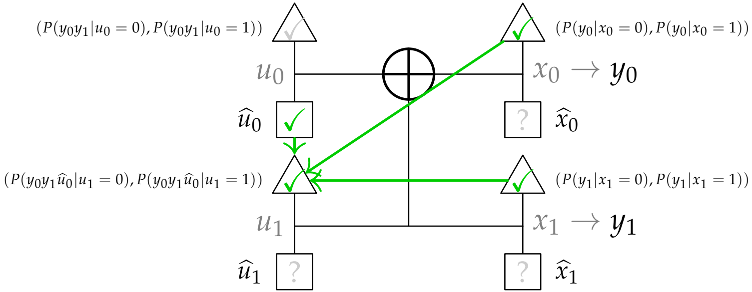

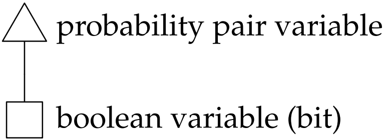

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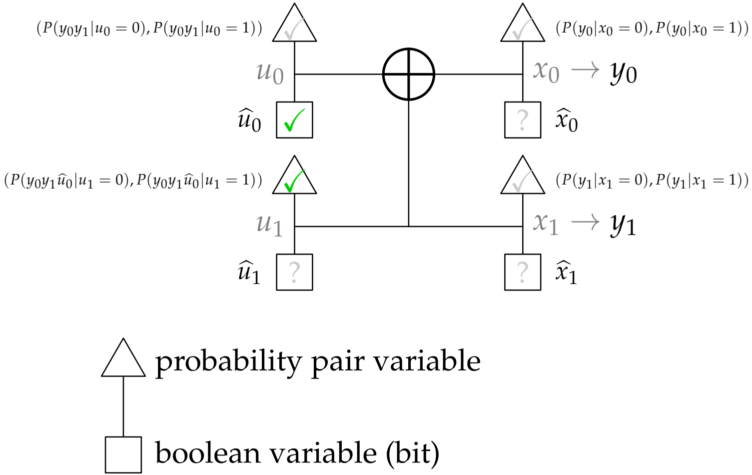

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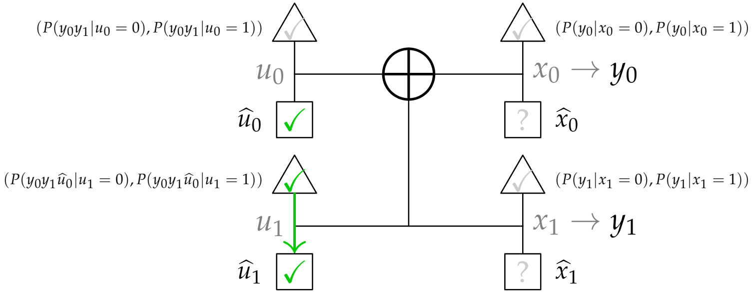

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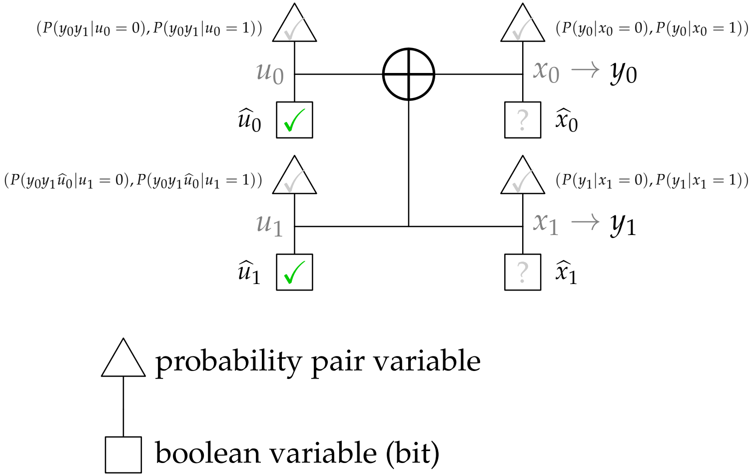

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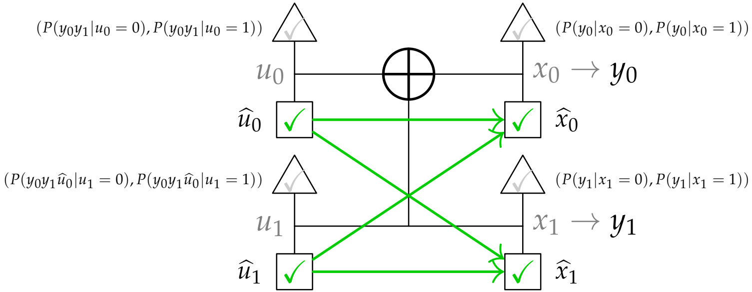

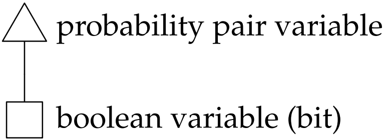

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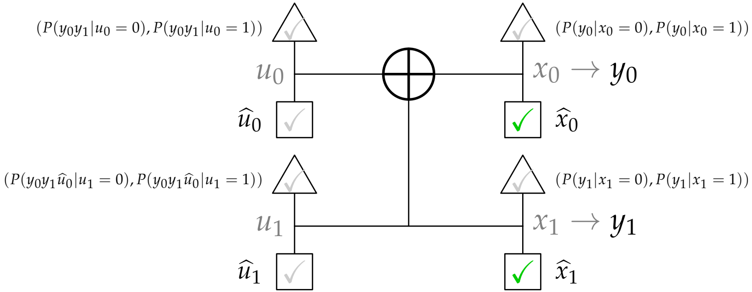

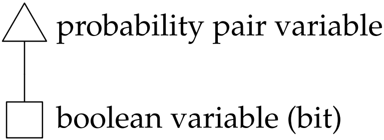

A larger example

Key point

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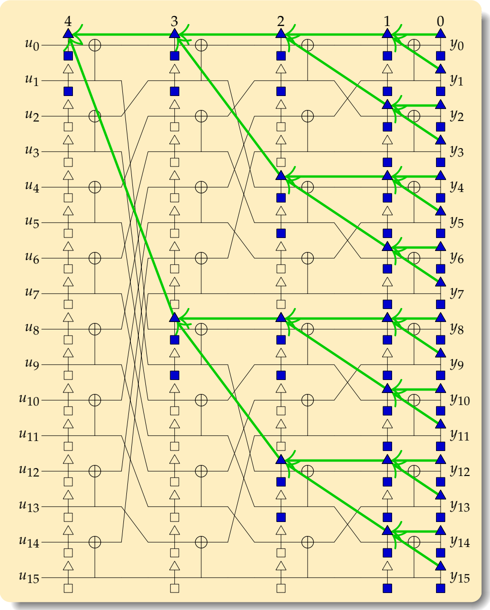

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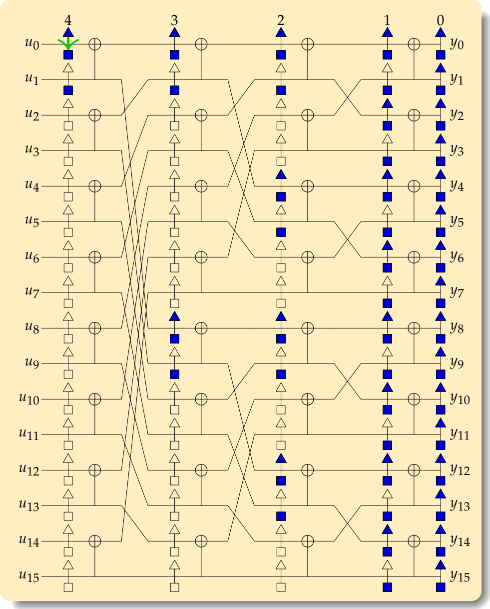

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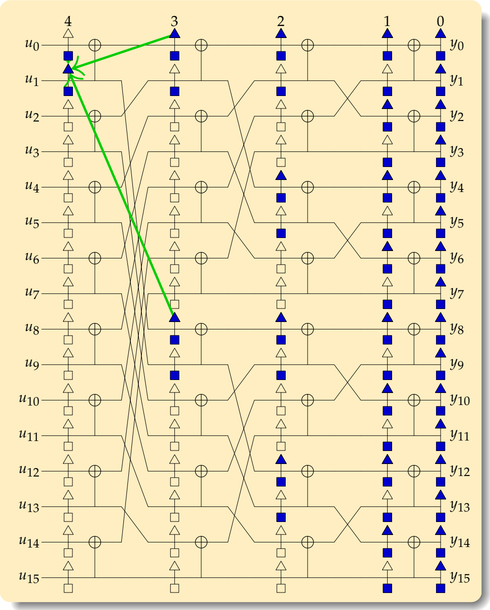

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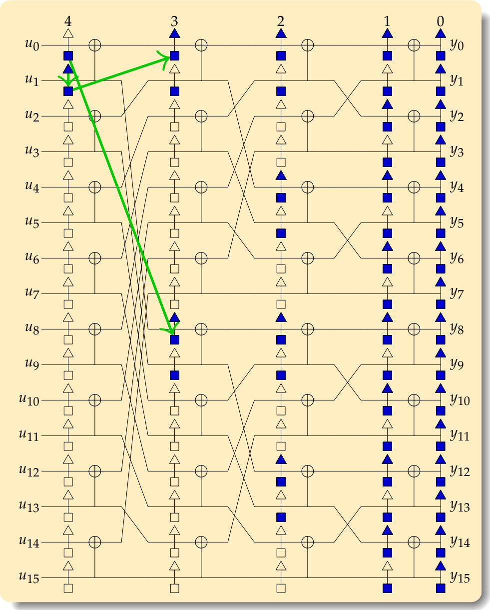

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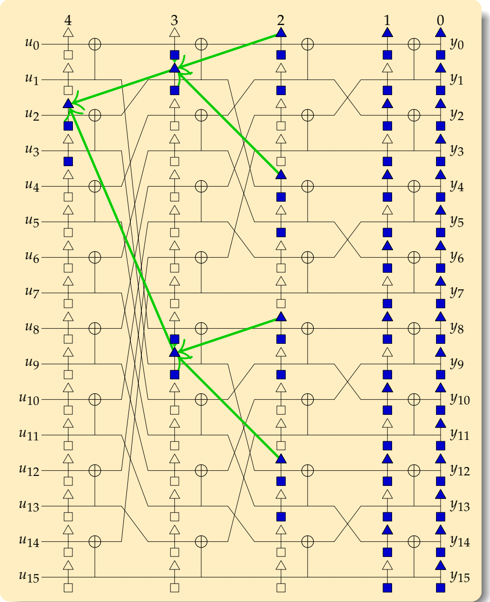

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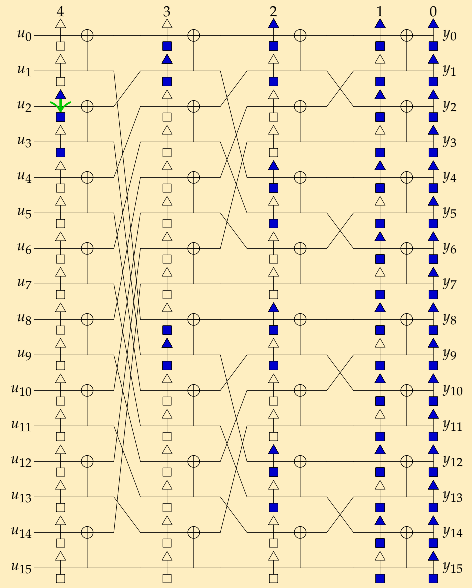

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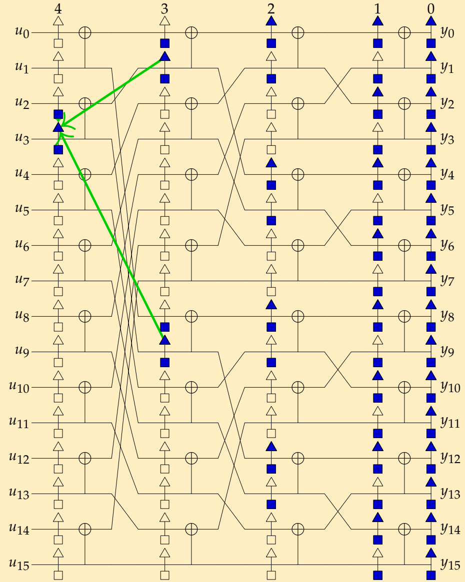

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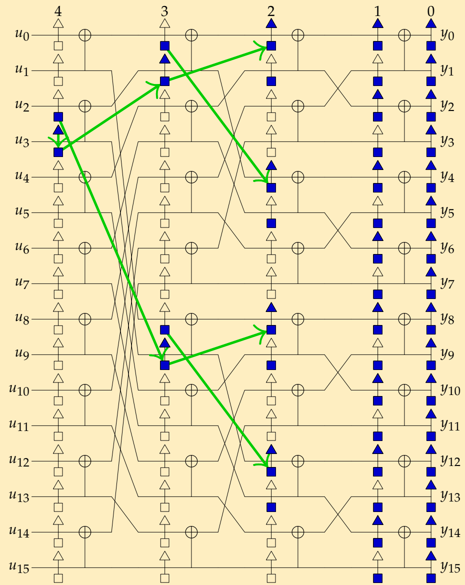

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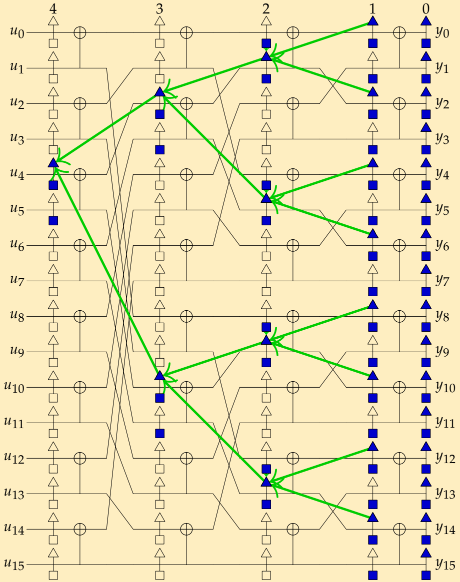

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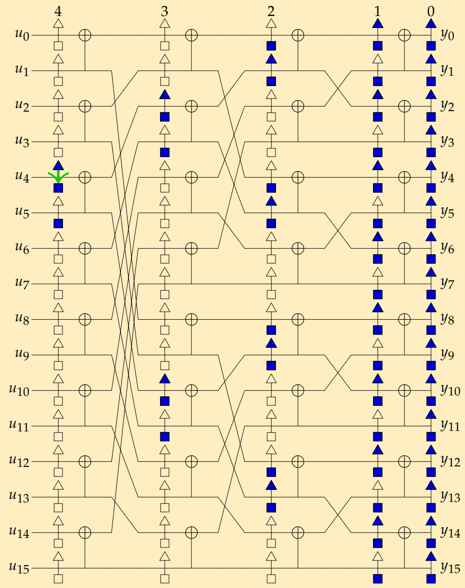

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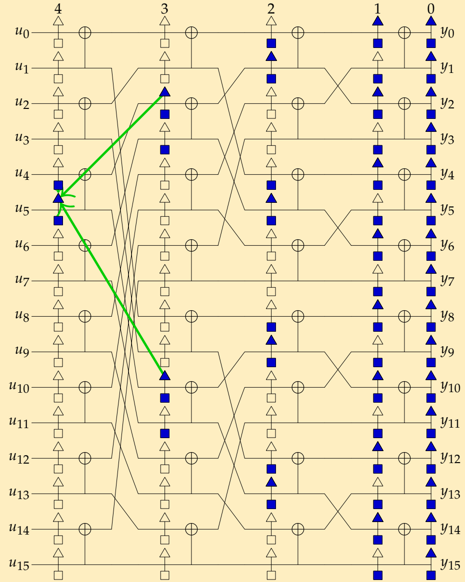

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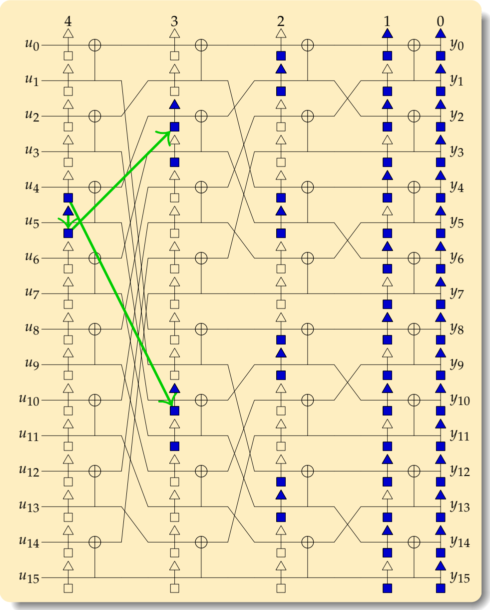

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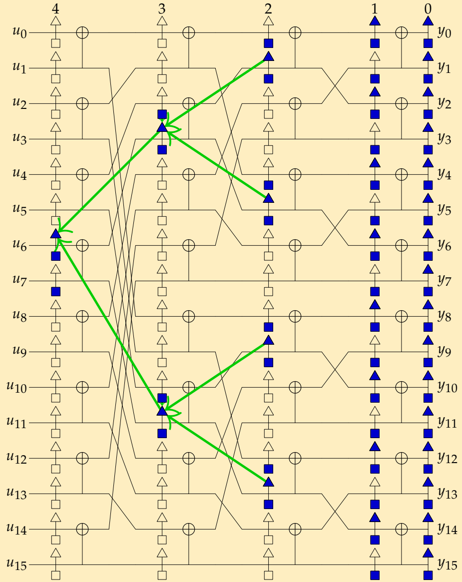

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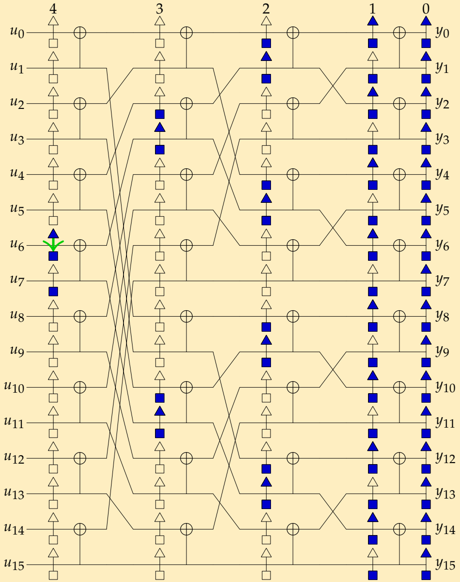

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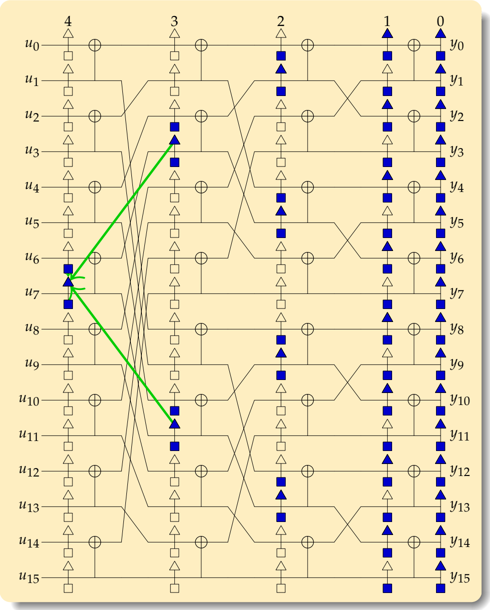

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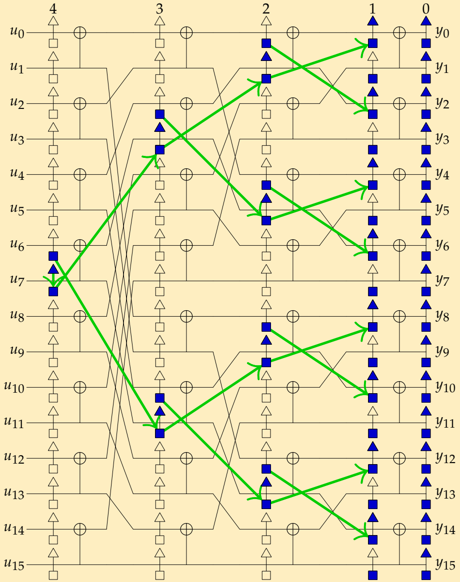

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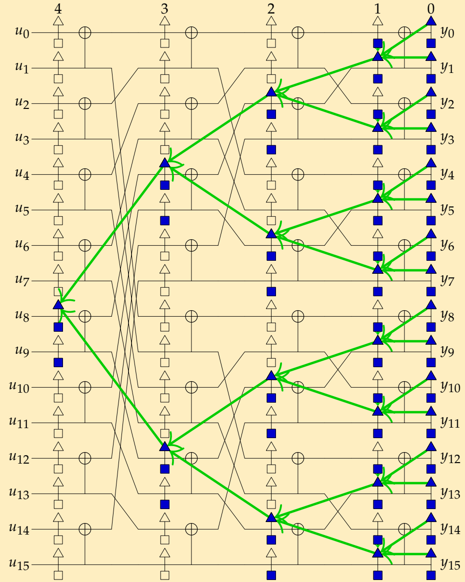

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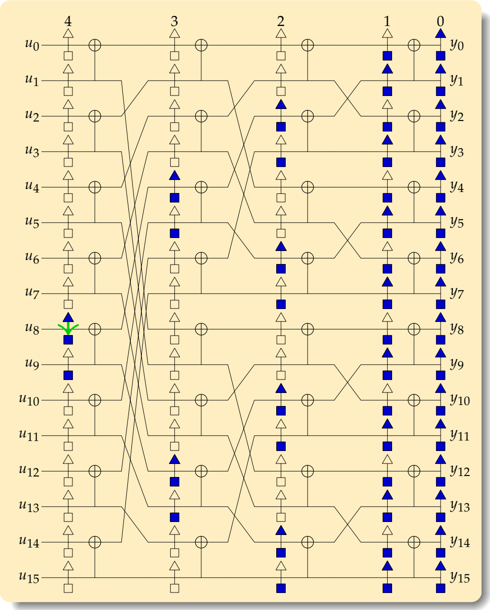

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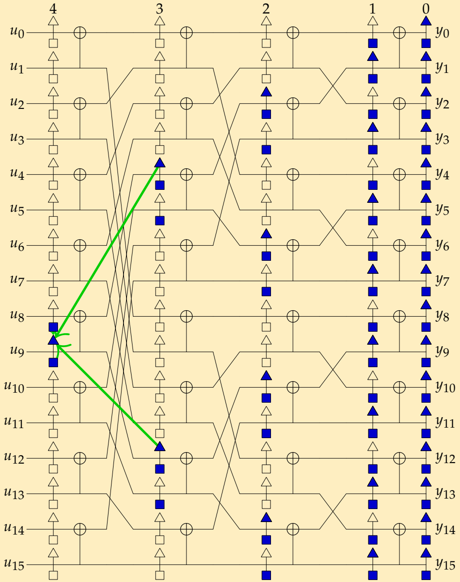

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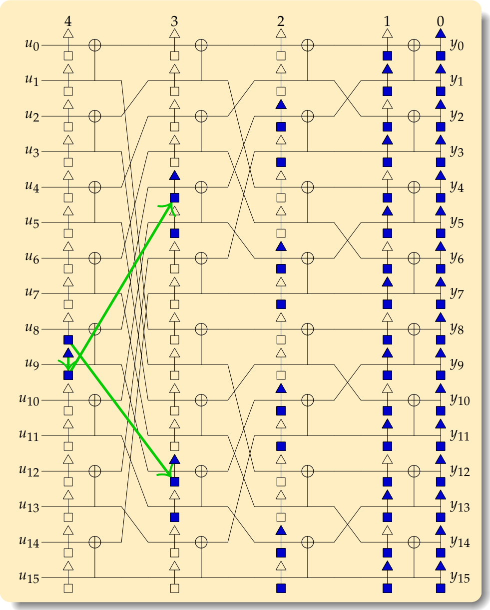

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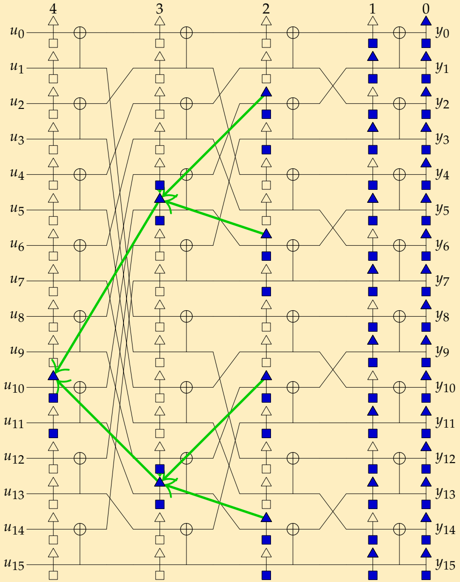

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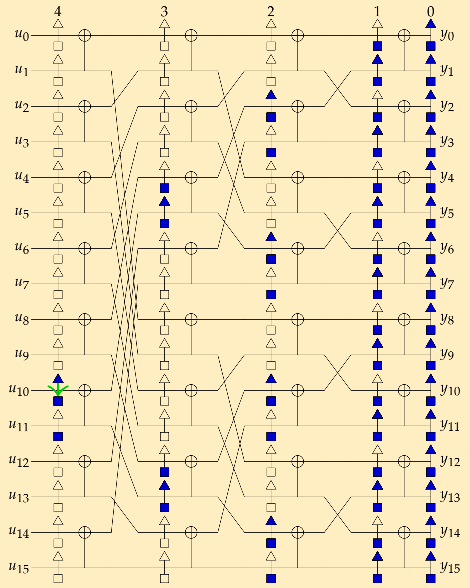

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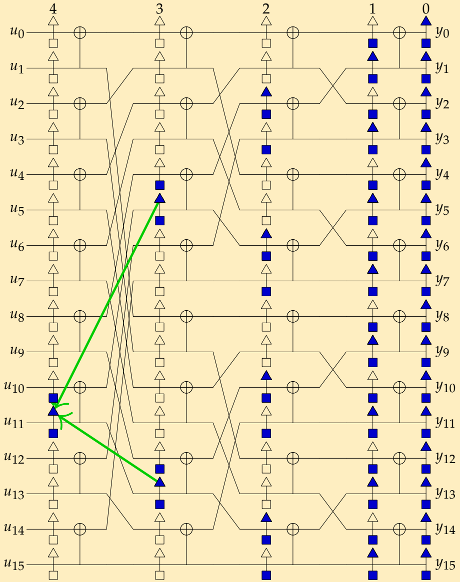

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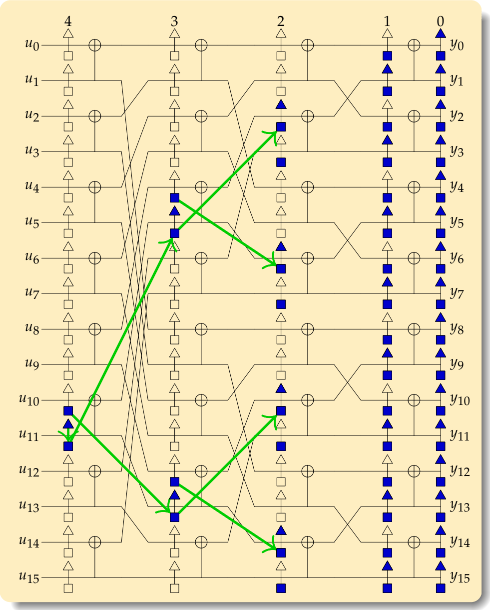

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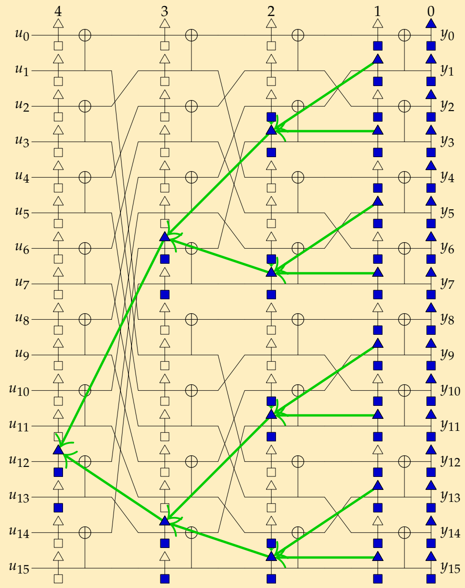

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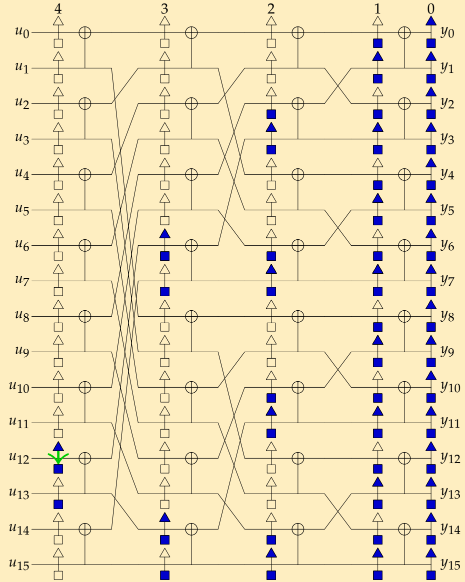

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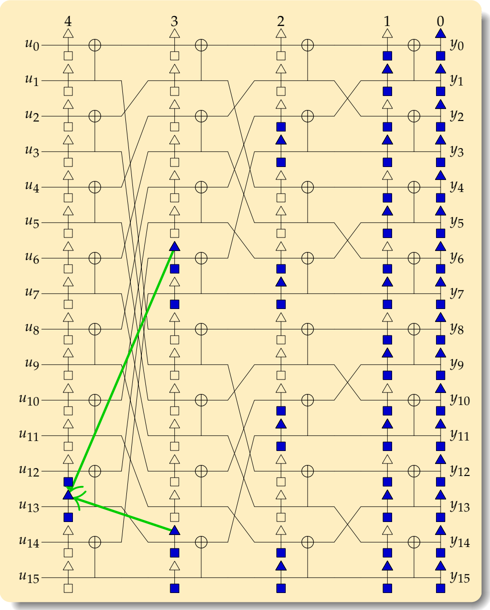

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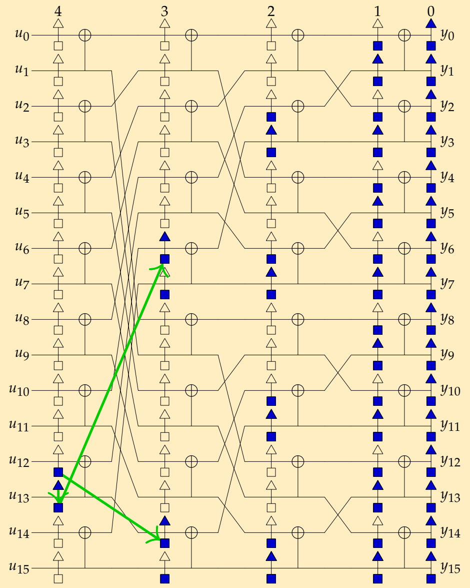

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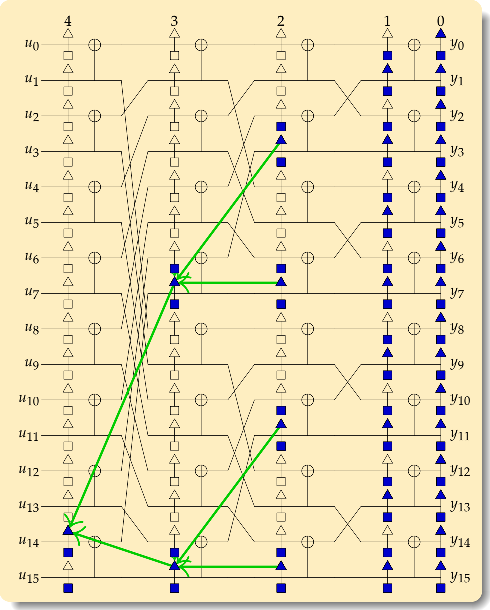

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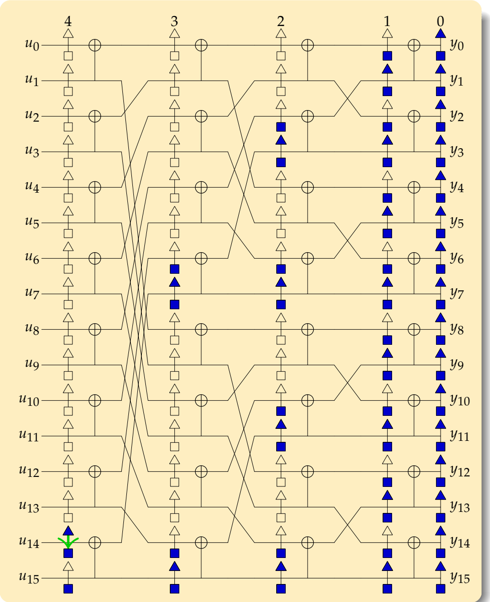

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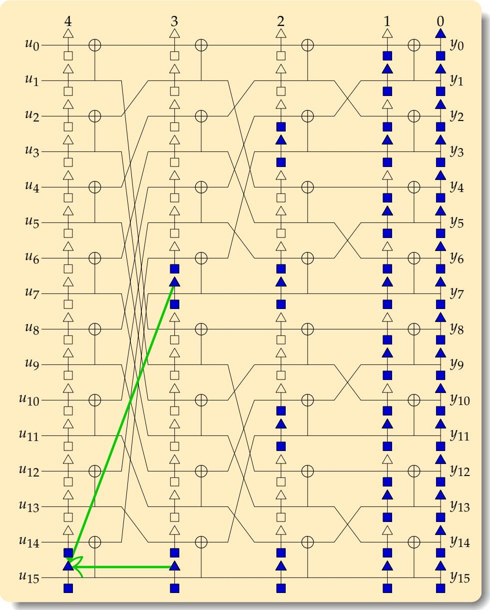

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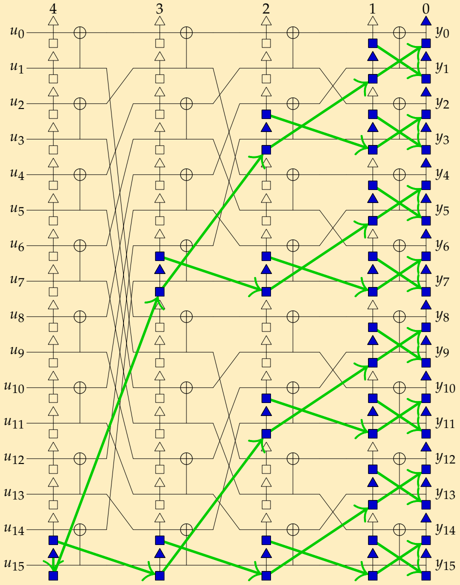

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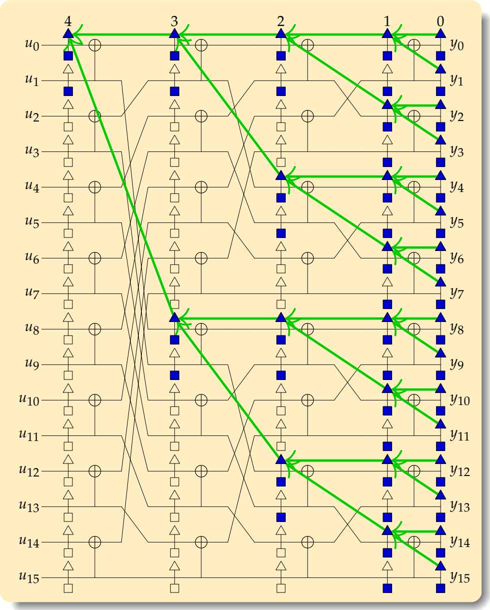

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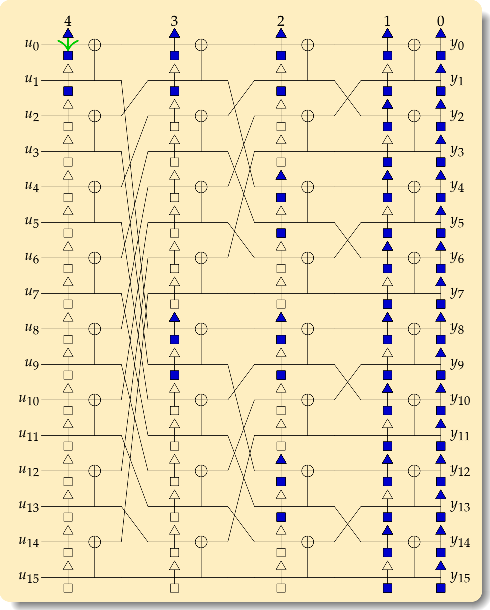

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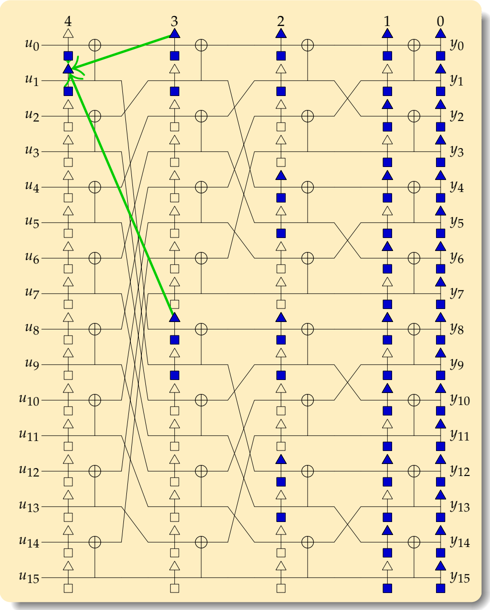

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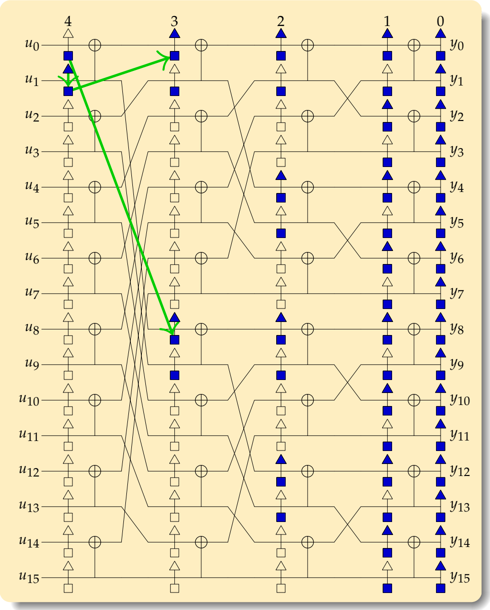

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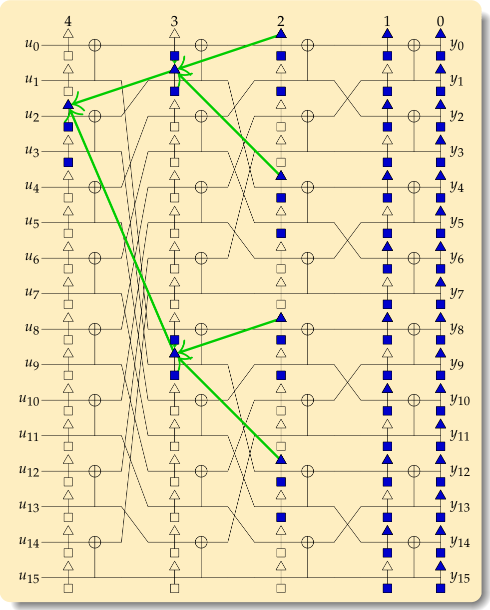

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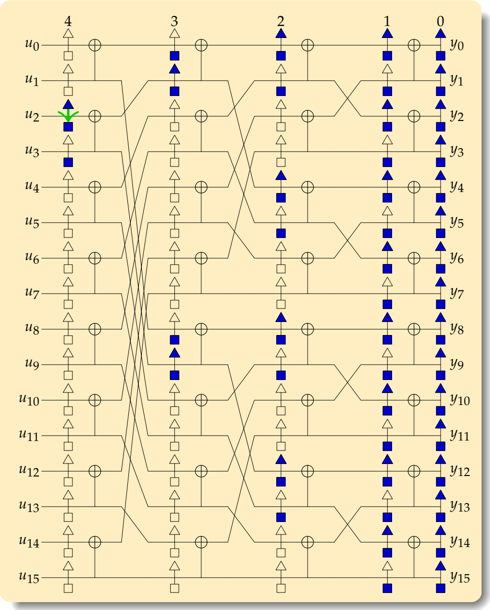

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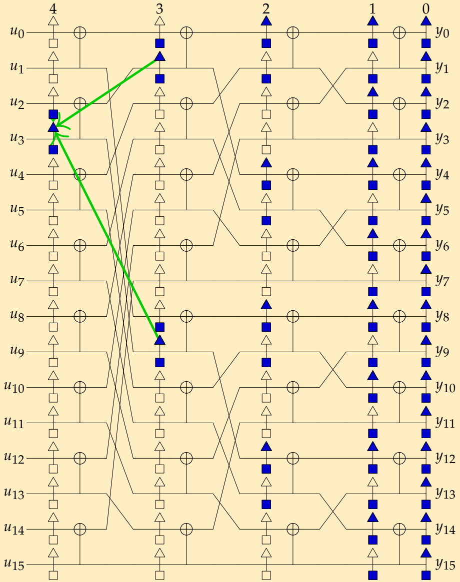

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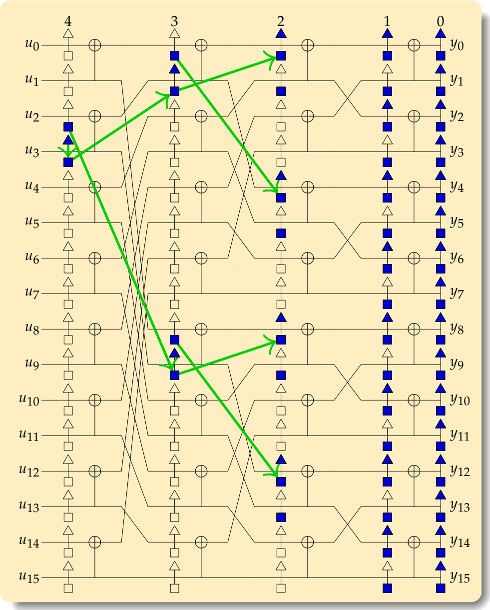

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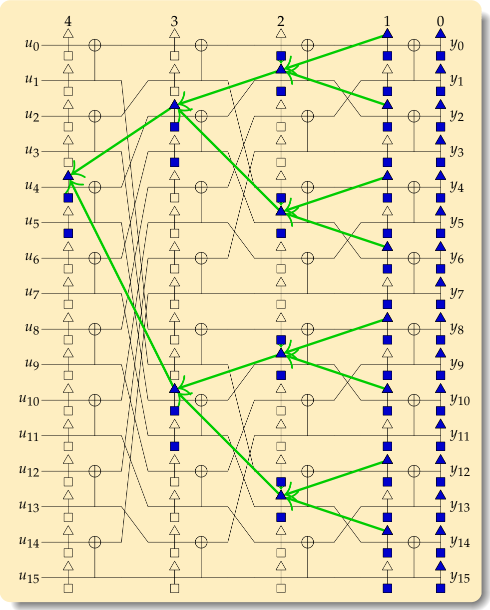

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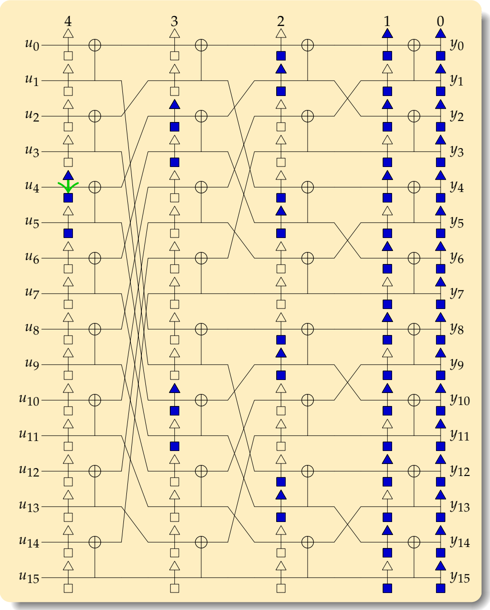

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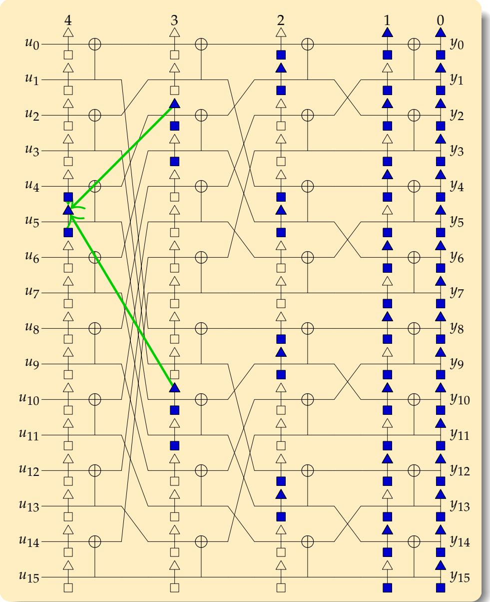

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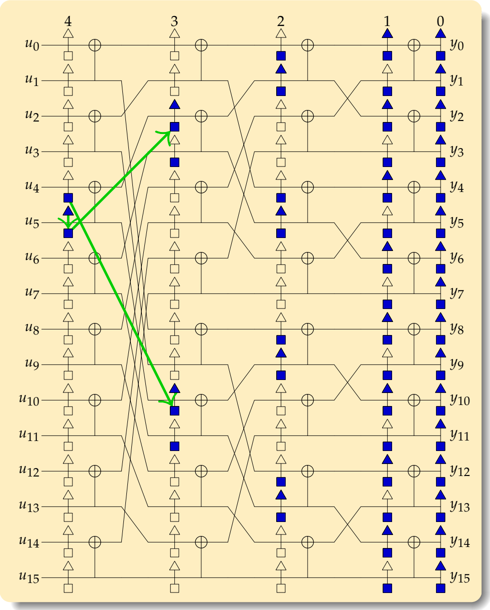

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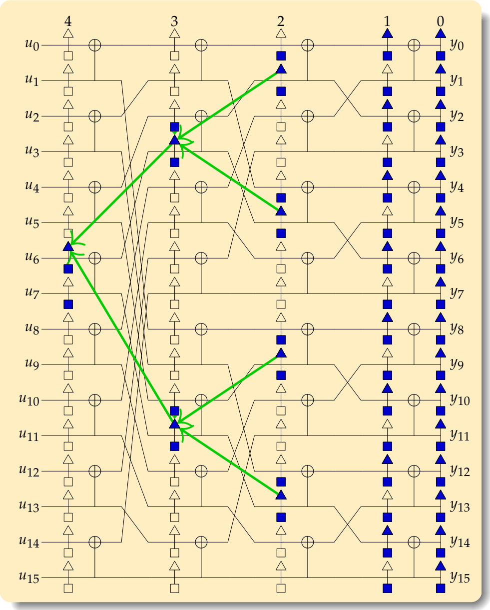

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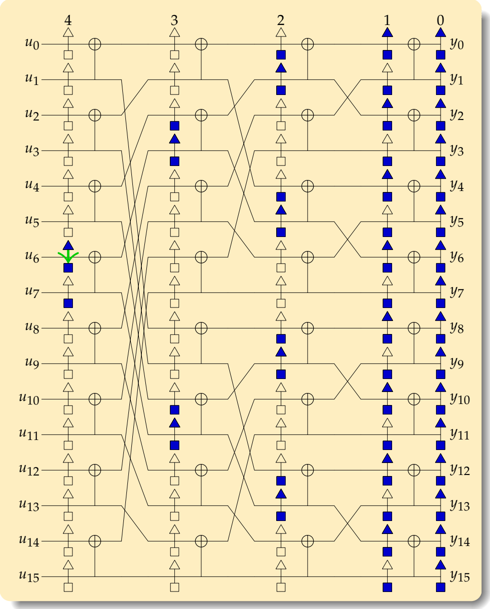

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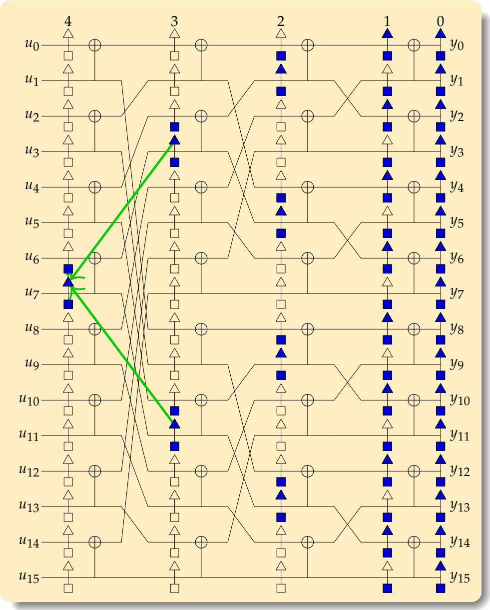

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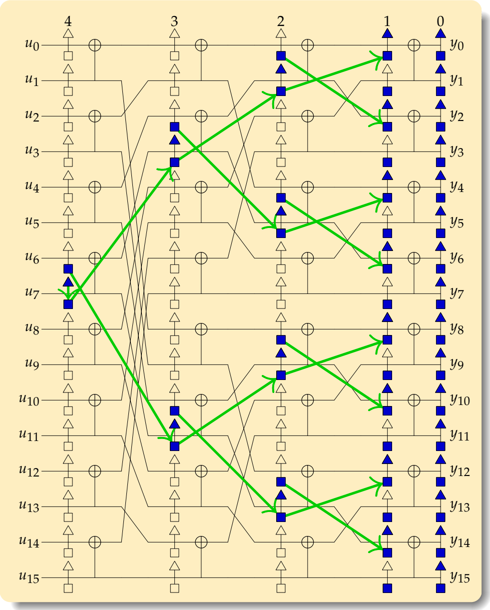

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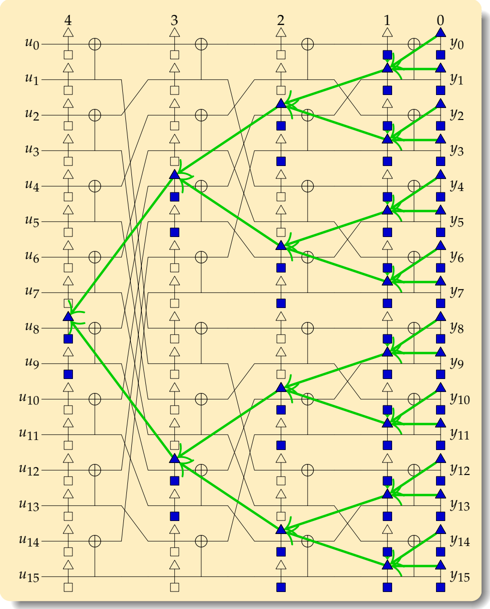

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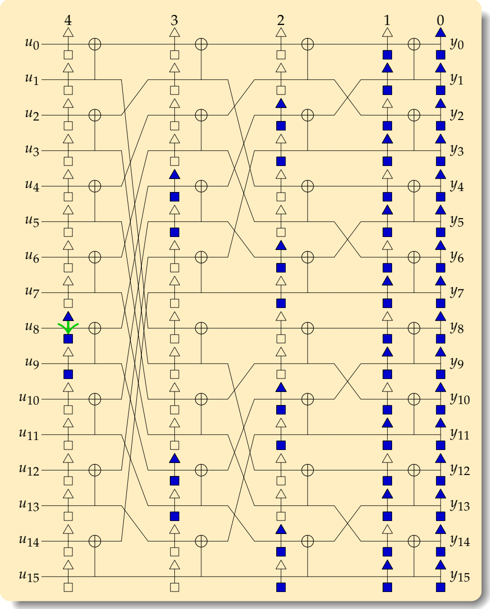

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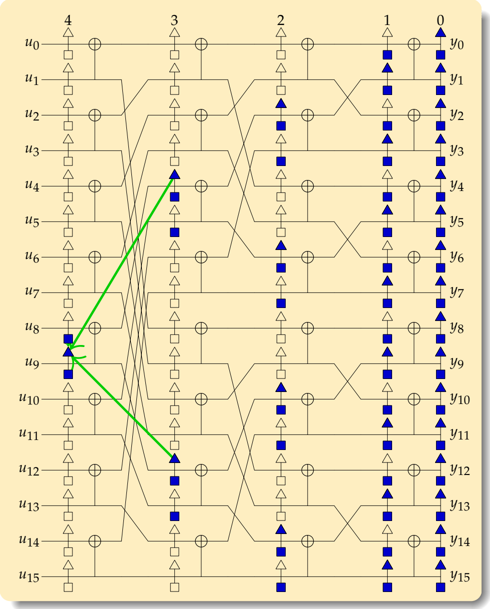

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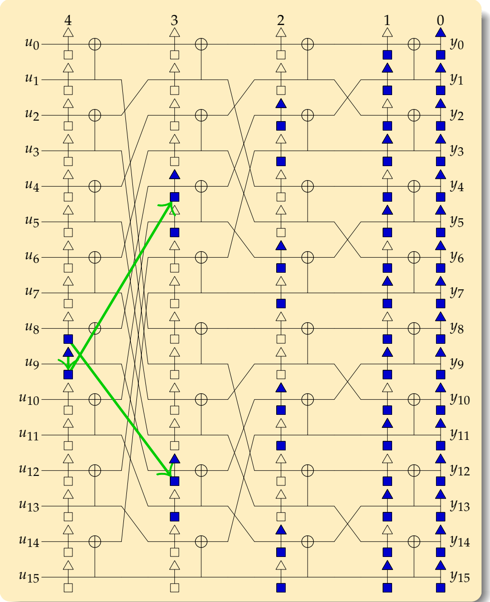

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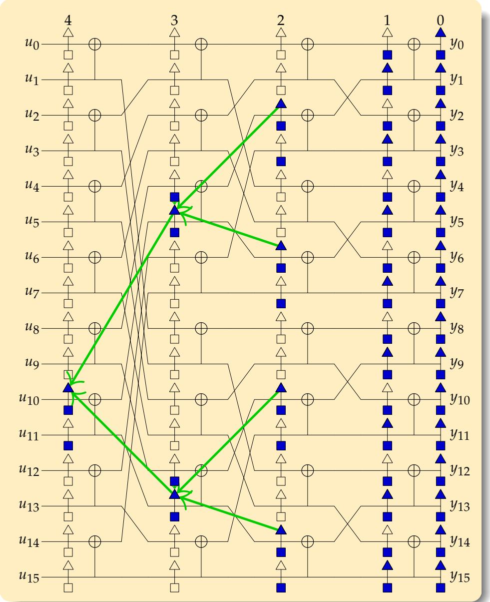

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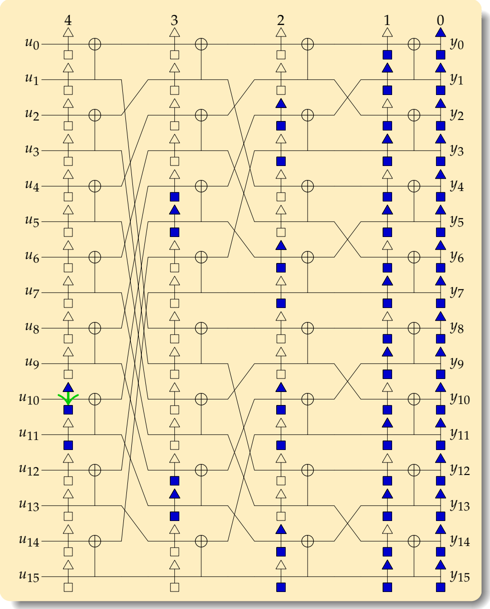

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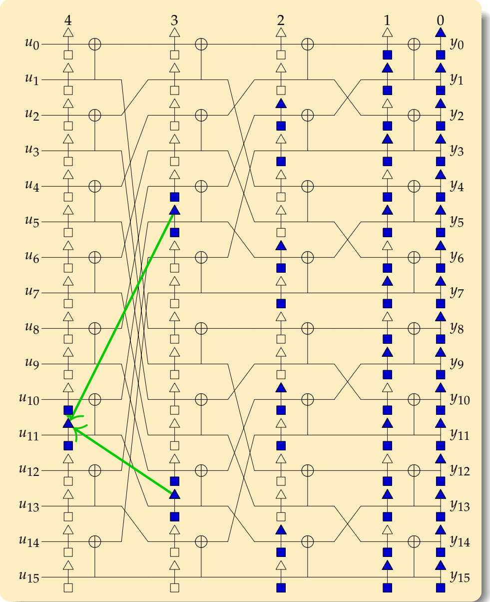

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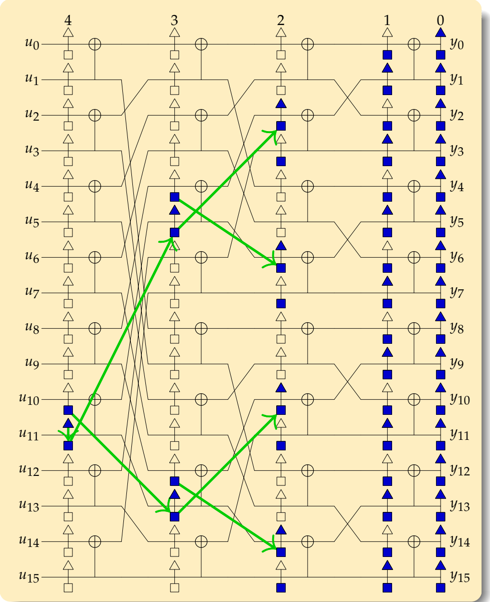

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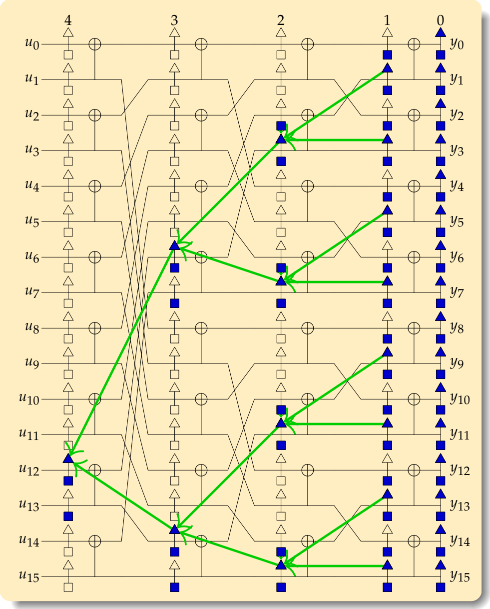

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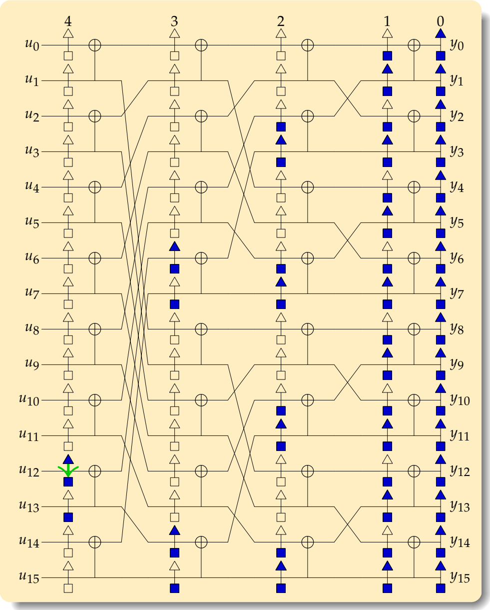

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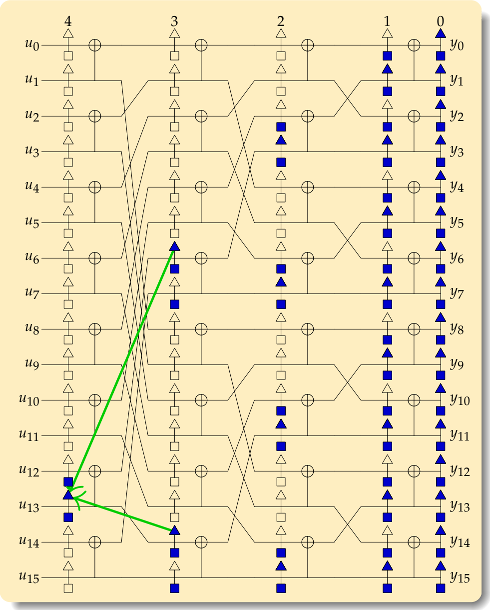

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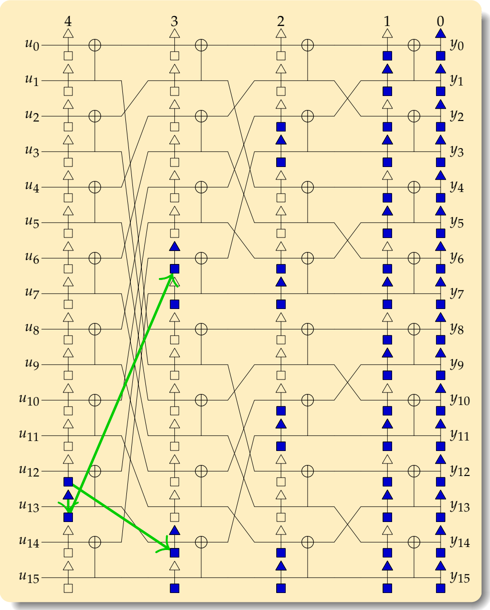

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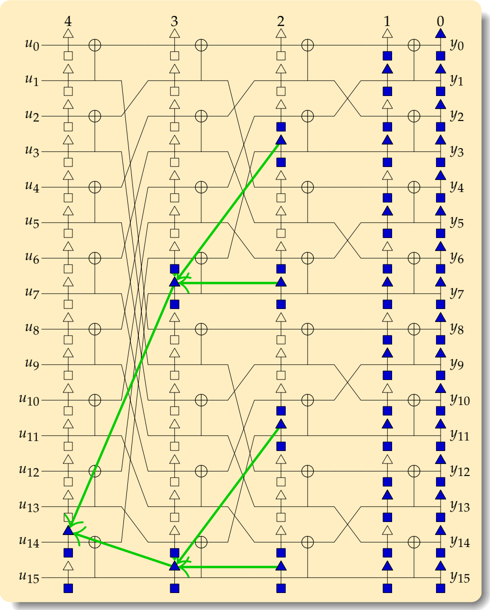

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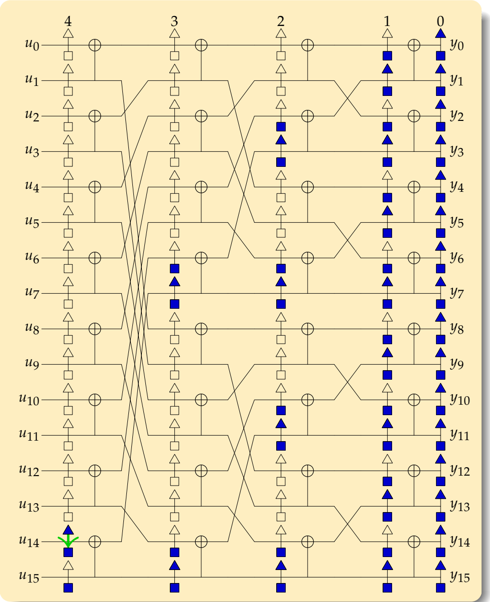

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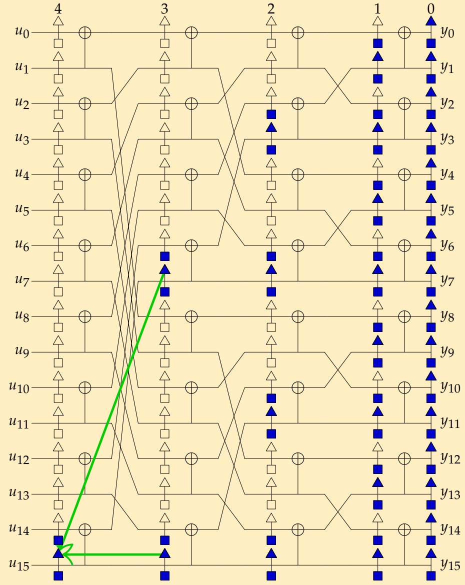

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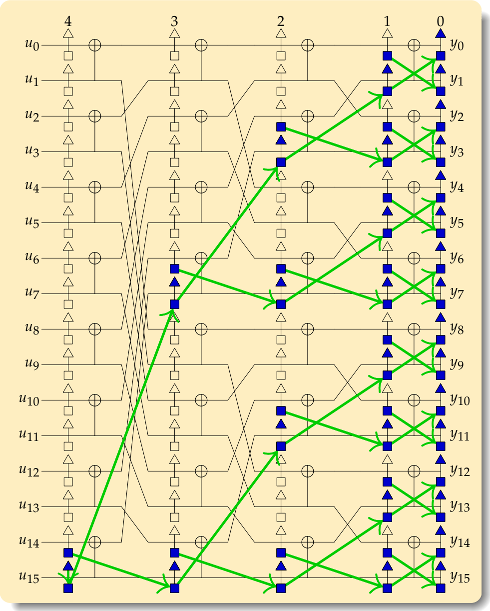


Application to list decoding

- In a naive implementation, at each split we make a copy of the variables.
- We can do better:
  - At each split, flag the corresponding variables as belonging to both paths.
  - Give each path a unique variable (make a copy) only before that variable will be written to.
  - If a path is killed, deflag its corresponding variables.
- Thus, instead of wasting a lot of time on copy operations at each stage, we typically perform only a small number of copy operations.

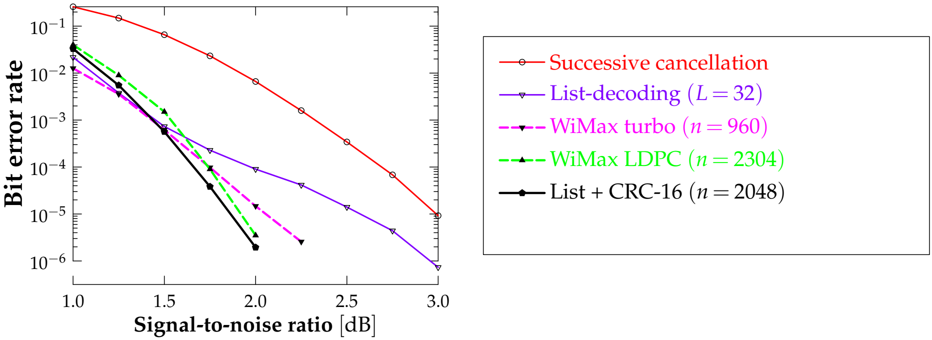
This was a mile high view, there are many details to be filled (book-keeping, data structures), but the end result is a running time of $O(L \cdot n \log n)$ with $O(L \cdot n)$ memory requirements.

Very recent results

Gross and Sarkis (MacGill University) have recently attained the following results.

- Full independent verification of our simulation data.
- Further improvement of performance using systematic polar codes.

E. Arıkan, Systematic polar codes, *IEEE Comm. Letters*, accepted for publication.



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