Instabilities and self-pulsation in a ring cavity with a photorefractive wave mixer

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We find instabilities and self-pulsation in an optical ring cavity with a photorefractive two-wave mixer. The mean-field limit is used to obtain a single criterion for instabilities that originate from a Hopf bifurcation.

Optical bistability and instabilities[1] in photorefractive wave mixing have been reported recently.[2–4] Some of these studies were based on competition between several coupled oscillators (or sets of gratings).[3] Bistability and self-pulsation in a photorefractive ring cavity (with two crystals in the cavity) were also demonstrated.[4]

We have shown[5] that photorefractive two-beam coupling in a ring cavity exhibits absorptivelike and dispersivelike bistabilities. With the application of a dc electric field and nondegenerate mixing, the behavior of the system resembles the Stark effect with gain splitting and broadening.[6,7] In this Letter we show that this configuration also displays optical instabilities and self-pulsation in the single-mode, mean-field limit.[7,8] We also obtain a single criterion for such instabilities.

The dynamics of two-beam coupling in a photorefractive material is described by the time-dependent Maxwell equations for the propagating waves and a material equation that governs the coupling process through the dynamical formation of the space-charge field and the gratings. This process is described by the band-transport model of Kukhtarev *et al.*[9] In the quasi-cw (steady) approximation[10] it is assumed that the density of the mean carrier number is independent of time. In this case the grating buildup (or erasure) is governed by a single (complex) time constant.

Figure 1 shows the two-beam-coupling configuration considered here. Our treatment allows for a moving grating and the application of an external dc electric field upon the crystal.[11] With the standard slowly varying amplitude and plane-wave approximations, and for negligible absorption, the equations for the time-dependent beam-coupling process are

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial z}\right)A_1 = -\sigma\frac{i\omega}{n_0}GA_4, \tag{1a}$$

$$\left(\frac{\partial}{\partial t} + c'\frac{\partial}{\partial z}\right)A_4^* = +\sigma\frac{i\omega}{n_0}GA_1^*, \tag{1b}$$

$$\frac{\partial G}{\partial t} = \nu_0\left[\frac{-in_1}{2}f(E_0)A_1A_4^* - G(I_0 + i\Delta_0)\right], \tag{1c}$$

where $A_j(z, t)$ are the beams' complex amplitudes, $I_j = |A_j|^2$, $I_0 = I_1 + I_4$, $G(z, t)$ is the grating's complex amplitude, $c' = (c\cos\theta)/n_0$, θ is the angle of the beams with respect to the normal to the crystal face (see Fig. 1), σ gives the direction of the energy transfer according to its sign ($\sigma = \pm 1$) and is dependent on the orientations of the crystal and the two beams, E_0 is the externally applied field, $\tau_0 = t_0/I_0$ is the time constant for $E_0 = 0$, and $t_0 \equiv \nu_0^{-1}$ is the time constant for the grating formation, normalized to intensity units, and is a constant of the material for a given geometry. The reciprocal dependence of the time constant on the total light intensity is valid only for intensities well below saturation. A more exact relation is given by $\tau_0 = t_0 \times I^{-x}$, where[12] $0.6 < x < 0.7$. $\Delta = (\omega_1 - \omega_4)\tau_0$ is the dynamical detuning of the grating. The static detuning is defined by $\Delta_0 \equiv (\omega_1 - \omega_4)t_0$ (=const.). We note that $\omega_1 - \omega_4 \ll \omega_1, \omega_4$, and therefore the phase mismatch is negligible. $f(E_0)$ is a complex function that describes the changes in the grating's complex amplitude due to E_0 and is given by[5] $f(E_0) = [(E_d + E_p)/E_d]\{(E_0 + iE_d)/[E_0 + i(E_d + E_p)]\} \equiv F + iF'$. n_1 is the maximum change of the index of refraction with $E_0 = \Delta = 0$ and is given by $n_1 = -r_{\text{eff}}n_0^3\{(E_dE_p)/[2(E_d + E_p)]\}$ such that its multiplication by $iA_1A_4^*/I_0$ gives the complex grating's amplitude, $G(x, t)$. r_{eff} is the effective electro-optic coefficient, and n_0 is the background index of refraction. E_d and E_p are material parameters[5] for a given geometry.

The boundary condition for the ring configuration of Fig. 1 is

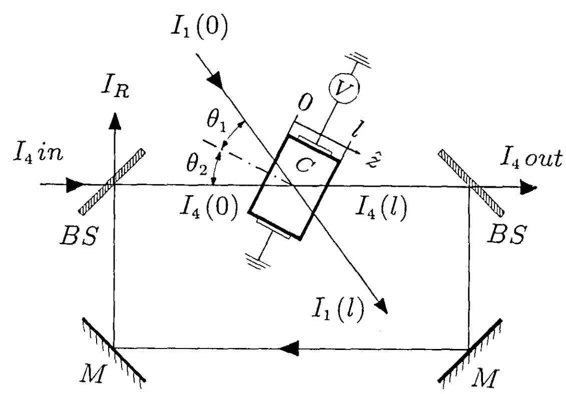

Fig. 1. Photorefractive unidirectional ring cavity with an injected signal. C, crystal; V, voltage source; BS's, beam splitters; M's, mirrors.

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$$A_4^*(0,t) = \sqrt{T}A_{4\text{in}} + R\,e^{-i\delta_0}A_4^*(l, t-\Delta t),$$
$$A_1(0,t) = \text{const.}, \tag{2}$$

where $\Delta t \equiv (\mathcal{L} - l)/c'$, $\mathcal{L}$ is the total effective cavity length, and l is the interaction length of the two beams in the crystal. T and R are the transmissivity and the reflectivity, respectively, of the input and output beam splitters ($T = 1 - R$). The cavity detuning δ_0 is defined by $\delta_0 \equiv (\omega_4 - \omega_c)/(c/\mathcal{L})$, where $\omega_c = q(2\pi c/\mathcal{L})$ is the cavity mode nearest to the frequency of the incident field, ω_4 (q is an integer).

The configuration shown in Fig. 1, without the injected signal and with $\sigma = +1$, has been previously studied in detail and is known as the photorefractive unidirectional ring oscillator.[13] With the inclusion of the injected signal, the case of $\sigma = +1$ is similar to that of a laser with an injected signal. The case of $\sigma = -1$ is similar to that of the optical bistability configuration. The steady-state solution of Eqs. (1) with the boundary conditions of Eq. (2) was given in Ref. 5. It was shown that this system exhibits absorptivelike and dispersivelike optical bistabilities. This solution, however, did not include power-broadening effects that originate from the photorefractive Stark effect.[7]

In the mean-field limit[8] the electromagnetic fields and the grating's amplitude inside the medium are taken to be almost uniform in space. This approximation is valid when the nonlinear interaction of the beams, the transmissivity of the beam splitters, and the cavity detuning are small[8]:

$$\gamma_0 l \to 0, \qquad T \to 0, \qquad \delta_0 \to 0, \tag{3}$$

where γ_0 is the steady-state resonant ($E_0 = \Delta_0 = 0$) coupling coefficient, $\gamma_0 \equiv (\omega n_1/2n_0c')$.

We follow the derivations of Ref. 8 using the mean-field limit. The steady-state quantities and the dynamical perturbations are treated separately and then recombined to give a new set of quantities: $\tilde{A}_1(z', t') \equiv \overline{A_{1\text{ss}}} + a_1(z', t')$, $\tilde{A}_4^*(z', t') \equiv \overline{A_{4\text{ss}}^*} + a_4^*(z', t')$, and $\tilde{G}(z', t') \equiv \overline{G_{\text{ss}}} + \delta G(z', t')$ with a new set of variables, $z' = z$ and $t' = t + \Delta t(z/l)$. These changes convert the boundary condition of Eq. (2) into a periodic one. The new set of equations is then given by

$$\left(\frac{\partial}{\partial t'} + c'\frac{l}{\mathcal{L}}\frac{\partial}{\partial z'}\right)\tilde{A}_1 = \tau_p^{-1}\left\{[A_{1\text{ss}}(l) - A_{1\text{ss}}(0)] - \sigma\frac{i\omega}{n_0}\frac{l}{c'}\tilde{G}\tilde{A}_4\right\}, \tag{4a}$$

$$\left(\frac{\partial}{\partial t'} + c'\frac{l}{\mathcal{L}}\frac{\partial}{\partial z'}\right)\tilde{A}_4^* = \tau_{\text{cav}}^{-1}\left[-(1+i\theta)\tilde{A}_4^* + \frac{A_{4\text{in}}}{\sqrt{T}} + \sigma\frac{i\omega}{n_0}\frac{l}{c'T}\tilde{G}\tilde{A}_1^*\right], \tag{4b}$$

$$\frac{\partial\tilde{G}}{\partial t'} = t_0^{-1}\left[\frac{-in_1}{2}f(E_0)\tilde{A}_1\tilde{A}_4^* - \tilde{G}(\tilde{I}_0 + i\Delta_0)\right], \tag{4c}$$

with $\tau_p \equiv \mathcal{L}/c'$, $\tau_{\text{cav}} \equiv k^{-1} \equiv \mathcal{L}/c'T$, $\theta \equiv \delta_0/T$, and $\tilde{I}_0 = \tilde{I}_1 + \tilde{I}_4$. We note that three different time constants govern the dynamical behavior of the system. Since the time response of the photorefractive effect is relatively slow owing to carrier-transport mechanisms, and since $T \to 0$, we have $\tau_p \ll \tau_{\text{cav}}, t_0$. Then Eq. (4a) can be adiabatically eliminated and $\tilde{A}_1$ can be regarded as a constant in space and time. This result is not surprising; photons that originate from beam 1 traverse the crystal only once and leave the system. Photons of beam 4, however, have a long cavity lifetime (τ_{cav} is reciprocally dependent on T). With the assumption of relations (3) of weak interaction, only these photons experience the nonlinear coupling effects. Thus we can assume that $\tilde{A}_1(0, t) \cong \tilde{A}_1(l, t) \cong$ constant. Because the photorefractive linewidth ν_0 is narrow with respect to the free spectral range of the cavity, we consider the single-mode equations only and therefore omit the spatial derivatives[8] from Eqs. (4). This means that the photorefractive gain mechanism cannot supply enough gain to other cavity modes besides the one with ω_c.

Then we obtain our basic equations of the single-mode, mean-field-limit model for the photorefractive ring cavity:

$$\dot{x} = -k[(1+i\theta)x - y + 2\sigma Cg], \tag{5a}$$

$$\dot{g} = \nu_0[(F + iF')x - (1 + |x|^2 + i\Delta_0)g], \tag{5b}$$

where $x \equiv \tilde{A}_4^*/\tilde{A}_1 = \tilde{A}_{4\text{out}}/[\sqrt{T}A_1(0)]$ is the normalized output amplitude, $y \equiv A_{4\text{in}}/[\sqrt{T}A_1(0)]$ is the normalized input amplitude, $C \equiv \gamma_0 l/2T$ is the bistability parameter, and $g \equiv 2i\tilde{G}/n_1$ is the normalized grating's amplitude. For normalization purposes we take $A_1(0) = 1$ and then $\tilde{I}_0 = 1 + |x|^2$. The normalized input and output intensities are defined by $Y \equiv y^2$ and $X \equiv |x|^2$, respectively, and $\mathcal{G} \equiv |g|^2$.

The steady-state solution of Eqs. (5) is given by

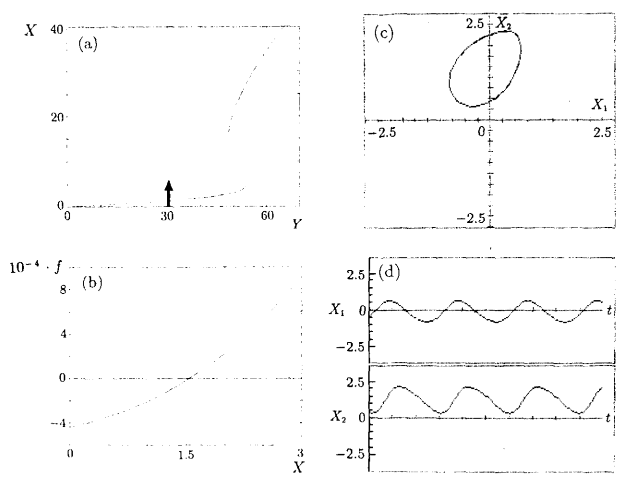

Fig. 2. Instabilities in photorefractive optical bistability. (a) The steady-state-normalized output intensity X versus the normalized input intensity Y for $C = 5$, $\sigma = -1$, $\Delta_0 = 4$, $\theta = 0.5$, $k = \nu_0 = 1$, and $E_0 = 2.5$ kV/cm. (b) The Hopf bifurcation condition f versus the normalized output intensity X. (c) The phase plane portrait of the same system when driven with normalized input intensity $y = 5.5$ ($Y = 30.25$); the variables are $x_1 = \text{Re}\{x\}$ and $x_2 = \text{Im}\{x\}$. (d) x_1 and x_2 versus (t/τ_{cav}); one division is $1(t/\tau_{\text{cav}})$.

$$Y = X\left\{\left[1 - \frac{2\sigma C(1+X)F}{(1+X)^2+\Delta_0{}^2}\right]^2 + \left[\theta + \frac{2\sigma C\Delta_0 F}{(1+X)^2+\Delta_0{}^2}\right]^2 + \frac{4\sigma CF'}{(1+X)^2+\Delta_0{}^2}[\sigma CF' - \Delta_0 - \theta(1+X)]\right\}. \quad (6)$$

With a zero applied field ($F = 1$, $F' = 0$) this result is identical to Eq. (21) of Ref. 5, when power-broadening effects are added [the substitution of $\Delta = \Delta_0/(1 + X)$].

We next study the local stability of the system through linear stability analysis. We assume small perturbations of the forms $\delta x = x - x_s$ and $\delta g = g - g_s$, where x_s and g_s are the steady-state solutions of Eqs. (5). Substituting these perturbations into Eqs. (5) and taking only the linear terms, we have

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta x}^* \\ \dot{\delta g} \\ \dot{\delta g}^* \end{bmatrix} = \begin{bmatrix} -k(1+i\theta) & 0 & 2\sigma Ck & 0 \\ 0 & -k(1-i\theta) & 0 & 2\sigma Ck \\ \nu_0[(F+iF') - x_s{}^*g_s] & -\nu_0 x_s g_s & -\nu_0(1+X_s+i\Delta_0) & 0 \\ -\nu_0 x_s{}^* g_s{}^* & \nu_0[(F-iF') - x_s g_s{}^*] & 0 & -\nu_0(1+X_s-i\Delta_0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta x^* \\ \delta g \\ \delta g^* \end{bmatrix}, \quad (7)$$

where $X_s = |x_s|^2$. The characteristic equation of Eq. (7) is given by

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \quad (8)$$

A transition of the system from a stable state to instability occurs[14] when a negative real root passes the origin (from left to right) or when two complex conjugate roots (with a negative real part) cross the imaginary axis of the complex plane. The first case (saddle-node bifurcation) occurs when a_4 changes its sign and becomes negative. This corresponds to the turning points of the negative-slope branch of the input/output curve of the photorefractive ring cavity. It can be shown that $dY/dX = [a_4/(k^2\nu_0{}^2)]\{1/[(1 + X_s)^2 + \Delta_0{}^2]\}$, which shows that the negative branch, as is usual in optical bistability, is always unstable. The second case corresponds to Hopf bifurcation and is of more interest to us because it can lead to positive-slope instabilities and self-pulsation. A single critical criterion for the Hopf bifurcation can be obtained as done in Ref. 14. By substituting $\lambda = i\mu$, $\mu > 0$ into Eq. (8), the polynomial breaks into real and imaginary parts: $\mu^4 - a_2\mu^2 + a_4 = 0$ and $a_1\mu^2 - a_3 = 0$. The frequency of oscillations on the instability boundaries is found to be $\mu = \sqrt{a_3/a_1}$ (with the requirement that $a_3/a_1 > 0$). Then the condition for Hopf bifurcation is derived:

$$f = a_1a_2a_3 - a_1{}^2a_4 - a_3{}^2 = 0. \quad (9)$$

In fact, this is the next to last principal subdeterminant of the Routh–Hurwitz criterion,[15] which is the first to change its sign in the case of Hopf bifurcation. This also means that $f < 0$ is a sufficient condition for instability.

In Fig. 2 we give a numerical example that demonstrates instabilities in photorefractive optical bistability. In Fig. 2(a) the steady-state solution [Eq. (6)] is drawn, where the Hopf bifurcation condition [Eq. (9)] is applied to obtain the unstable regions. The parameters used are $C = +5$, $\sigma = -1$, $\Delta_0 = 4$, $\theta = 0.5$, $k = \nu_0 = 1$, and $E_0 = 2.5$ kV/cm. The unstable regions are labeled by dashed curves. In Fig. 2(b) we give f, the instability criterion, as a function of X for this example. As can be seen, instabilities appear in the lower branch of the bistability curve, starting from the zero input field. In Figs. 2(c) and 2(d) we drive the same system in the unstable region, near its boundary, with a normalized input amplitude $y = 5.5$ ($Y = 30.25$). The driving intensity is marked by an arrow in Fig. 2(a). Figure 2(c) is a phase portrait of Eqs. (5) for the two variables $x_1 = \text{Re}\{x\}$ and $x_2 = \text{Im}\{x\}$. These two variables are also drawn as a function of time in Fig. 2(d). All the numerical calculations were done with a fourth-order Runge–Kutta algorithm, with a step size of 0.05 (smaller step sizes gave the same attractor), and all the transients were allowed to decay before any data were taken. A more detailed explanation for these instabilities, through the gain-feedback approach,[16] is given in Ref. 7.

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