Motion Aided Sampling and Reconstruction

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Abstract

Motivated by motion compensated filtering in image processing we consider the problem of sampling and reconstruction of signals with sampling rates below the Nyquist rate. It is assumed that temporal dependence can be induced via motion. This way, the data consists of both spatial and temporal sampling and we analyze here the conditions for reconstruction for a number of typical motions. Extensive simulation experiments are also provided which further support the analysis.

1 Introduction

Motivated by the problem of motion compensated filtering in video processing and super resolution problems (see e.g. [1]), we consider the following problem: Let $I_0(x)$ be a W_x- band limited signal, namely

$$\widehat{I}_0(\omega_x) = 0 \quad \text{for} \quad |\omega_x| > \frac{W_x}{2} \tag{1}$$

where $\widehat{I}_0(\omega_x)$ denotes the Fourier transform of $I_0(x)$. Suppose that $I_0(x)$ is sampled at intervals Δx where $\Delta x > \frac{2\pi}{W_x}$. It is well known that, in this case, $I_0(x)$ cannot be reconstructed from the data $\{I_0(\ell\Delta x)\}_{\ell\in\mathbb{Z}}$. Here we assume that the signal can be 'moved' and through this motion a temporal dimension is added to the problem. This motion can be achieved by moving the sampling device rather than the sampled signal. As an example one may consider moving the 1D scanning sensor array in a direction orthogonal to the scanning direction (a patent application for this application is under review).

By applying this motion we generate a *two dimensional* signal given by

$$I(x,t) = I_0(x - f(t)) \tag{2}$$

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where $f(t)$ represents the motion. Initially, we consider general motion. All we require is that $f(0) = 0$. Suppose now the data we generate results from sampling both in the x direction (with the same sampling interval Δx as before) and in the temporal direction with sampling interval Δt. Hence the data we have now is $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$.

The problem we address in this paper is, under *what conditions* can one reconstruct $I_0(x)$ from the data $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$. We provide a general reconstruction formula which is applicable to all cases treated here. As it turns out, this is basically, a direct application of Papoulis generalized sampling expansion (GSE). Our main thrust in the current paper is related to the question of *existence* and not to the associated question of practical reconstruction. Note that, since we assume knowledge of $f(t)$, once $I_0(x)$ is reconstructed, $I(x,t)$ can be reconstructed as well by using (2). The main contribution of this paper is to provide thorough analysis of a number of specific motions. In each case we specify the conditions on the motion parameters required to achieve the desired reconstruction.

Previous work on this problem includes the special case of constant velocity motion [5] and [4]. Also, some results have been reported on constant acceleration motion in [5] . However, in the latter case, the question of whether or not reconstruction is actually possible has not been addressed. Here we give necessary and sufficient conditions for reconstruction with general global motion. Constant velocity and acceleration motions are special cases of our result.

The paper is organized as follows: After presenting, in Section 2, the general case, we proceed to analyze some specific types of motion in Section 3 - namely, constant velocity, constant acceleration and two periodic (oscillatory) motions. Next, in Section 4, we present some simulation results which demonstrate the validity of our analysis and the feasibility of the proposed reconstruction process. Section 5 contains some concluding remarks.

2 The General Case

We establish first the result for the general case as posed in the previous section and then relate the results to some specific types of motions.

Before proceeding, we introduce some notation which will be used in the sequel (This notation is common in Number Theory, see e.g. [6]). For $m, n \in \mathbb{Z}$, $m|n$ means that m *divides* into n, $\gcd(m,n)$ refers to the *greatest common devisor* of m and n. The relationship $m \equiv n \,(\mathrm{mod}\, Q)$, for $Q \in \mathbb{N}$, means that $Q|(m-n)$ and is called *congruence relationship*. Given $Q \in \mathbb{N}$, $\overline{n}$ denotes the set of all integers congruent to n and the set $\{0, 1, ..., Q-1\}$ is called the *set of least (nonnegative) residues modulo* Q and the set $\{\overline{0}, \overline{1}, ..., \overline{Q-1}\}$ is the *least residue system modulo* Q.

Let us also denote

$$N = \left\lceil \frac{W_x \Delta x}{2\pi} \right\rceil \tag{3}$$

$$c = \frac{2\pi}{\Delta x} \tag{4}$$

and the mapping $F : Z \to [0, 1)$ defined by

$$F[n] = \left\lceil \frac{f(n\Delta t)}{\Delta x} \right\rceil - \frac{f(n\Delta t)}{\Delta x} \tag{5}$$

where $\lfloor a \rfloor$ denotes the largest integer smaller than a and $\lceil a \rceil$ denotes the smallest integer larger than a. Then, clearly

$$0 \leq \Delta x F[n] < \Delta x \quad \text{for all} \quad n \in \mathbb{Z} \tag{6}$$

We next introduce:

Definition 1 Denote by $\overline{n_F}$ the set of integers which result in the same value under the mapping F (namely, $F[n] = F[n_1]$ for all $n, n_1 \in \overline{n_F}$) and let $\mathcal{N}_F$ denote the set $\mathcal{N}_F = \{n_m\}_{m=1}^{M}$ such that $\overline{(n_m)_F}$ are all disjoint and $\bigcup\limits_{m=1}^{M} \overline{(n_m)_F} = \mathbb{Z}$. Then, the Resolution Gain Factor (RGF) M_F is the number of elements in $\mathcal{N}_F$.

We note that by the definition of $f(t)$ we can always assume that $0 \in \mathcal{N}_F$ and arbitrarily choose $n_1 = 0$.

Let

$$x_m = \Delta x F[n_m] \text{ for every } n_m \in \mathcal{N}_F \tag{7}$$

We can then state the general result:

Theorem 2 *$I_0(x)$ can be reconstructed from the data $\{I(\ell\Delta x, n\Delta t)\}_{(\ell,n)\in\mathbb{Z}\times\mathbb{Z}}$ if and only if*

$$M_F \geq N \tag{8}$$

The reconstruction formula is:

$$\begin{aligned} I_0(x) &= \sum_{k=-\infty}^{\infty} \sum_{m=1}^{N} I(k\Delta x, n_m\Delta t)\, \varphi_m(x - k\Delta x) \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=1}^{N} I_0(k\Delta x + x_m)\, \varphi_m(x - k\Delta x) \end{aligned} \tag{9}$$

where $x_m = \Delta x F[n_m]$, $n_m \in N_F$,

$$\varphi_m(x) = \frac{1}{c} \int_{-\frac{cN}{2}}^{-\frac{c(N-2)}{2}} \Phi_m(\omega_x, x)\, e^{jx\omega_x} d\omega_x \tag{10}$$

and $\{\Phi_m(\omega_x, x)\}_{m=1}^{N}$ are the solutions of the following set of linear equations

$$\sum_{m=1}^{N} e^{j(\omega_x + rc)x_m} \Phi_m(\omega_x, x) = e^{jrcx} \quad \textit{for} \quad r = 1, ..., N \tag{11}$$

in which x is arbitrary and $\omega_x \in \left(-\frac{cN}{2}, -\frac{c(N-2)}{2}\right)$.

Proof. : First we note that (8) ensures that (11) can indeed be written with distinct x_m's. Furthermore, the matrix of coefficients of the equations in (11) has the form

$$\begin{bmatrix} 1 & e^{j\omega_x x_2} & e^{j\omega_x x_3} & \cdots & e^{j\omega_x x_{N-}} \\ 1 & e^{j(\omega_x+c)x_2} & e^{j(\omega_x+c)x_3} & \cdots & e^{j(\omega_x+c)x_N} \\ 1 & e^{j(\omega_x+2c)x_2} & e^{j(\omega_x+2c)x_3} & \cdots & e^{j(\omega_x+2c)x_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(\omega_x+Nc)x_2} & e^{j(\omega_x+Nc)x_3} & \cdots & e^{j(\omega_x+Nc)x_N} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{jcx_2} & e^{jcx_3} & \cdots & e^{jcx_N} \\ 1 & e^{j2cx_2} & e^{j2cx_3} & \cdots & e^{j2cx_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{jNcx_2} & e^{jNcx_3} & \cdots & e^{jNcx_N} \end{bmatrix} \cdot diag\left\{1, e^{j\omega_x x_2}, ..., e^{j\omega_x x_N}\right\}$$

Since $0 \leq x_m \neq x_r < \Delta x$, from (4) we have $e^{jcx_m} \neq e^{jcx_r}$. Then, recognizing that the first matrix is a Vandermonde matrix, this implies that it is nonsingular and so is the whole coefficient matrix. Hence, (8) ensures the existence of $\{\Phi_m(\omega_x, x)\}_{m=1}^N$.

Once this is established we have converted the problem to a special case of a result due to Papoulis (see e.g. [3] or [2]) and (9) follows. ■

An immediate observation from (9) is that one may not need the whole data set in order to reconstruct the signal. Furthermore, when we look at the data available as samples of $I_0(x)$ we note that in fact we have generated a periodic (or recurrent) sampling pattern with M_F irregularly spaced (in general) samples in each period. As the above theorem states, in order to be able to reconstruct $I_0(x)$ we need to make sure that $f(t)$ and Δt are such that (8) is satisfied. From an implementation point of view, if $M_F > N$, one would, all else being equal, want to choose the subset of N values $\{x_m\}$ which are closest to being uniformly spaced in the interval $[0, \Delta x)$ as this will result in the best conditioned matrix of coefficients in (11).

Remark 3 *The reconstruction functions $\varphi_m(x - k\Delta x)$ in (9) can be viewed as impulse responses of reconstruction filters. This approach has been described in [8] where these same functions have been derived in a somewhat different way. However, the conditions for their existence, and hence, the conditions for reconstruction, are the same. These conditions are our main interest in this paper.*

Next, we investigate some special cases.

3 Special Cases

In this section we look at motions with constant velocity or constant acceleration which have been considered elsewhere in the literature. We also look at periodic motions, which we feel are of practical interest. For each motion we determine the conditions on the motion parameters in relation to the sampling rates $\Delta x, \Delta t$ so that (8) is satisfied.

3.1 Motion with constant velocity

Consider the case

$$f(t) = Vt \tag{12}$$

Then, clearly, if $F[n_1] = F[n_2]$ for some $n_1 < n_2$ we must have

$$\frac{V\Delta t}{\Delta x} = \frac{m}{n_2 - n_1}$$

for some integer m. Hence, if $\frac{V\Delta t}{\Delta x}$ is an irrational number, $F[n_1] \neq F[n_2]$ for any $n_1 \neq n_2$, which means that $\mathcal{N}_F = \mathbb{Z}$. Hence, $M_F = \infty$ and reconstruction (at least theoretically) is possible for any bandwidth signal.

Let us assume now that $\frac{V\Delta t}{\Delta x}$ is a rational number. Then we make the claim:

Claim 4 *Let* $\frac{V\Delta t}{\Delta x} = \frac{R}{Q}$ *such that* $\gcd(R, Q) = 1$. *Then*

$$M_F = Q \tag{13}$$

Proof. : As we already observed, $F[n_2] = F[n_1]$ iff

$$\frac{m}{n_2 - n_1} = \frac{V\Delta t}{\Delta x} = \frac{R}{Q} \tag{14}$$

Since R and Q are coprime integers this will hold iff $Q | (n_2 - n_1)$ (namely, Q divides $n_2 - n_1$). Or

$$F[n_2] = F[n_1] \Leftrightarrow n_2 \equiv n_1 \pmod{Q} \tag{15}$$

This means that $\mathcal{N}_F$ is the *set of least residues* (as mentioned earlier), namely, is equal to $\{0, 1, ..., Q-1\}$ which completes the proof. ■

Remark 5 *This type of motion has been extensively considered in the literature and the reconstruction methods presented, typically use filters which are referred to as 'Motion Compensated Filters' (see e.g. [5]). The corresponding reconstruction formulae make use of the whole data set hence, they are not unique. Indeed, there is some freedom in the choice of these reconstruction filters. It can be shown that there exists a choice of filter which results in exactly the formula given by (9). However, in most of the existing literature, while recognizing that there are some 'critical velocities' for which reconstruction is not possible, no conditions have been presented. As far as we know, these type of reconstruction conditions appeared first in [4] for the constant velocity case only.*
Using our terminology here, the condition given in [4] is

$$\min_{0 < n \leq \left\lfloor \frac{W_x \Delta x}{2\pi} \right\rfloor} F[n] > 0 \tag{16}$$

and can readily be shown to be equivalent to (13).

3.2 Motion with constant acceleration.

The study of this case is motivated, again, by related work in video processing (see e.g. [5] and [1]). In these references the authors show that frequency domain insights are not helpful here and propose to use short time Fourier transforms for the reconstruction. However, the question whether reconstruction is possible at all, has not been addressed.

We have here

$$f(t) = at^2 \tag{17}$$

where a is constant and, without loss of generality, we assume it is positive. Then,

$$F[n] = \left\lceil \frac{a(\Delta t)^2}{\Delta x} n^2 \right\rceil - \frac{a(\Delta t)^2}{\Delta x} n^2 \tag{18}$$

and we observe again, that if $\frac{a(\Delta t)^2}{\Delta x}$ is irrational $F[n_1] \neq F[n_2]$ whenever $n_1 \neq n_2$, which in turn means that $M_F = \infty$ (recall that M_F is the count of elements in $\mathcal{N}_F$). Let us then consider the case when $\frac{a(\Delta t)^2}{\Delta x} = \frac{R}{Q}$ is a rational number ($\gcd(Q\ ,R) = 1$). We can readily show that $F[n_1] = F[n_2]$ for any integer value R if and only if it is true for $R = 1$, so from here on, without loss of generality, we will assume $\frac{a(\Delta t)^2}{\Delta x} = \frac{1}{Q}$.

We make the following claim:

Claim 6 *Let* $\frac{a(\Delta t)^2}{\Delta x} = \frac{1}{Q}$, $Q > 0$ *be a given integer. Its unique factorization (see e.g. [6]) is given by*

$$Q = 2^{m_o} \prod_{i=1}^{I} p_i^{m_i} \tag{19}$$

where $p_i > 2$ *are distinct prime numbers* $m_o \geq 0$ *and* $m_i > 0$, . *Then*

$$M_F = M_0 \prod_{i=1}^{I} \left(\left(\frac{p_i - 1}{2} \right) \left(p^{m_i - 1} + \left\lfloor \frac{m_i - 1}{2} \right\rfloor \right) + 1 \right) \tag{20}$$

where

$$M_0 = \begin{cases} 1 & for\, m_o = 0 \\ \frac{4+2^{m_o-1}}{3} & for\; m_o\; even \\ \frac{5+2^{m_o-1}}{3} & for\; m_o\; odd \end{cases} \tag{21}$$

Proof. : (See Appendix A). ■

As an illustration, say $Q = 360 = 2^3 3^2 5$, using the above formula, the RGF is $M_F = 36$. We recall that this result means that if $\frac{a(\Delta t)^2}{\Delta x} = \frac{R}{360}$ (R any natural number coprime with 360) we could use the data generated to reconstruct a signal of bandwidth up to $36 \frac{2\pi}{\Delta x}$.

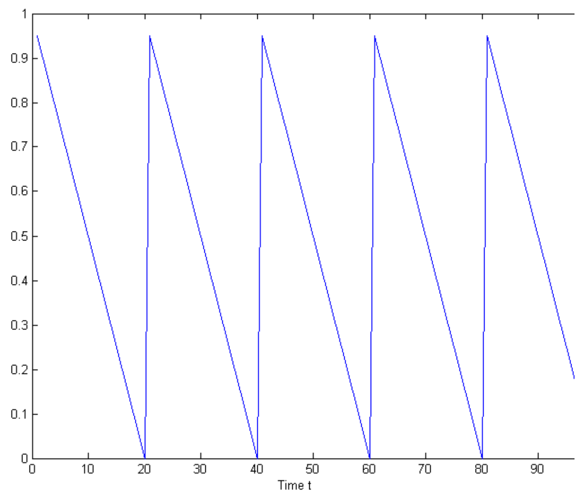

Figure 1: Periodic Motion - Case 1 (T=20).

3.3 Periodic Motion

Perhaps the most interesting type of motion to consider for practical applications is different types of periodic motion where we assume that there exists a $T > 0$ for which $f(t+T) = f(t)$. As is well known, sampling a periodic function does not necessarily result in a periodic sequence unless $\frac{\Delta t}{T} = \frac{R}{Q}$ - a rational number. Obviously, in this case, $f((n+Q)\Delta t) = f(n\Delta t)$ and if the resulting period Q, is less than N, reconstruction will be impossible. Hence, $\frac{\Delta t}{T}$ irrational or $Q \geq N$ is a necessary condition for reconstruction in this case. To generate necessary and sufficient conditions we need to consider more specific possible choices for the periodic function $f(t)$.

Case 7 $f(t) = V\left(\left\lceil \frac{t}{T} \right\rceil - \frac{t}{T}\right)$ *(See Figure 1).*

For this motion we obtain

$$F[n] = \left\lceil \frac{V\left(\left\lceil \frac{\Delta t}{T} n \right\rceil - \frac{\Delta t}{T} n\right)}{\Delta x} \right\rceil - \frac{V\left(\left\lceil \frac{\Delta t}{T} n \right\rceil - \frac{\Delta t}{T} n\right)}{\Delta x} \tag{22}$$

We can then prove the following claim:

Claim 8 *Let* $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ *and* $\frac{V}{\Delta x} = \frac{R_2}{Q_2}$ *with* $\gcd(Q_i, R_i) = 1$, $i = 1, 2$, *and let* $g = \gcd(Q_1, R_2)$. *Then the RGF is given by*

$$M_F = \min\left(Q_1, \frac{Q_1 Q_2}{g}\right) \tag{23}$$

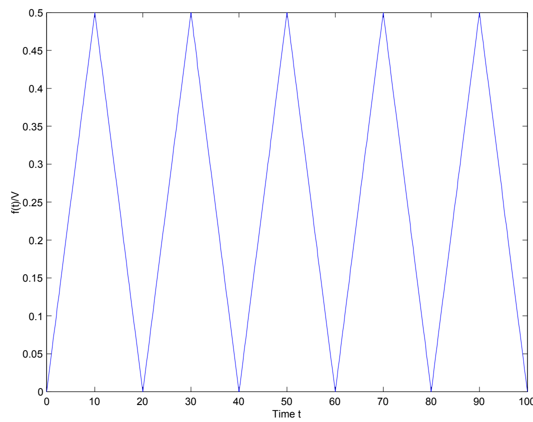

Figure 2: Periodic Motion - Case 2 (T=20).

Proof. : Define

$$F_1[n] = \left\lceil \frac{R_1}{Q_1} n \right\rceil - \frac{R_1}{Q_1} n \tag{24}$$

then $0 \leq m_n = Q_1 F_1[n] < Q_1$ are integers and since, $F_1[n_1] = F_1[n_2] \Leftrightarrow R_1 n_1 \equiv R_1 n_2 \pmod{Q_1} \Leftrightarrow n_1 \equiv n_2 \pmod{Q_1}$, we have a one to one correspondence between the sets$\{0, 1, ..., Q_1 - 1\}$ and $\{m_0, m_1, ..., m_{Q_1-1}\}$. Furthermore, for any $n_1, n_2 \in \{0, 1, ..., Q_1 - 1\}$, $n_1 \equiv n_2 \pmod{Q_1} \Rightarrow n_1 = n_2$.

We can now rewrite $F[n]$ as

$$\begin{aligned} F[n] &= \left\lceil \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \right\rceil - \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \\ &= \left\lceil \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \right\rceil - \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \end{aligned}$$

where, by definition, $\gcd\left(\frac{Q_1 Q_2}{g}, \frac{R_2}{g}\right) = 1$. Hence, $F[n_1] = F[n_2] \Leftrightarrow m_{n_1} \equiv m_{n_2} \left(\bmod \frac{Q_1 Q_2}{g}\right) \Rightarrow$ the set of $n \in \{0, 1, ..., Q_1 - 1\}$ for which $F[n_1] \neq F[n_2]$ (namely, the set $\mathcal{N}_F$) is given by $\{0, 1, ..., Q_1 - 1\} \cap \left\{n : 0 \leq m_n < \frac{Q_1 Q_2}{g} - 1\right\} \Rightarrow M_F = \min\left(Q_1, \frac{Q_1 Q_2}{g}\right)$ as claimed. ■

It can be observed from this claim that, if $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ while $\frac{V}{\Delta x}$ is irrational $M_F = Q_1$. On the other hand, if $\frac{\Delta t}{T}$ is irrational $M_F = \infty$ no matter what $\frac{V}{\Delta x}$ is.

Case 9 *$f(t) = V \left| \left\lceil \frac{t}{T} - \frac{1}{2} \right\rceil - \frac{t}{T} \right|$ (See Figure 2).*

Here

$$F[n] = \left\lceil \frac{V\left|\left\lceil \frac{\Delta t}{T}n - \frac{1}{2}\right\rceil - \frac{\Delta t}{T}n\right|}{\Delta x} \right\rceil - \frac{V\left|\left\lceil \frac{\Delta t}{T}n - \frac{1}{2}\right\rceil - \frac{\Delta t}{T}n\right|}{\Delta x} \tag{25}$$

For this case, we prove the following claim:

Claim 10 *Let* $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ *and* $\frac{V}{\Delta x} = \frac{R_2}{Q_2}$ *with* $\gcd(Q_i, R_i) = 1$, $i = 1, 2$, *and let* $g = \gcd(Q_1, R_2)$. *Then the RGF is given by*

$$M_F = \min\left(\left\lfloor \frac{Q_1}{2} \right\rfloor + 1, \frac{Q_1 Q_2}{g}\right) \tag{26}$$

Proof. : The proof is quite similar to the proof of Claim 3. Define

$$F_1[n] = \left|\left\lceil \frac{R_1}{Q_1}n - \frac{1}{2}\right\rceil - \frac{R_1}{Q_1}n\right| \tag{27}$$

then $0 \leq m_n = Q_1 F_1[n] \leq \left\lfloor \frac{Q_1}{2} \right\rfloor$ are integers and since, $F_1[n_1] = F_1[n_2]$ $\Leftrightarrow R_1 n_1 \equiv R_1 n_2 \,(\operatorname{mod} Q_1)$ or $R_1 n_1 \equiv -R_1 n_2 \,(\operatorname{mod} Q_1) \Leftrightarrow n_1 \equiv n_2 \,(\operatorname{mod} Q_1)$ or $n_1 \equiv -n_2 \,(\operatorname{mod} Q_1)$, we have a one to one correspondence between the sets $\left\{0, 1, ..., \left\lfloor \frac{Q_1}{2} \right\rfloor\right\}$ and $\left\{m_0, m_1, ..., m_{\lfloor \frac{Q_1}{2} \rfloor}\right\}$. Furthermore, for any $n_1, n_2 \in \left\{0, 1, ..., \left\lfloor \frac{Q_1}{2} \right\rfloor\right\}$, $n_1 \equiv n_2 \,(\operatorname{mod} Q_1)$ *or* $n_1 \equiv -n_2 \,(\operatorname{mod} Q_1) \Rightarrow n_1 = n_2$.

We can now rewrite $F[n]$ as

$$\begin{aligned} F[n] &= \left\lceil \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \right\rceil - \frac{R_2}{Q_1 Q_2} Q_1 F_1[n] \\ &= \left\lceil \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \right\rceil - \frac{\frac{R_2}{g}}{\frac{Q_1 Q_2}{g}} m_n \end{aligned}$$

where, by definition, $\gcd\left(\frac{Q_1 Q_2}{g}, \frac{R_2}{g}\right) = 1$. Hence, $F[n_1] = F[n_2] \Leftrightarrow m_{n_1} \equiv m_{n_2} \left(\operatorname{mod} \frac{Q_1 Q_2}{g}\right) \Rightarrow$ the set of $n \in \left\{0, 1, ..., \left\lfloor \frac{Q_1}{2} \right\rfloor\right\}$ for which $F[n_1] \neq F[n_2]$ (namely, the set $\mathcal{N}_F$) is given by $\left\{0, 1, ..., \left\lfloor \frac{Q_1}{2} \right\rfloor\right\} \cap \left\{n : 0 \leq m_n < \frac{Q_1 Q_2}{g} - 1\right\} \Rightarrow$ $M_F = \min\left(\left\lfloor \frac{Q_1}{2} \right\rfloor + 1, \frac{Q_1 Q_2}{g}\right)$ as claimed. ■

Here too, it can be observed from the claim above that, if $\frac{\Delta t}{T} = \frac{R_1}{Q_1}$ while $\frac{V}{\Delta x}$ is irrational, $M_F = \left\lfloor \frac{Q_1}{2} \right\rfloor + 1$. On the other hand, if $\frac{\Delta t}{T}$ is irrational, $M_F = \infty$ no matter what $\frac{V}{\Delta x} \neq 0$ is.

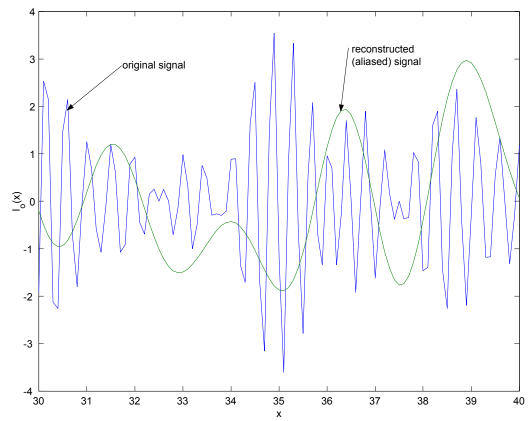

Figure 3: The original signal $I_o(x)$ and its reconstructed version from the undersampled data.

4 Simulation results

To illustrate the analysis we have applied the above mentioned motion aided sampling to the following signal

$$I_o(x) = \sum_{i=1}^{4} \sin(2\pi f_i x) \tag{28}$$

where $f_1 = 2.1,\ f_2 = 2.36,\ f_3 = 2.56,\ f_4 = 2.7$. This is clearly a bandlimited signal with bandwidth $W_x = 4\pi f_4 = 10.8\pi$. Suppose that we are constrained to sample this signal at $\Delta x = 1$. This clearly represents undersampling (since, the Nyquist sampling interval is $\frac{2\pi}{W_x} = \frac{10}{54}$). In Figure 3 we see a section of the original signal and its reconstruction from the undersampled data using lowpass filter of bandwidth W_x - we observe that, in this case, the aliasing is quite distinct.

We next introduce motion and experiment with the various types of motions described above. In each case, we sample in time at $\Delta t = 1$ and then reconstruct the signal from the sampled data. The reconstruction is done in two stages. First, we convert the continuous filters of eqn. (10) into discrete filters using the result in [9]. This results in a uniformly sampled version of the signal at intervals $\Delta\widetilde{x} = 0.1$ (which is considerably higher than the Nyquist rate of the signal). Then, using a lowpass filter on this uniformly sampled signal the original signal is reconstructed.

For each experiment we evaluate the Signal to Distortion Ratio (SDR) defined by

$$SDR = \frac{\int [I_o(x)]^2\, dx}{\int [I_o(x) - I_{rec}(x)]^2\, dx} \tag{29}$$

The results are summarized in Table 1. In this table, for each experiment, we enter the type of motion used, the motion parameters, the RGF (in *all cases* the actual RGF was *identical* to the one calculated using our analysis) as well as the resulting SDR.

Motion Type	V	a	T	RGF	Uniformity Ratio	SDR (dB)
Const. velocity	1/6			6	1	603
	5/6			6	1	628
	7/6			6	1	628
Const. acceler.		1/36		8	9	113
		1/22		12	4	112
		1/40		9	11	95
		1/20		6	7	84
Periodic 1	1/6		10/13	10	51	104
	5/6		10/13	10	3	131
	7/6		10/13	10	2.3	94
	1/6		6/13	6	31	62
	5/6		6/13	6	2.2	95
	7/6		6/13	6	7	75
	1/6		6/7	6	31	62
Periodic 2	7/6		6/13	4	-	-
	7/6		8/13	5	-	-
	7/6		9/13	5	-	-
	1/6		10/13	6	55	61
	5/6		10/13	6	7	72
	7/6		10/13	6	3.5	84

Table 1: Summary of experiment results.

We have stated earlier in the paper that, intuitively, the closer the resulting additional samples are to uniform sampling the more robust the reconstruction should be. To illustrate this claim, for each experiment we have calculated a measure of this 'uniformity' by taking the ratio of the maximal and minimal distances between adjacent sampling points.

To test the sensitivity we have also carried further experiments on the two periodic motions above. We have introduced errors in the motion parameters and used the resulting SDR as a measure of performance. From our extensive experiments we make the following observations:

1. The reconstruction is more sensitive to errors in the period T than to errors in the velocity V.

2. Periodic motion Type 1 is more robust than Type 2.

3. For identical errors, in both motions, the robustness increases as one decreases of the Uniformity Ratio (as was predicted above).

In Figure 4 we present a sample of the results. For this experiment we obtain Uniformity Ratio=3 and SDR=35.

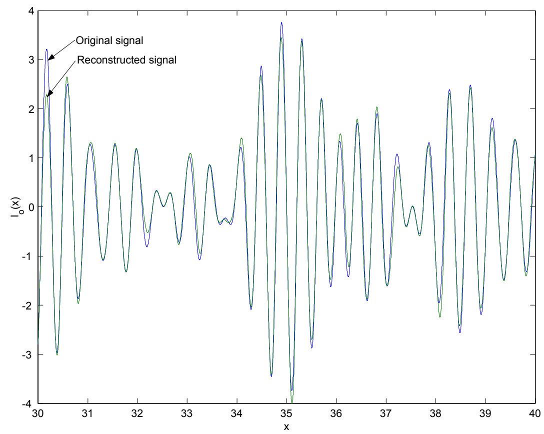

Figure 4: Comarison of original and reconstructed signal for periodic motion type 1, with $V = 5/6$, $T = 10/13$ and error in the T.

5 Conclusion

We have addressed the problem of using motion as a temporal enhancement for spatial sampling rates. While a number of algorithms for reconstruction of signals from their combined spatial and temporal samples have been previously described in the literature, most current results do not address the question 'when is this reconstruction possible?'. In this paper we analyze a number of typical motions, each with its own parameters, and derive necessary and sufficient conditions which guarantee, in each case, the feasibility of signal reconstruction. To demonstrate the validity of our analysis we have carried out extensive simulations. The paper has given a representative set of these results. In the experiments we have also tested the sensitivity of the reconstruction to errors in the motion parameter measurements. We have observed that, in the case of periodic motions, the reconstruction is quite robust to errors in the velocity and considerably more sensitive to errors in the period.

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The proof of Claim 2

Proof. To prove Claim 2 we first define a set of integers $\widetilde{\mathcal{N}}$

$$\widetilde{\mathcal{N}} = \left\{ 0 \leq \widetilde{n}_j < Q : (n_j)^2 \equiv \widetilde{n}_j \pmod{Q}, n_j \in \mathcal{N}_F \right\} \tag{30}$$

Note that, from Definition 1 of $\mathcal{N}_F$, we can readily see that for every $n \in \mathbb{Z}$, $(n)^2$ is congruent modulo Q to exactly one element in $\widetilde{\mathcal{N}}$. This set is commonly referred to as the *set of (least ,nonnegative) quadratic residues* (see e.g. [7]). Furthermore, since by definition, for any $n_j, n_k \in \mathcal{N}_F$, $j \neq k$, $(n_j)^2$ and $(n_k)^2$are not congruent modulo Q, we have $\widetilde{n}_j \neq \widetilde{n}_k$. Hence, there is a one-to-one correspondence between $\mathcal{N}_F$ and $\widetilde{\mathcal{N}}$ (namely, the two sets have the same number of elements, M_F). Next we need to prove the following three preliminary results: ■

Claim 11 *Let $Q_1, Q_2, Q \in N$, $\widetilde{\mathcal{N}_1}, \widetilde{\mathcal{N}_2}, \widetilde{\mathcal{N}}$ the corresponding sets of quadratic residues and $M_{1,F}, M_{2,F}, M_F$ the respective element counts of these sets. Then, if Q_1, Q_2 are (positive) coprime and $Q = Q_1 Q_2$ we obtain*

$$M_F = M_{1,F} \cdot M_{2,F} \tag{31}$$

Proof. Let $\widetilde{n}_j^1 \in \widetilde{\mathcal{N}_1}$ and $\widetilde{n}_k^2 \in \widetilde{\mathcal{N}_2}$. Consider the following system of congruences:

$$\begin{aligned} (x)^2 &\equiv \widetilde{n}_j^1 \,(\mathrm{mod}\, Q_1) \\ (x)^2 &\equiv \widetilde{n}_k^2 \,(\mathrm{mod}\, Q_2) \end{aligned} \tag{32}$$

Then, by the Chinese Remainder Theorem (see e.g. [6] or [7]) and (30) we know that this system always has solutions and the squares of any two solutions differ by a multiple of Q. Define $\widetilde{n}_{j,k}$ as

$$\widetilde{n}_{j,k} = \min_{r\in\mathbb{Z}} \left\{ (x)^2 - rQ \geq 0 \right\} \tag{33}$$

then $0 \leq \widetilde{n}_{j,k} < Q$ and, since $\widetilde{n}_j^1$ is not congruent to any other element of $\widetilde{\mathcal{N}_1}$ neither is $\widetilde{n}_k^2$ to any other element of $\widetilde{\mathcal{N}_2}$, $\widetilde{n}_{j,k} \neq \widetilde{n}_{\ell,m}$ whenever $(j,k) \neq (\ell,m)$. Hence, the set $\{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$ contains exactly $M_{1,F} \cdot M_{2,F}$ elements. Next we show that

$$\widetilde{\mathcal{N}} = \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\} \tag{34}$$

From (32) and (33) we have $0 \leq \widetilde{n}_{j,k} < Q$ and $(x)^2 \equiv \widetilde{n}_{j,k} \,(\mathrm{mod}\, Q)$ so $\widetilde{n}_{j,k} \in \widetilde{\mathcal{N}} \Rightarrow \widetilde{\mathcal{N}} \subseteq \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$.

Let $\widetilde{n}_r \in \widetilde{\mathcal{N}}$. Hence, there exists $n \in \mathbb{Z}$ such that $(n)^2 \equiv \widetilde{n}_r \,(\mathrm{mod}\, Q)$. On the other hand, by definition of $\widetilde{\mathcal{N}_1}$ and $\widetilde{\mathcal{N}_2}$, there exist $\widetilde{n}_\ell^1 \in \widetilde{\mathcal{N}_1}$ and $\widetilde{n}_m^2 \in \widetilde{\mathcal{N}_2}$ such that

$$\begin{aligned} (n)^2 &\equiv \widetilde{n}_\ell^1 \,(\mathrm{mod}\, Q_1) \\ &\equiv \widetilde{n}_m^2 \,(\mathrm{mod}\, Q_2) \end{aligned}$$

Hence, for the corresponding $\widetilde{n}_{\ell,m}$ we have $(n)^2 \equiv \widetilde{n}_{\ell,m} \,(\mathrm{mod}\, Q) \equiv \widetilde{n}_r \,(\mathrm{mod}\, Q)$ and since $0 \leq \widetilde{n}_r, \widetilde{n}_{\ell,m} < Q$ we must have $\widetilde{n}_r = \widetilde{n}_{\ell,m}$. Hence, $\widetilde{\mathcal{N}} \supseteq \{\widetilde{n}_{j,k} : 0 \leq j < M_{1,F}, 0 \leq k < M_{2,F}\}$, which establishes (34). This completes the proof. ■

Claim 12 *Let $Q = 2^{m_o}$. Then*

$$M_o \,(= M_F) = \begin{cases} 1 & for\, m_o = 0 \\ \frac{4+2^{m_o-1}}{3} & for\, m_o \; even \\ \frac{5+2^{m_o-1}}{3} & for\, m_o \; odd \end{cases} \tag{35}$$

Proof. Considering the congruence relationship $x^2 \equiv \widetilde{n}_i \,(\mathrm{mod}\, Q)$ we use a result in [6] which states that for odd n_i an x satisfying the the congruence relationship exists iff $\widetilde{n}_i \equiv 1 \,(\mathrm{mod}\, g)$ where g is the greatest common devisor of $Q = 2^{m_o}$ and 8. This means that for the set $\{1, 3, ..., 2^{m_o} - 1\}$ the number of integers for which an x exists is given by

$$\begin{aligned} 1 \text{ for } m_o &= 1, 2 \\ 2^{m_o-3} \text{ for } m_o &\geq 3 \end{aligned} \tag{36}$$

Next we observe that we can write

$$\{0,1,...,2^{m_o}-1\} = \{0\} \bigcup_{r=1}^{m_o} 2^{m_o-r}\{1,3,...,2^r-1\} \tag{37}$$

and that only the sets where $m_o - r$ is even contain elements for which an x exists. Hence we consider separately odd and even m_o. For m_o odd only the sets with odd $r\,(=2r_o+1)$ are counted and we get (note that $\{0\}$ counts for 1)

$$\begin{aligned} M_o &= 2 + \sum_{r_o=1}^{\frac{m_o-1}{2}} 2^{(2r_o+1)-3} \\ &= 2 + \frac{2^{m_o-1}-1}{3} \\ &= \frac{2^{m_o-1}+5}{3} \end{aligned}$$

For m_o even only the sets with even $r\,(=2r_o)$ are counted and we get

$$\begin{aligned} M_o &= 2 + \sum_{r_o=2}^{\frac{m_o}{2}} 2^{2r_o-3} \\ &= 2 + 2\frac{2^{m_o-2}-1}{3} \\ &= \frac{2^{m_o-1}+4}{3} \end{aligned}$$

which completes the proof. ∎

Claim 13 *Let $Q = p^m$ where $p > 2$ is prime. Then*

$$M_F = 1 + \frac{p-1}{2}\left(\left\lfloor \frac{m-1}{2} \right\rfloor + p^{m-1}\right)$$

Proof. We consider the set $\{0,1,...,p^m-1\}$ and want to find for how many of its elements $\widetilde{n}_i$, the congruence $x^2 \equiv \widetilde{n}_i \,(\mathrm{mod}\, Q)$ has solutions. Since we can write

$$\{0,1,...,p^m-1\} = \left\{0,p,2p,...,p^2,...,p^m-p\right\} \bigcup \{1,2,...,p-1,p+1,...,p^m-1\}$$

where one set contains all the elements which are devisable by p, there are p^{m-1} of them, and the second are the remaining elements, $p^{m-1}(p-1)$ of them. Or

$$\{0,1,...,p^m-1\} = \{0\} \bigcup_{r=1}^{m-1} p^r\{1,2,..,p-1\} \bigcup_{k=0}^{p^{m-1}-1} \{kp+1,...,kp+p-1\}$$

As stated earlier, whether an element $0 \le \widetilde{n} < p^m$ belongs to $\widetilde{\mathcal{N}}$ is equivalent to whether the congruence $x^2 \equiv \widetilde{n}\,(\mathrm{mod}\, p^m)$ has a solution. From a result in

[7] we have that if p does not divide into $\widetilde{n}$ the above congruence has a solution iff the congruence $x^2 \equiv \widetilde{n} \pmod{p}$ has a solution. Furthermore, in the set $\{1, 2, ..., p-1\}$ there are exactly $\frac{p-1}{2}$ elements for which the above congruence has a solution. Noting also that $x^2 \equiv (\widetilde{n}p^r) \pmod{p^m}$ can have a solution iff r is even and if the congruence $x^2 \equiv \widetilde{n} \pmod{p^{m-r}}$ has a solution and that $kp + \widetilde{n} \equiv \widetilde{n} \pmod{p}$ we can conclude that (including the element 0 for which the congruence has a trivial solution)

$$\begin{aligned} M_F &= 1 + \left\lfloor \frac{m-1}{2} \right\rfloor \frac{p-1}{2} + p^{m-1} \frac{p-1}{2} \\ &= 1 + \frac{p-1}{2} \left(\left\lfloor \frac{m-1}{2} \right\rfloor + p^{m-1} \right) \end{aligned}$$

which completes the proof. ∎

The proof of Claim 2 then consists of recalling the factorization of a general Q as given in (19) and applying the three claims above.