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Robust hybrid control incorporating over-saturation

José A. De Doná[a,*], S.O. Reza Moheimani[a], Graham C. Goodwin[a], Arie Feuer[b]

[a] *Department of Electrical and Computer Engineering, The University of Newcastle, Callaghan 2308, Australia*
[b] *Department of Electrical Engineering, Technion, Israel Institute of Technology, Technion City, Haifa 32000, Israel*

Abstract

A switching controller aimed at dealing with a class of uncertain systems subject to input saturation is presented. The switching strategy is such that a pre-specified level of over-saturation is allowed, forcing the input deliberately into saturation. The goal of this method is to use the full authority of the available control. The proposed scheme allows for, up to, 100% over-saturation. Under these conditions, robust stability of the hybrid scheme is established. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Systems composed of logic-based switching controllers together with the processes they are intended to control are concrete examples of hybrid dynamical systems [5]. Within the last decade, a number of analytical studies of such systems has emerged (see e.g. references cited in [5]). Stability of these schemes has been studied in, for example, [8,9], where the robust stabilizability of hybrid systems is considered.

Methods that employ controller switching to deal with input constraints have been proposed in, for example, [3,10,12]. A key idea encapsulated in these methods is that of *maximal constraint admissible sets* that ensure subsequent satisfaction of the constraints. A number of controllers in a sequence of increasing performance is precomputed and, at each time, the best controller considered "safe" (i.e. such that the constraints are not violated) is used. A common feature of these schemes is that the input constraints are avoided, and thus over-saturation is never reached. By the term "over-saturation", we mean a situation in which a controller initially demands an input level greater than the available range, followed by truncation via a simple saturation operation.

In the context of controllers that avoid saturation, it is recognised that with such cautious designs, the control capacity of the system is not fully utilised leading to performance degradation in terms of speed of response and disturbance rejection (see e.g. [4]). Thus, a key design improvement would be to make better use of the available input authority. In [1] we have presented a logic-based controller for nominal plants that allows some level of over-saturation, providing a better utilisation of the full power of the available control.

In this paper we present the design of a switching controller for a class of uncertain systems, namely, systems described by state equations which depend on time-varying unknown-but-bounded uncertain

* Corresponding author.
E-mail addresses: eejose@ee.newcastle.edu.au (J.A. De Doná), reza@ee.newcastle.edu.au (S.O. Reza Moheimani), eegcg@cc.newcastle.edu.au (G.C. Goodwin), feuer@ee.technion.ac.il (A. Feuer)

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parameters. The controller is based on a switching strategy that allows a pre-specified level of over-saturation, thus rendering an improvement in performance. The design takes into account system uncertainties and the algorithm guarantees asymptotic stability for the complete family of uncertain systems with allowed over-saturation up to 100%. The construction of the stabilizing switching controller involves the solution of algebraic Riccati equations. Conditions for the existence of these solutions are provided. We address the SISO case, although the extension to the multivariable case appears straightforward.

2. System and definitions

We consider a class of uncertain linear systems described by state-space equations of the form

$$\dot{x}(t) = (A + D\Delta(t)E)x(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\Delta(t) \in \mathbb{R}^{p\times q}$ is a matrix of uncertain parameters satisfying the bound $\Delta^{\mathrm{T}}(t)\Delta(t) \leqslant I$, $u(t) = \mathrm{sat}(\tilde{u}(t))$, $\tilde{u}(t) \in \mathbb{R}$ is the control input and $\mathrm{sat}(\cdot)$ is the saturation function with saturation level $\bar{U}$, namely, $\mathrm{sat}(s) = \mathrm{sgn}(s)\min\{|s|, \bar{U}\}$.

The switching controller consists of a bank of N precomputed gains $\{K_i\}_{i=1}^N$, in a sequence of increasing levels of performance. Each gain K_i is computed such that the control $u(t) = -K_i x(t)$ quadratically stabilizes the uncertain system (1) when no saturation is present. We next define the notion of *quadratic stabilizability* and we provide an algorithm from the literature to compute a suitable stabilizing control law $u(t) = -K_i x(t)$ (Definition 2.1 and Theorem 3.1 in [6]).

Definition 2.1. The uncertain system (1) is said to be quadratically stabilizable if there exists a linear feedback control law $u(t) = -K_i x(t)$, a positive definite symmetric matrix $P_i \in \mathbb{R}^{n\times n}$ and a constant $\alpha_i > 0$ such that the following condition holds: Given any admissible uncertainty $\Delta(\cdot)$, the Lyapunov derivative corresponding to the resulting closed-loop system and the Lyapunov function $V(x) = x^{\mathrm{T}}P_i x$ satisfies the bound

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P_i + P_iA)x + 2x^{\mathrm{T}}P_iD\Delta(t)Ex - 2x^{\mathrm{T}}P_iBK_ix \leqslant -\alpha_i\|x\|^2, \tag{2}$$

for all pairs $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. In this inequality, $\|\cdot\|$ denotes the standard Euclidean norm.

If inequality (2) is satisfied, it is straightforward to verify that the corresponding closed-loop system is uniformly and asymptotically stable; e.g., see [11].

We then have the following theorem from the literature (see [6]), which applies to a single controller:

Theorem 2.1 (Petersen [6]). *Let $Q \in \mathbb{R}^{n\times n}$ and $r_i \in \mathbb{R}$ be a given positive-definite symmetric weighting matrix and a given positive weighting constant respectively, and suppose there exists a constant $\varepsilon_i > 0$ such that the algebraic Riccati equation*

$$A^{\mathrm{T}}P + PA - PBr_i^{-1}B^{\mathrm{T}}P + \varepsilon_i PDD^{\mathrm{T}}P + \frac{1}{\varepsilon_i}E^{\mathrm{T}}E + Q = 0 \tag{3}$$

has a positive-definite symmetric solution, denoted P_i. Then the uncertain system (1) is quadratically stabilizable. Furthermore, a suitable stabilizing control law is given by $u(t) = -K_i x(t)$, where

$$K_i = r_i^{-1}B^{\mathrm{T}}P_i. \tag{4}$$

Next, consider the bank of gains $\{K_i\}_{i=1}^N$, computed using (3)–(4) for a sequence of weights $r_1 > r_2 > \cdots > r_N > 0$ (as explained later). The controller has a switching strategy that selects which linear gain K_i is in use at each time in the control law $\tilde{u}(t) = -K_i x(t)$. In selecting each gain, we have noticed that it is desirable to allow moderate over-saturation of the control signal. That is, to allow the control to effectively exceed the saturation level. In order to measure the magnitude of the control saturation we define an *over-saturation index* as follows.

Definition 2.2. Given a saturation function $\mathrm{sat}(\cdot)$ and a scalar control signal $\tilde{u}(t)$ we define a function $\beta(t)$ as

$$\beta(t) = \begin{cases} \dfrac{\tilde{u}(t) - \mathrm{sat}(\tilde{u}(t))}{\mathrm{sat}(\tilde{u}(t))} & \text{for } \tilde{u}(t) \neq 0, \\ 0 & \text{for } \tilde{u}(t) = 0. \end{cases} \tag{5}$$

Clearly $\beta(t) = 0$ whenever $\tilde{u}(t)$ is not saturated, and $\beta(t)$ is the relative value of the excess of the control with respect to the saturation bound $\bar{U}$ when $\tilde{u}(t)$ is saturated. The "over-saturation index" is then defined as a constant $\bar{\beta}$ such that the allowed supremum of the control signal satisfies $\|\beta(t)\|_\infty \leqslant \bar{\beta}$. Based on reasonably extensive simulations we have found that allowing each controller to have up to $\bar{\beta} = 1$ (i.e. "allowed over-saturation" =100%) gives a good trade-off between overshoot and speed of response for a variety of systems and operating points. Moreover, a crucial point to be made below is that with an allowed over-saturation of up to 100%, the switching scheme can be shown to be robustly stable, see Theorem 4.2.

3. Preliminary results: comparison of algebraic Riccati equations

Before presenting the logic-based switching controller design, we introduce the following results.

Theorem 3.1. *Consider the algebraic Riccati equation*

$$A^{\mathrm{T}}P + PA - PBr^{-1}B^{\mathrm{T}}P + \varepsilon PDD^{\mathrm{T}}P + \frac{1}{\varepsilon}E^{\mathrm{T}}E + Q = 0, \tag{6}$$

where Q is a positive-definite symmetric matrix and $r > 0$. Assume that there exists a constant $\varepsilon > 0$ such that the algebraic Riccati equation (6) *has a positive-definite solution, denoted P_+.*

Now, consider a second algebraic Riccati equation corresponding to a different positive-definite weighting matrix $\tilde{Q} < Q$, (*meaning that $Q - \tilde{Q}$ is a positive-definite matrix*), *and the same constants r and ε:*

$$A^{\mathrm{T}}\tilde{P} + \tilde{P}A - \tilde{P}Br^{-1}B^{\mathrm{T}}\tilde{P} + \varepsilon \tilde{P}DD^{\mathrm{T}}\tilde{P} + \frac{1}{\varepsilon}E^{\mathrm{T}}E + \tilde{Q} = 0. \tag{7}$$

Then, the algebraic Riccati equation (7) *has a positive-definite solution $\tilde{P}_+$ such that $\tilde{P}_+ < P_+$.*

Proof. Denote by M the inverse of P in the first algebraic Riccati equation (6) (assuming it exists), namely $M \triangleq P^{-1}$. Then, pre- and post-multiplying Eq. (6) by M we have

$$MA^{\mathrm{T}} + AM + M\left(\frac{1}{\varepsilon}E^{\mathrm{T}}E + Q\right)M - (Br^{-1}B^{\mathrm{T}} - \varepsilon DD^{\mathrm{T}}) = 0. \tag{8}$$

Then, clearly, $M_+ \triangleq P_+^{-1}$ is a positive-definite solution of (8) (note that M_+ exists and $M_+ > 0$ since $P_+ > 0$; e.g., see [2]).

Corresponding to the algebraic Riccati Eq. (8) we can form a matrix $\mathscr{K}$ (see e.g. [7]) as follows:

$$\mathscr{K} = \begin{bmatrix} Br^{-1}B^{\mathrm{T}} - \varepsilon DD^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & -\frac{1}{\varepsilon}E^{\mathrm{T}}E - Q \end{bmatrix}. \tag{9}$$

Now, consider the following algebraic Riccati equation:

$$\tilde{M}A^{\mathrm{T}} + A\tilde{M} + \tilde{M}\left(\frac{1}{\varepsilon}E^{\mathrm{T}}E + \tilde{Q}\right)\tilde{M} - (Br^{-1}B^{\mathrm{T}} - \varepsilon DD^{\mathrm{T}}) = 0 \tag{10}$$

and its associated matrix $\tilde{\mathscr{K}}$

$$\tilde{\mathscr{K}} = \begin{bmatrix} Br^{-1}B^{\mathrm{T}} - \varepsilon DD^{\mathrm{T}} & -A \\ -A^{\mathrm{T}} & -\frac{1}{\varepsilon}E^{\mathrm{T}}E - \tilde{Q} \end{bmatrix}. \tag{11}$$

We then have that $\tilde{\mathscr{K}} - \mathscr{K} \geqslant 0$, and we can conclude from Theorem 2.2 of Ran and Vreugdenhil [7] that there exists a solution $\tilde{M}_+$ to the algebraic Riccati equation (10) such that $\tilde{M}_+ \geqslant M_+ > 0$. Also, it can be shown that the first inequality is a strict inequality, i.e. $\tilde{M}_+ > M_+ > 0$. This, in turn, implies that there exists a positive-definite matrix $\tilde{P}_+ \triangleq \tilde{M}_+^{-1}$ such that $0 < \tilde{P}_+ < P_+$. Now, by pre- and post-multiplying Eq. (10) (with $\tilde{M}$ replaced by the positive-definite solution $\tilde{M}_+$) by $\tilde{P}_+$, we conclude that $\tilde{P}_+$ is the solution to (7). □

Corollary 3.1. *Consider now two algebraic Riccati equations of the form* (3) *corresponding to the same weighting matrix $Q > 0$ and two different weights r_i and r_{i+1}, such that $r_i > r_{i+1} > 0$. To simplify the notation we will call these algebraic Riccati equations $\mathscr{R}_i(P, \varepsilon_i) = 0$ and $\mathscr{R}_{i+1}(P, \varepsilon_{i+1}) = 0$ respectively. Assume that there exists a constant $\varepsilon_{i+1} > 0$ such that $\mathscr{R}_{i+1}(P, \varepsilon_{i+1}) = 0$ has a positive-definite solution P_{i+1}. Then, a constant ε_i satisfying*

$$\varepsilon_i r_i = \varepsilon_{i+1} r_{i+1} \tag{12}$$

implies the existence of a positive-definite solution P_i to equation $\mathscr{R}_i(P, \varepsilon_i) = 0$. Furthermore,

$$\frac{P_{i+1}}{r_{i+1}} > \frac{P_i}{r_i}. \tag{13}$$

Proof. Divide equation $\mathscr{R}_{i+1}(P, \varepsilon_{i+1}) = 0$ by r_{i+1} to obtain

$$A^{\mathrm{T}}P^* + P^*A - P^*BB^{\mathrm{T}}P^* + \varepsilon_{i+1}^* P^*DD^{\mathrm{T}}P^* + \frac{1}{\varepsilon_{i+1}^*}E^{\mathrm{T}}E + Q_{i+1}^* = 0, \tag{14}$$

where $P^* \triangleq P/r_{i+1}$, $\varepsilon_{i+1}^* \triangleq \varepsilon_{i+1}r_{i+1}$ and $Q_{i+1}^* \triangleq Q/r_{i+1}$.

Notice now that the assumption that there exists a constant $\varepsilon_{i+1} > 0$ such that $\mathscr{R}_{i+1}(P, \varepsilon_{i+1}) = 0$ has a positive-definite solution P_{i+1} implies that $P_{i+1}^* \triangleq P_{i+1}/r_{i+1} > 0$ is a solution of (14) with the constant $\varepsilon_{i+1}^* \triangleq \varepsilon_{i+1}r_{i+1}$.

Now, we can form the following algebraic Riccati equation by dividing equation $\mathscr{R}_i(P,\varepsilon_i)=0$ by r_i:

$$A^{\mathrm{T}}P^* + P^*A - P^*BB^{\mathrm{T}}P^* + \varepsilon_i^* P^*DD^{\mathrm{T}}P^* + \frac{1}{\varepsilon_i^*} E^{\mathrm{T}}E + Q_i^* = 0, \tag{15}$$

where $P^* \triangleq P/r_i$, $\varepsilon_i^* \triangleq \varepsilon_i r_i$ and $Q_i^* \triangleq Q/r_i$.

It is easy to see that $Q_i^* < Q_{i+1}^*$ and we can then use Theorem 3.1 to conclude that Eq. (15) has a positive-definite solution P_i^* which satisfies

$$P_i^* < P_{i+1}^* \tag{16}$$

when a constant $\varepsilon_i^* = \varepsilon_{i+1}^*$, is used. Eq. (12) follows from the last equality and the definitions of ε_i^* and ε_{i+1}^*. The existence of a positive-definite solution P_i^* to Eq. (15) with a constant ε_i^* clearly implies that the matrix $P_i \triangleq P_i^* r_i > 0$ is a positive-definite solution to $\mathscr{R}_i(P,\varepsilon_i)=0$ with the constant $\varepsilon_i = \varepsilon_i^*/r_i$. Inequality (13) follows, then, from (16). □

4. Switching controller

The switching controller we propose consists of a bank of gains $\{K_i\}_{i=1}^N$, a partition of the state space into N cells $\{\mathscr{C}_i\}_{i=1}^N$ and a switching strategy that selects a gain K_i when the state $x(t) \in \mathscr{C}_i$, in such a way that a predefined over-saturation index $\bar{\beta}$ is not exceeded. In this section we present these elements together with a design methodology. We then show that the resulting hybrid scheme is asymptotically stable.

4.1. Bank of gains

Consider the uncertain system (1), which is assumed to be quadratically stabilizable (Definition 2.1), subject to input saturation. We propose the following controller. Let $\{r_i\}_{i=1}^N$ be a sequence of N values such that

$$r_1 > r_2 > \cdots > r_N > 0 \tag{17}$$

and choose a $n\times n$ symmetric, positive-definite matrix Q. For the smallest weight r_N in (17) find a constant $\varepsilon = \varepsilon_N$ such that the algebraic Riccati equation (3) has a positive-definite solution. The existence of this solution is guaranteed, independently of the choice of the weights Q and r_N, by the assumption of quadratic stabilizability (see, e.g., Theorem 3.3 in [6]). Denote by P_N the positive-definite solution corresponding to r_N and ε_N. Now, for each of the remaining weights $r_i \in \{r_1,\ldots,r_{N-1}\}$, compute $\varepsilon_i = \varepsilon_N r_N / r_i$ according to (12). With the pairs (r_i,ε_i), $i=1,\ldots,N-1$, thus obtained, find the positive-definite solutions P_i from equation (3). Note, from Corollary 3.1, that these solutions exist and that the sequence of solutions $\{P_i\}_{i=1}^N$, satisfies inequality (13). Finally, compute the sequence of gains $\{K_i\}_{i=1}^N$ using (4).

4.2. Partition of the state space

We consider ellipsoids defined by

$$\{x\colon x^{\mathrm{T}}P_i x \leqslant R_i\} \tag{18}$$

which will be shown (Corollary 4.1) to be positively invariant sets under the control $u(t) = -\mathrm{sat}(K_i x(t))$, where P_i and K_i are solutions of (3) and (4) for given r_i and ε_i.

In Eq. (18), R_i is given by

$$R_i = \frac{(1+\bar{\beta})^2 \bar{U}^2 r_i^2}{B^{\mathrm{T}}P_i B}, \quad i=1,\ldots,N, \tag{19}$$

where $\bar{\beta} > 0$ is the over-saturation index (Definition 2.2) and $\bar{U}$ is the saturation level.

The constants R_i are such that $|K_i x| \leqslant (1+\bar{\beta})\bar{U}$ whenever $x^{\mathrm{T}}P_i x \leqslant R_i$, i.e. they define ellipsoids such that an over-saturation of $\bar{\beta}$ is not exceeded.

We next prove that the ellipsoids in the sequence $\{x\colon x^{\mathrm{T}}P_i x \leqslant R_i\}_{i=1}^N$ are nested.

Theorem 4.1. *The ellipsoids* $\{x\colon x^{\mathrm{T}}P_i x \leqslant R_i\}_{i=1}^N$, *are nested. Namely,*

$$\{x\colon x^{\mathrm{T}}P_{i+1} x \leqslant R_{i+1}\} \subset \{x\colon x^{\mathrm{T}}P_i x \leqslant R_i\}, \quad i=1,2,\ldots,N-1. \tag{20}$$

Proof. Note that (20) holds if and only if for any x such that $x^{\mathrm{T}}P_i x = R_i$, there exists a constant $0<\alpha<1$ such that $\alpha^2 x^{\mathrm{T}}P_{i+1}x = R_{i+1}$, which in turn holds if and only if

$$x^{\mathrm{T}}\left(\frac{P_{i+1}}{R_{i+1}} - \frac{P_i}{R_i}\right)x = \frac{1}{\alpha^2} - 1 > 0. \tag{21}$$

So, a necessary and sufficient condition for the nesting property is

$$\frac{P_{i+1}}{R_{i+1}} - \frac{P_i}{R_i} > 0, \quad i=1,2,\ldots,N-1. \tag{22}$$

Now, from (13), we have that

$$\frac{P_{i+1}}{r_{i+1}} \frac{B^{\mathrm{T}}P_{i+1}B}{r_{i+1}(1+\bar{\beta})^2\bar{U}^2} > \frac{P_i}{r_i} \frac{B^{\mathrm{T}}P_i B}{r_i(1+\bar{\beta})^2\bar{U}^2} \tag{23}$$

which, from (19) yields

$$\frac{P_{i+1}}{R_{i+1}} > \frac{P_i}{R_i}, \quad i=1,2,\ldots,N-1. \tag{24}$$

Hence, by (22), the ellipsoids defined by (18), together with (3) and (19), are nested. □

This nesting property allows us to perform the following partition of the state-space region, contained in the largest ellipsoid, in N cells $\{\mathscr{C}_i\}_{i=1}^N$ defined as

$$\begin{aligned}\mathscr{C}_i &= \{x\colon x^{\mathrm{T}}P_i x \leqslant R_i \text{ and } x^{\mathrm{T}}P_j x > R_j \\ &\qquad j=i+1,\dots,N\}, \quad i=1,\dots,N-1, \\ \mathscr{C}_N &= \{x\colon x^{\mathrm{T}}P_N x \leqslant R_N\}.\end{aligned} \tag{25}$$

4.3. Switching strategy — stability

Given the state-space partition (25), the controller is defined by the switching strategy

$$\tilde{u} = -K_i x \quad \text{for } x \in \mathscr{C}_i,\ i=1,\dots,N. \tag{26}$$

Since the ellipsoids which define the state partition are nested (Theorem 4.1), the control (25)–(26) is well defined, in the sense that, for each point of the state space (contained in the largest ellipsoid considered), there exists a unique control gain K_i.

We next prove robust stability of the hybrid system formed by the process and the switching controller.

Theorem 4.2. *The uncertain system* (1) *with input saturation and the proposed controller* (25)–(26) *with* $\bar{\beta} \leqslant 1$ *is asymptotically stable for all* x *such that* $x^{\mathrm{T}}P_1 x \leqslant R_1$ (*i.e. in the state-space region covered by the outermost ellipsoid considered*).

Proof. Given the hybrid nature of the control system, we propose a piecewise quadratic candidate Lyapunov function of the form $V(x) = x^{\mathrm{T}}P_i x$, for $x \in \mathscr{C}_i$, $i = 1,\dots,N$.

From (1), (3), (4), (26) and Claim 1 of Petersen [6], we have that the time derivative of the Lyapunov function inside cell $\mathscr{C}_i$ has the following upper bound:

$$\begin{aligned}\dot{V}(x) &\leqslant x^{\mathrm{T}}\left(A^{\mathrm{T}}P_i + P_i A + \varepsilon_i P_i D D^{\mathrm{T}} P_i + \frac{1}{\varepsilon_i}E^{\mathrm{T}}E\right)x \\ &\quad -2x^{\mathrm{T}}P_i B \operatorname{sat}(K_i x) \\ &= -x^{\mathrm{T}}Qx + r_i|K_i x|\,(|K_i x| - 2\operatorname{sat}(|K_i x|)) \\ &\quad \text{for } x \in \mathscr{C}_i,\ i=1,\dots,N.\end{aligned} \tag{27}$$

It follows from (19) that for any $x(t)$ such that

$$x(t)^{\mathrm{T}}P_i x(t) \leqslant R_i \tag{28}$$

(and, in particular, for $x \in \mathscr{C}_i$) we have $|K_i x(t)| \leqslant (1+\bar{\beta})\bar{U} \leqslant 2\bar{U}$, hence, in (27) we obtain

$$|K_i x(t)| - 2\operatorname{sat}(|K_i x(t)|) \leqslant 0, \quad \forall x \in \mathscr{C}_i. \tag{29}$$

Clearly then, since Q is positive-definite, $\dot{V}$ is a negative-definite function, i.e.

$$\dot{V}(x) < 0 \quad \text{for } 0 \neq x \in \mathscr{C}_i,\ i=1,\dots,N. \tag{30}$$

We conclude that the trajectories in each cell $\mathscr{C}_i$ approach the origin with a monotonic decrease in V along the trajectory. Since the ellipsoids are nested and all contain the origin, this means that the trajectories will cross the cell boundaries as they approach the origin. Eventually, the trajectories will enter the smaller ellipsoid of radius R_N where asymptotic convergence to the origin is assured by (30). □

Corollary 4.1. *The ellipsoids* $\{x\colon x^{\mathrm{T}}P_i X \leqslant R_i\}$, *with* $\bar{\beta} \leqslant 1$, *are positively invariant sets under the control* $u(t) = -\operatorname{sat}(K_i x(t))$, *i.e. for any initial condition* $x_0 = x(t_0)$ *such that* $x_0^{\mathrm{T}}P_i x_0 \leqslant R_i$, *then for all* $t \geqslant t_0$, $x(t)^{\mathrm{T}}P_i x(t) \leqslant R_i$, *where* $x(t)$ *is the solution of* (1) *with control* $u(t) = -\operatorname{sat}(K_i x(t))$.

To see this, note that for any $x(t)$ *along a trajectory that satisfies* (28), *inequality* (30) *is satisfied, which means that the trajectories will never leave the ellipsoid* $\{x\colon x^{\mathrm{T}}P_i x \leqslant R_i\}$.

Remark 4.1. Note that the inequality (30) also eliminates the possibility of chattering when switching at the boundaries $V(x) = x^{\mathrm{T}}P_i x = R_i$, since all the trajectories will head away from the boundaries as they approach the origin.

Observation 4.1. If the uncertain system (1) is quadratically stable with $u = 0$, it can be shown that as $r_1 \to \infty$, the solution to the algebraic Riccati equation (3) exists and is positive-definite. As a result, it follows that in this case the outermost ellipsoid can be made arbitrarily large by letting $r_1 \to \infty$. However, if system (1) is not quadratically stable with $u = 0$, it can be shown, with a technique similar to that used to prove Theorem 3.1, that $P_1/r_1 \to \bar{M}$ as $r_1 \to \infty$, where $\bar{M}$ is a symmetric positive-definite matrix. Hence, the outermost ellipsoid will approach a limiting ellipsoid, and cannot be expanded to cover an arbitrarily large set of initial conditions.

5. Illustrative example

To illustrate the switching controller design procedure and the resulting hybrid system performance, we have chosen a simple pendulum as an example of the system to be controlled. The equation for this system

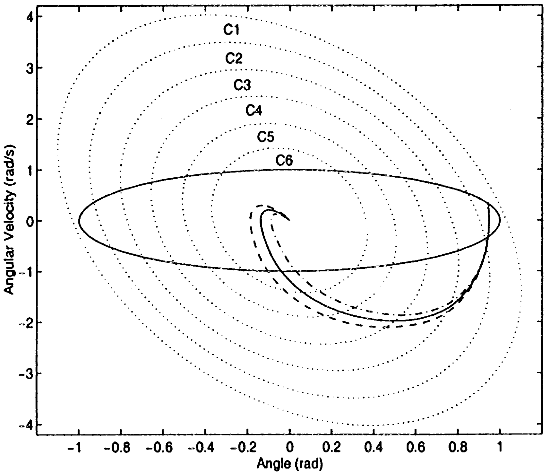

Fig. 1. State trajectories.

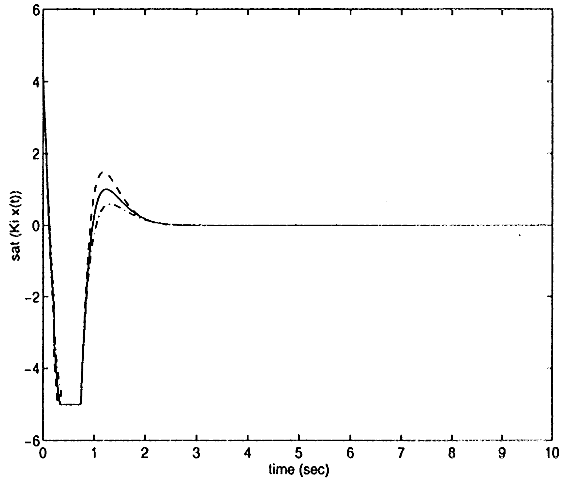

Fig. 2. Control signal.

is $\ddot{\theta} = -(g/l)\sin\theta + \tau/ml^2$, where θ is the angle, m the mass, l the pendulum length and τ the torque input. We assume that $g/l = 10$.

The system is linearised around an angle of zero radians, by using $\sin\theta \approx \theta$. We then obtain the state equations $\dot{x}_1 = x_2$, $\dot{x}_2 = -10x_1 + u$, with angle and angular velocity as state variables and normalised input $u \triangleq \tau/ml^2$, which is assumed to be subject to a saturation level of $\bar{U} = 5$. This system was used in [12] as an example of the piecewise-linear LQ control (PLC). In [1] we presented simulations for this example which show the performance improvements obtained by using a design that allows over-saturation.

We will now introduce some level of uncertainty $\Delta \in \mathbb{R}$ into the model to take into account linearization errors. We then consider a model

$$\dot{x} = \left(\begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta [\, 1 \;\; 0 \,] \right) x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$\triangleq (A + D\Delta E)x + Bu \tag{31}$$

with $|\Delta| \leqslant 1$, i.e., we introduce an uncertainty of up to $\pm 10\%$ in the element a_{21} of the A-matrix. The set of possible initial states is chosen as in the examples of De Doná et al. [1], Wredenhagen and Belanger [12] to be the unit circle.

For this example we have used a set of $N = 6$ weights: $r_1 = 10$, $r_2 = 0.5$, $r_3 = 0.2$, $r_4 = 0.1$, $r_5 = 0.05$ and $r_6 = 0.025$, and $Q = I_{2\times 2}$. Note that we have chosen r_1 so that the set of all possible initial states is included in the outermost ellipsoid (see Fig. 1). Starting from the smallest weight r_6, we found a positive-definite solution to (3) corresponding to a constant $\varepsilon_6 = 0.5$. The remaining ε_i, $i = 1, \ldots, 5$, are found, according to (12), by making $\varepsilon_i = \varepsilon_6 r_6 / r_i$. Then, the positive-definite solutions P_i, $i = 1, \ldots, 5$, corresponding to the pairs (r_i, ε_i), are computed from (3). Finally, the design of the controller requires the computation of the controller gains K_i from (4) and the ellipsoid radii R_i from (19). In the computation of R_i, an over-saturation index of 100% was used ($\bar{\beta} = 1$).

In Fig. 1 we show the state trajectories in the phase plane for an initial condition $(\theta, \dot{\theta}) = (54^\circ, 20^\circ s^{-1})$. Three different values of Δ were used for the model (31), namely, $\Delta = -1$ (dashed-plot), $\Delta = 0$ (solid-plot) and $\Delta = 1$ (dashdot-plot). Notice that although constant values were used, Δ could be time varying as in (1). We also show the switching cells $\mathscr{C}_i$ contours (dotted-plot) and the unit circle, surface of all initial conditions, (solid-plot).

In Fig. 2 we show the controls generated by the hybrid controller for the three cases: $\Delta = -1$ (dashed-plot), $\Delta = 0$ (solid-plot) and $\Delta = 1$ (dashdot-plot). Note that in all cases the control effectively reaches saturation (compare, for example, with the control generated by the PLC in [12], where saturation is avoided).

6. Conclusions

This paper has described a logic-based switching controller aimed at enhancing performance in the presence of saturating actuators by allowing over-saturation. Robust stability of the scheme is established.

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