Tracking Maneuvering Targets with a Soft Bound on the Number of Maneuvers

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***Abstract*—We revisit the problem of tracking the state of a hybrid system capable of performing a bounded number of mode switches. In a previous paper we have addressed a version of the problem where we have assumed the existence of a deterministic, known hard bound on the number of mode transitions. In addition, it was assumed that the system can possess only two modes, e.g., the maneuvering and non-maneuvering regimes of a tracked target. In the present paper we relax both assumptions: we assume a soft, stochastic bound on the number of mode transitions, and altogether remove the restriction on the number of modes of the system (thus, e.g., the target can have multiple different maneuvering modes, in addition to the non-maneuvering one). Similarly to the case where the number of transition was deterministically hard-bounded, the existence of the bound renders the mode switching mechanism non-Markov. Thus, the two formulations address similar, though not identical, problems, that cannot be solved by direct application of standard algorithms for hybrid systems. The novel solution approach is based on transforming the non-Markovian mode switching mechanism to an equivalent Markovian one, at the price of augmenting the mode definition. A standard interacting multiple model (IMM) filter is then applied to the transformed problem in a straightforward manner. The performance of the new method is demonstrated via a simulation study comprising three examples, in which the new method is compared with 1) the filter for hard-bounded mode transitions, and 2) a standard IMM filter directly applied to the original problem. The study shows that even when working outside its operating envelope, the new filter closely approximates the best filter for the scenario.**

Keywords: Multiple model estimation, target tracking, hybrid systems, fault detection and isolation.

I. Introduction

Hybrid systems are systems characterized by a continuously varying state vector and a discretely varying (switching) parameter vector, that takes values in some finite (but arbitrarily large) set [1]. The switching parameter vector is usually referred to as the system mode. The hybrid system framework is frequently used to model multi-sensor fault-prone systems. Typical examples are navigation systems using the signals of global navigation satellite systems (that are prone to jamming and spoofing) and inertial sensors (accelerometers and rate gyros) that are, frequently, of low grade. Another typical application of the hybrid system framework is in maneuvering target tracking, where it is assumed that the target can maneuver, at any point in time, in one of a finite set of maneuvers (modes).

Because many hybrid systems are characterized by stochastic state and/or parameter vector, much work has been devoted to the simultaneous estimation of both state and parameters in such systems. As is well known, the optimal, in the mean squared error (MSE) sense, filtering algorithm for hybrid systems requires infinite computation resources [2]. Therefore, a variety of suboptimal estimation techniques was proposed [3]–[8]. In all of these cases, the switching parameter vector was assumed to obey a Markovian transition law, and the system was capable of performing an infinite number of mode transitions.

When the number of mode switches permitted by the system is finite, with probability one, the assumption that the parameter vector is Markovian no longer holds. Typical problems featuring a bounded number of mode switches are encountered in maneuvering target tracking, particularly during the end-game phase of the interception scenario. Thus, [9] analyzes a stochastic ballistic missile interception scenario. Because the (incoming) blind theatre ballistic missile cannot implement an optimal deterministic evasion strategy, it executes a random bang-bang maneuver consisting of a maximal maneuver in one direction, followed by a randomly-timed single switch to a maximal maneuver in the opposite direction. In [10], optimal differential games-based strategies are derived, and it is shown that for both adversaries these strategies are of the bang-bang type, with possibly a single direction change if the target has non-minimum phase dynamics. In [11] two possible target strategies are assumed: a constant maximum maneuver, and a square wave maneuver with a small number (one or two) of direction changes. Constant target maneuvers have also been assumed in [12], [13] and a square wave maneuver with a small number of switches (one or two) has been assumed in [14].

In [15] we addressed the problem of tracking the state of a hybrid system capable of performing a bounded number of mode switches. A known hard bound on the number of maneuvers was assumed. It was shown that the optimal tracking algorithm requires the implementation of a polynomially growing number of primitive Kalman filters, thus calling for the implementation of some suboptimal estimation algorithm. On the other hand, the system's switching dynamics is not Markov because of the a priori bounded number of model switches, thus ruling out the use of popular estimation schemes such as the IMM [6] or GPB [2] algorithms. An efficient estimation scheme, that uses a number of primitive Kalman filters that is linear in the number of possible maneuvers, was derived. The scheme resembles the IMM algorithm in that it uses interaction between some of the primitive filters before every estimation cycle, thus reducing the number of such filters required by the optimal scheme.

In the present paper we take a conceptually different approach to handle a hybrid system capable of performing a finite number of mode transitions. Instead of assuming a deterministically known hard bound on the number of switches between different system modes [15], we assume herein that the system can, at any time, perform a terminating transition, with a known probability (we define a transition at time t_1 to be terminating if the system does not switch modes for $t > t_1$). This formulation renders the number of possible mode transitions, though still unbounded, finite with probability one.

Somewhat resembling the formulation of [15], the novel formulation adopted herein differs from it in several important aspects. First, a soft bound is imposed on the number of mode transitions, as opposed to the hard bound assumed in [15]. Thus, one does not need to know in advance the exact number of possible maneuvers, but only the stochastic law governing the switching mechanism. Second, the non-Markov transitions between different modes may be easily transformed into Markovian ones by augmenting the mode set with terminating modes (i.e., modes reachable by terminating transitions). This permits direct utilization of the IMM and related methods. The proposed approach also circumvents two other shortcomings of the approach utilized in [15], in that the number of possible maneuvering modes may be arbitrarily large, and the system can start from any initial mode. In contradistinction, in [15] the system is allowed to have a binary mode system (e.g., maneuvering and non-maneuvering modes, in maneuvering target tracking applications, or faulty and nominal operational modes, in fault tolerant estimation applications), and the algorithm cannot naturally handle arbitrary mode initiation (a special heuristic approach was used there to alleviate this problem).

The remainder of this paper is organized as follows. In Section II we formulate the problem at hand. In Section III we describe the augmentation of the mode set and the resulting Markov chain, and outline the resulting algorithm. The algorithm is tested in simulation and compared to a standard IMM and to the algorithm for hard-bounded number of maneuvers in Section IV. Concluding remarks are made in Section V.

II. Problem Formulation

Consider the following state space representation of a stochastic dynamical system:

$$x_{k+1} = A_k x_k + w_k \tag{1}$$
$$z_k = H_k x_k + v_k. \tag{2}$$

Here $\{w_k\}$ and $\{v_k\}$ are mutually independent, zero-mean white Gaussian process and measurement noise sequences with covariances $\{Q_k\}$ and $\{R_k\}$, respectively. These driving processes are assumed to be independent of the initial state x_0, that is assumed to be Gaussian with mean $\bar{x}_0$ and covariance P_0.

The system (1)–(2) is specified by four matrix sequences $\{A_k\}$, $\{H_k\}$, $\{Q_k\}$, and $\{R_k\}$. At each time k the set $\mathcal{M}_k \triangleq \{A_k, H_k, Q_k, R_k\}$ comprises the *mode* of the system. Different values of the mode correspond to, for example, different flight regimes (e.g., maneuvering/non-maneuvering) of an aircraft, or different sensor conditions (e.g., nominal/faulty). The state x_k may be estimated optimally in the mean-square sense using a standard Kalman filter [16] provided the mode sequence evolves in time in a deterministic manner, namely, the exact value of $\mathcal{M}_k$ is known for each k.

We consider the case where at time k the mode $\mathcal{M}_k$ may assume one of r possible values $m_1, \ldots, m_r$. In addition, the evolution of the sequence $\{\mathcal{M}_k\}$ does not occur in a deterministic manner. Instead, we consider the following stochastic switching mechanism.

Let $\{N_k\}$ be a Bernoulli Markov chain with the following transition probability matrix

$$P = \begin{pmatrix} p_0 & 1-p_0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

and $\mathbb{P}\{N_0 = 0\} = 1$. Thus, the process $\{N_k\}$ performs a single transition from state 0 to state 1, and remains there forever w.p. 1. Using the transition behavior of $\{N_k\}$, the stochastic mode switching mechanism of the sequence $\{\mathcal{M}_k\}$ is captured by the following conditional probability:

$$\begin{aligned}\mathbb{P}\{\mathcal{M}_k = m_j \mid \mathcal{M}_{k-1} = m_i, N_k = \alpha, \mathcal{M}_{k-2}, N_{k-1}, \ldots\} \\ = \begin{cases} p_{ij}, & \alpha = 0 \\ \delta_{ij}, & \alpha = 1 \end{cases}, \quad i,j \in \{1,\ldots,r\}, \, k = 1,2,\ldots\end{aligned} \tag{4}$$

where $\{p_{ij}, \ i,j \in \{1,\ldots,r\}\}$ are known probabilities and δ_{ij} is Kronecker's delta.

Equations (1), (2) and (4) define a hybrid stochastic system, as the continuous uncertainty associated with the state vector x_k is accompanied by a discretely varying uncertainty associated with the (discrete) random mode transition law. This system is not a standard Markov-Jump Linear System (MJLS), because the mode transition law is Markov only given N_k, as formally stated in the following lemma.

Lemma 1. *The stochastic process $\{\mathcal{M}_k\}_{k=0}^{\infty}$ described by the transition dynamics of Eq.* (4) *is not Markov.*

Proof: We prove the lemma by a counterexample. Let $r = 2$ and assume $p_{12} = p_{21} = 1$. Then,

$$\begin{aligned}&\mathbb{P}\{\mathcal{M}_{k+1} = m_1 \mid \mathcal{M}_k = m_1, \mathcal{M}_{k-1} = m_1\} \\ &\quad = \mathbb{P}\{\mathcal{M}_{k+1} = m_1 \mid \mathcal{M}_k = m_1, N_{k+1} = 1\} \\ &\quad = 1.\end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\{\mathcal{M}_{k+1}=m_1 \mid \mathcal{M}_k=m_1\}\\
&\quad=\mathbb{P}\{\mathcal{M}_{k+1}=m_1 \mid \mathcal{M}_k=m_1, N_{k+1}=1\}\\
&\qquad\times\mathbb{P}\{N_{k+1}=1 \mid \mathcal{M}_k=m_1\}\\
&\qquad+\mathbb{P}\{\mathcal{M}_{k+1}=m_1 \mid \mathcal{M}_k=m_1, N_{k+1}=0\}\\
&\qquad\times\mathbb{P}\{N_{k+1}=0 \mid \mathcal{M}_k=m_1\}\\
&=\mathbb{P}\{N_{k+1}=1 \mid \mathcal{M}_k=m_1\}\\
&\neq 1.
\end{aligned} \tag{6}$$

■

Lemma 1 provides the reason for the fact that the IMM algorithm and related methods cannot be applied in a straightforward manner to the problem at hand. The IMM algorithm assumes that modes switch according to a Markov law, with transition probabilities that are stated in a known transition probability matrix (TPM). In the present case, since the mode switching process is not Markov, applying the IMM and related algorithms can only be done in an approximate manner by defining a pseudo transition probability matrix that aims at capturing the main switching characteristics of the process in some approximate sense.

Our goal in the present work is to obtain an estimation algorithm, having moderate computational resource requirements, that is capable of tracking the state of a hybrid system that may perform a finite number of mode transitions that follow the transition law of Eqs. (3) and (4).

III. Filter Derivation

In this section we describe the proposed method for state estimation in hybrid systems with modes evolving according to (4). The main idea is to transform the system's non-Markovian mode set into an equivalent one, governed by a Markovian switching law. This, then, permits a straightforward utilization of a standard IMM algorithm on the equivalent Markovian mode set. Thus, the exposition of the new method essentially reduces to deriving the mode set transformation. For completeness, nevertheless, we first outline the principles and mode of operation of the IMM filtering algorithm.

A. Background on IMM

The IMM filter assumes that the mode sequence $\{\mathcal{M}_k\}$ constitutes a Markov chain on a finite state space $\{m_1,\ldots,m_r\}$ with known transition probability matrix (TPM) $P_{r\times r}=(p_{ij})$. The main idea underlying the IMM algorithm is to maintain a bank of primitive Kalman filters, each matched to a different model in the given model set. At step k of the algorithm, the j-th filter produces a local estimate $\hat{x}_j(k)$ with an associated error covariance $P_j(k)$ using its initial estimate $\hat{x}_j^0(k-1)$ and the associated covariance $P_j^0(k-1)$, which are generated externally, and the current measurement z_k, which gets processed by all KFs in the bank. In addition, each filter produces a current value of its own (model-matched) likelihood function $\Lambda_j(k)$. The key element of the IMM scheme is the interaction block that generates, using all local estimates, covariances, and likelihoods from the previous cycle, individual initial conditions for each of the primitive filters in the bank. The steps of the algorithm are summarized as follows.

a) Mixing Probabilities: For $i,j=1,\ldots,r$ compute

$$\begin{aligned}
\mu_{i|j}(k-1) &\triangleq \mathbb{P}\{\mathcal{M}_{k-1}=m_i \mid \mathcal{M}_k=m_j, Z_{k-1}\}\\
&=\frac{1}{c_j}p_{ij}\mu_i(k-1),
\end{aligned} \tag{7}$$

where c_j is a normalizing constant and $\mu_i(k) \triangleq \mathbb{P}\{\mathcal{M}_k=m_i \mid Z_k\}$.

b) Mixing Step: For $j=1,\ldots,r$ compute the initial state estimate for the filter matched to m_j

$$\hat{x}_j^0(k-1)=\sum_{i=1}^{r}\hat{x}_i(k-1)\mu_{i|j}(k-1) \tag{8}$$

and the corresponding covariances.

c) Mode-Matched Filtering: For $j=1,\ldots,r$, using (8) and the corresponding covariance compute the mode-matched estimate $\hat{x}_j(k)$ and $P_j(k)$ as well as the likelihood $\Lambda_j(k)$ which is approximated as Gaussian

$$\Lambda_j(k)=\mathcal{N}(z_k;\hat{z}_j(k),S_j(k)), \tag{9}$$

where $\hat{z}_j(k)$ and $S_j(k)$ are the predicted measurement and innovation covariance computed by the j-th filter using the initial conditions (8).

d) Mode Probability Update: For $j=1,\ldots,r$

$$\mu_j(k)=\frac{1}{c}\Lambda_j(k)\sum_{i=1}^{r}p_{ij}\mu_i(k-1), \tag{10}$$

where c is a normalization factor.

e) Output Computation: The algorithm output at time k is obtained as a fused version of the local estimates:

$$\hat{x}(k)=\sum_{j=1}^{r}\hat{x}_j(k)\mu_j(k). \tag{11}$$

The associated covariance is computed in a similar manner.

B. Mode Set Transformation

We have already observed that the system's natural mode sequence, $\{\mathcal{M}(k)\}$, is not a Markov process. However, the joint sequence $\{\mathcal{M}(k), N_k\}$ *is* Markov, as the following lemma states.

Lemma 2. *The joint sequence $\{\mathcal{M}(k), N_k\}$, with $\{N_k\}$ defined in* (3)*, is Markov with transition probability matrix having the following entries*

$$\begin{aligned}
&\mathbb{P}\{\mathcal{M}_{k+1}=m_j, N_{k+1}=\alpha \mid \mathcal{M}_k=m_i, N_k=\beta\}\\
&\quad=\begin{cases}
p_{ij}p_0, & \alpha=0,\ \beta=0\\
\delta_{ij}(1-p_0), & \alpha=1,\ \beta=0\\
0, & \alpha=0,\ \beta=1\\
\delta_{ij}, & \alpha=1,\ \beta=1
\end{cases}.
\end{aligned} \tag{12}$$

Proof: Rewrite the distribution of the supermode at time $k+1$ conditioned on the entire history as in (13), where the

$$
\begin{aligned}
&\mathbb{P}\left\{\mathcal{M}_{k+1}=m_j, N_{k+1}=\alpha \mid \mathcal{M}_k=m_i, N_k=\beta, \mathcal{M}_{k-1}=m_{i_1}, N_{k-1}=\beta_1, \ldots\right\} \\
&\quad=\mathbb{P}\left\{\mathcal{M}_{k+1}=m_j \mid N_{k+1}=\alpha, \mathcal{M}_k=m_i, N_k=\beta, \mathcal{M}_{k-1}=m_{i_1}, N_{k-1}=\beta_1, \ldots\right\} \\
&\qquad \times \mathbb{P}\left\{N_{k+1}=\alpha \mid \mathcal{M}_k=m_i, N_k=\beta, \mathcal{M}_{k-1}=m_{i_1}, N_{k-1}=\beta_1, \ldots\right\} \\
&\quad=\mathbb{P}\left\{\mathcal{M}_{k+1}=m_j \mid N_{k+1}=\alpha, \mathcal{M}_k=m_i\right\} \mathbb{P}\left\{N_{k+1}=\alpha \mid \mathcal{M}_k=m_i, N_k=\beta\right\}
\end{aligned} \tag{13}
$$

second transition follows from the distribution of the mode sequence $\{\mathcal{M}_k\}$ and the auxiliary Markov chain $\{N_k\}$. The right hand side of (13) depends on m_i, m_j, α, and β, rendering the joint sequence $\{\mathcal{M}(k), N_k\}$ Markov. The calculation of the TPM (12) is straightforward. ■

Equipped with Lemma 2 we construct an augmented mode set by defining a supermode as the pair $\{\mathcal{M}(k), N_k\}$. Transitions between different supermodes occur according to a Markovian dynamics, with TPM calculated in (12). Clearly, the number of modes in the augmented set is twice the original number of modes. This duplication is the cost of transforming the non-Markovian transition dynamics into an equivalent Markovian one.

Example 1. *To illustrate the structure of the augmented Markov chain consider again the case of $r = 2$. The state transition diagram depicted in Fig. 1 captures the transition dynamics of the augmented process. Notice that essentially the same set of primitive Kalman filters will be used for states corresponding to $N = 0$ and $N = 1$. This is obvious, since top and bottom states in Fig. 1 do not differ in their dynamical model, but rather by the fact that no more mode transitions will occur for $N = 1$.*

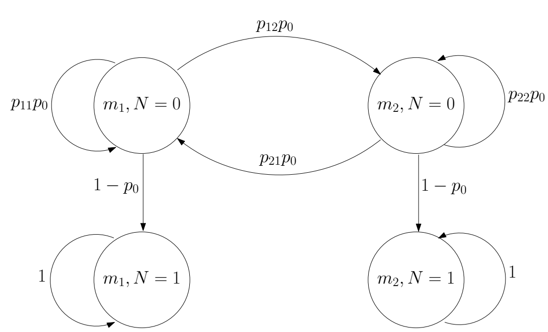


Figure 1: State transition diagram of the Markov chain for the augmented mode set of Example 1.

C. Discussion: Soft vs Hard Bounds on the Number of Mode Transitions

The proposed algorithm is closely related to the filter devised in [15], that tackles the problem of tracking a maneuvering target with a deterministically known hard bound on the number of possible maneuvers (mode transitions). Obviously, the dissimilarities between the two different problem formulations imply corresponding differences between the resulting estimation algorithms. We highlight here two such differences, that constitute two clear advantages of the novel formulation introduced herein over the one presented in [15].

The first advantage stems from the fact that the present formulation naturally and straightforwardly adapts to systems with an arbitrary number of dynamical models, corresponding to, e.g., different maneuvering regimes. In clear contradistinction, the approach of [15] can only handle the case of a system possessing just two dynamical models, corresponding to, e.g., the maneuvering and non-maneuvering states of the system. The obvious enhanced flexibility of the present approach is a direct consequence of using the IMM algorithm on the augmented Markovian model set, as the IMM algorithm can handle a model set of arbitrary size.

The second difference between the two approaches has to do with the system's mode initialization. The approach of [15] makes a rather restrictive assumption on the initial mode of the system, by constraining it to be deterministically known. Although several methods are proposed there to relax this assumption, they are either ad-hoc, or require unjustified increase in the method's computational complexity. The current algorithm, on the other hand, inherits again from the IMM algorithm its elegant treatment of the unknown initial conditions, by setting a prior probability distribution on the set of possible initial modes.

IV. Simulation Study

In this section we test the performance of the proposed algorithm and compare it with that of two existing algorithms: 1) the method for hard-bounded number of maneuvers, proposed in [15], and 2) the state-of-the-art IMM filter, adapted in an ad-hoc manner to the problem at hand. We first describe the experimental setup, consisting of the models m_1 and m_2, and the measurement generation model. We then present three examples, differing in their mode evolution mechanisms. In the first example, the mode sequence evolves according to the TPM of (12). In the second example, the mode sequence evolution follows the problem setting of the filter for hard-bounded number of maneuvers. Finally, we test the algorithm in a Markovian scenario, where the number of mode transitions is unbounded.

Simulation Setup

We consider the following state equation

$$x_{k+1} = A_k x_k + w_k, \tag{14}$$

describing the dynamics of a maneuvering target, where the state vector $x_k = [p_k \; v_k \; a_k]^T$ comprises the target's position, velocity, and acceleration, and $\mathrm{cov}(w_k) = Q_k$.

At time k, the pair $\{A_k, Q_k\}$ may assume one of two values, $m_1 = \{A^1, Q^1\}$ or $m_2 = \{A^2, Q^2\}$. In the (nominal) regime m_1, the system obeys the dynamics of the discrete white noise acceleration (DWNA) model [17], specified by the following matrices:

$$A^1 = \begin{pmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q^1 = \begin{pmatrix} T^4/4 & T^3/2 & 0 \\ T^3/2 & T^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sigma_{w_1}^2, \tag{15}$$

where $\sigma_{w_1}^2$ is some nominal process noise intensity. In the (abnormal) regime, m_2, the corresponding model is chosen to be the discrete Wiener process acceleration (DWPA) model [17], specified by the following matrices:

$$A^2 = \begin{pmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} T^4/4 & T^3/2 & T^2/2 \\ T^3/2 & T^2 & T \\ T^2/2 & T & T \end{pmatrix} \sigma_{w_2}^2, \tag{16}$$

where $\sigma_{w_2}^2$ is the abnormal process noise intensity.

The measurements are generated according to

$$z_k = H_k x_k + v_k, \tag{17}$$

where, irrespective of the current motion regime, $H_k = [1\ 0\ 0]$, and $\{v_k\}$ is a zero mean white Gaussian sequence with (constant) covariance $R_k = \sigma_v^2$.

Notice that, in the present case, at time k, the mode $\mathcal{M}_k = \{A_k, H_k, Q_k, R_k\}$, defined in Section II, has a deterministic part, $\{H_k, R_k\}$, and a stochastic one $\{A_k, Q_k\}$. Thus, for the sake of brevity, we shall refer to the subset $\{A_k, Q_k\}$ as the actual mode. The stochastic transition law between the possible mode values, m_1 and m_2, is described in the sequel.

In all examples below the following common parameters were used: $\sigma_{w_1} = 0.3\ \text{m/s}^2$, $\sigma_{w_2} = 6\ \text{m/s}^2$, $\sigma_v = 140$ m, $T = 10$ s. These parameters correspond to a maneuvering index $\lambda = \sigma_w T^2/\sigma_v$ of about 0.2 for the nonmaneuvering regime, and about 4.2 for the maneuvering one, meaning that the problem cannot be solved to a satisfactory level of accuracy using a single, non-adaptive KF (see, e.g., the discussion in [18]) and the use of adaptive filters of the multiple model variety is inevitable.

In addition, in all examples below, the initial state is $x_0 = [0\ 0\ 0]$ and P_0 is an all-zero matrix. Each filter is initialized in a perfect manner and the inial model is set to be the nonmaneuvering one.

Performance Evaluation

In all experiments in the sequel we generate a sequence of $N = 200$ state vectors using, at each time, one of the models m_1 or m_2 defined in the previous subsection. We use different transition mechanisms for generating transitions between the two modes.

In all cases we compare the performance of the proposed filter, termed here Soft Bound Filter (SBF), the filter of [15] for hard-bounded number of maneuvers, termed Hard Bound Filter (HBF), and a standard IMM adapted in an ad-hoc (yet reasonable) manner to the problem at hand, termed here A-H/IMM. In addition, we generate a "genie" Kalman filter, that possesses perfect information on the mode transitions times. The latter, ideal (but non-realistic), filter serves as an overall performance bound, indicating for each of the compared algorithms how far is it from the (theoretical and unachievable) optimal performance.

Three experiments are presented. In the first one, transitions are generated according to the TPM of the proposed filter defined in (12). In the second experiment, we generate the transitions according to the model assumed by the filter for hard-bounded number of maneuvers of [15]. Finally, we generate a random scenario according to a standard Markovian dynamics. We elaborate on each of the experiments next.

Example 1: In this experiment the transitions between m_1 and m_2 are generated according to the TPM of (12) with

$$p_0 = 0.98 \text{ and } p_{11} = p_{22} = 0.92, \tag{18}$$

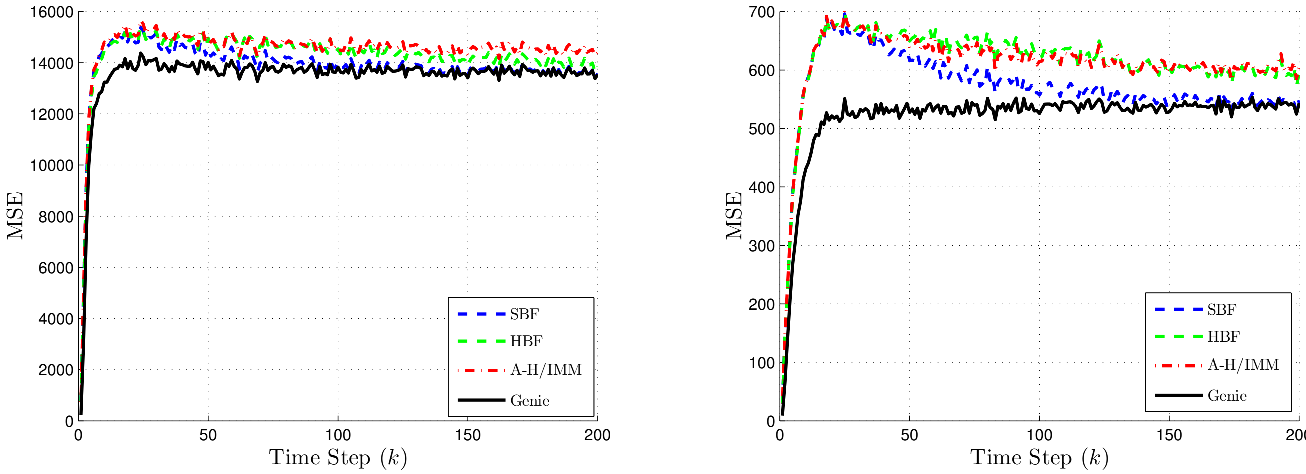
meaning that the expected time until the terminal transition is $\frac{1}{1-p_0} = 50$, the expected time between transitions (before the terminal transition) is $\frac{1}{1-p_{11}} = 12.5$, and the expected number of transitions (before the terminal transition is made) is $\frac{50}{12.5} = 4$.

To make a fair comparison, we set the maximal number of transitions for the hard-bounded algorithm to 4, and the probability for a mode switch to 0.08. Likewise, the IMM algorithm comprises two models, with $\left(\begin{smallmatrix} 0.92 & 0.08 \\ 0.08 & 0.92 \end{smallmatrix}\right)$ serving as the transition probability matrix.

The corresponding squared position and velocity errors of all three algorithms, averaged over $20,000$ independent Monte Carlo runs, are presented in Fig. 2.

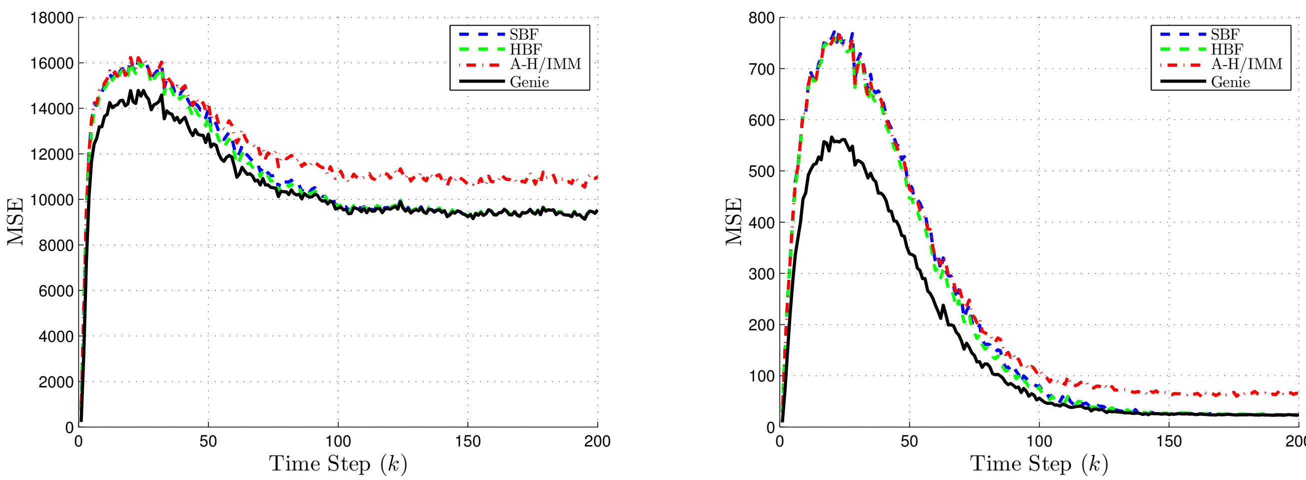
Not surprisingly, the proposed filter attains the best performance, almost coinciding with the "genie" KF in the second half of the time interval. Indeed, since, on average, the system is allowed to perform 4 transitions, after the corresponding time these are exhausted and a single model is the true one until the end of the scenario. Knowing this fact, the proposed filter generates its estimates using the correct model, thus attaining nearly optimal performance. The filter for hard-bounded number of transitions operates under a mismatched model regime, since the number of transition is 4 on average, whereas the true number may be larger or smaller. Thus, it is not surprising that its performance is inferior to the that of the SBF algorithm. However, since at the end the transitions are exhausted, with significant probability the HBF algorithm eventually follows the correct model, which results in good estimates. This explains the fact that in the second half of the interval the HBF attains slightly better performance than the ad hoc IMM filter, which, assuming Markovian transitions, operates here in a mismatched model regime.

Example 2: In this experiment the transitions between m_1 and m_2 are generated according to the modeling mechanism underlying the HBF algorithm, such that the system may perform exactly 4 mode transitions with probability 0.08 for a switch between models. The TPM of the proposed filter, as well as that of the ad hoc IMM filter, remain unchanged. The



(a) Position **(b) Velocity**

Figure 2: Squared position and velocity estimation errors of Example 1.



(a) Position **(b) Velocity**

Figure 3: Squared position and velocity estimation errors of Example 2.

corresponding squared position and velocity errors of all the algorithms, averaged over $20,000$ independent Monte Carlo runs, are presented in Fig. 3.

Operating under nominal conditions, the HBF algorithm attains superior performance, coinciding with the genie KF, in the second half of the interval, after all 4 transitions have been exhausted. Remarkably, the SBF algorithm performs only slightly worse than the HBF algorithm, though operating in non-nominal conditions. Specifically, the two filters nearly coincide at the beginning, before the maximum number of transitions has been performed (since the transition dynamics then looks the same for both), and towards the end, when both filters follow the true model. The ad hoc IMM filter operates with acceptable accuracy (relatively to HBF and SBF) as long as the system alternates between m_1 and m_2, since these alternations resemble then Markovian transitions, but, relatively to these two filters, its performance degrades significantly after the system follows a constant model (approximately after 50 time steps, which is the mean time for performing 4 mode transitions). The poor performance of the ad hoc IMM filter is explained by the fact that the filter is not aware of the fact that, after the hard bound on mode transitions has been achieved, no more transitions can occur; therefore, it maintains a mismatched primitive KF, thus increasing the overall MSE.

Note that in the present example, since $r_{\max}$ is even, the final target dynamics is dictated by the nonmaneuvering model. This explains the smaller steady-state errors obtained by all filters in comparison to the previous and the following examples.

Example 3: In the final example we test all algorithms in a true Markovian scenario, such that the underlying assumptions of both HBF and SBF do not hold. We generate a random scenario using the TPM of the IMM filter, such that transitions occur with probability 0.08. As in the previous cases, the HBF algorithm assumes that there are 4 possible transitions, and the SBF algorithm assumes that the terminal transition would occur within 50 time units on average. The corresponding squared position and velocity errors of all the algorithms, averaged over $20,000$ independent Monte Carlo runs, are presented in Fig. 4.

As could be expected, the ad hoc IMM filter attains the best performance, as it operates under precisely-matched, nominal conditions. Nevertheless, it is still significantly sub-optimal, as is observed by comparing its performance with that of the "genie" KF. The performance of the SBF algorithm appears to be robust, being only slightly inferior to that of the IMM. Interestingly, the HBF algorithm is much more sensitive to modeling errors, and develops significant errors after 50 time steps, since after that time, on average, unmodeled transitions occur. Notice that all filters perform equally well before $k = 50$. This is true because before 4 transitions have been performed, violations in the modeling assumptions cannot be sensed. This observation does not hold for Example 1, since there the terminating transition may occur after more or less than 4 transitions. Thus the SBF becomes superior before $k = 50$.

V. Conclusions

We have revisited the problem of tracking the state of a hybrid system characterized by a bounded number of mode transitions. Contrary to a previously addressed version of the problem, where the bound on the number of mode transitions was assumed to be deterministically known, in the present formulation we assume a soft, stochastic bound, permitting an unlimited number of mode transitions yet rendering this number finite with probability 1. The new formulation leaves intact the major problem addressed also in the previous formulation, in that, given the bound (whether deterministic or stochastic), the mode switching mechanism is non-Markovian. This precludes the use of state-of-the-art estimators devised for hybrid systems with Markovian mode sequences, such as those belonging to the GPB variety or the interacting multiple model (IMM) algorithm. The solution approach, adopted herein, is based on a novel redefinition of the system mode, leading to a more complex modal state space, but rendering the augmented mode sequence Markov. A standard IMM algorithm is then applied straightforwardly to the transformed problem, to yield a sub-optimal solution to the original problem with performance comparable to that of an IMM filter when applied to a Markov hybrid system.

The new formulation enjoys two advantages over the hard-bound version of the problem. First, the number of modes the system is permitted to possess is unlimited; in the hard-bound version the system could only possess two modes, e.g., nominal and anomalous. Second, whereas in the hard-bound formulation the system was assumed to deterministically start from a known mode, the present formulation removes this assumption and naturally addresses stochastic mode initialization (some solutions were proposed to alleviate this limitation in the hard-bound formulation, but these either were ad hoc or were associated with an unreasonable increase in computational complexity).

A simulation study demonstrates the performance of the new filter in three examples, comparing it with the filter for hard-bounded number of mode transitions and a standard IMM filter applied to the original (non-Markovian) system. It is shown that when applied to a system with a soft bound on the number of mode transitions, the new filter outperforms both alternatives and reaches almost optimal performance. Even when a hard bound is put on the number of transitions, the new algorithm performs nearly as good as the filter devised for hard-bounded number of mode switches. When the filters are applied to a system with truly Markov (unbounded) mode transitions, the new filter's performance closely approximates the performance of the IMM filter, and clearly outperforms the filter devised for a hard-bounded number of mode switches.

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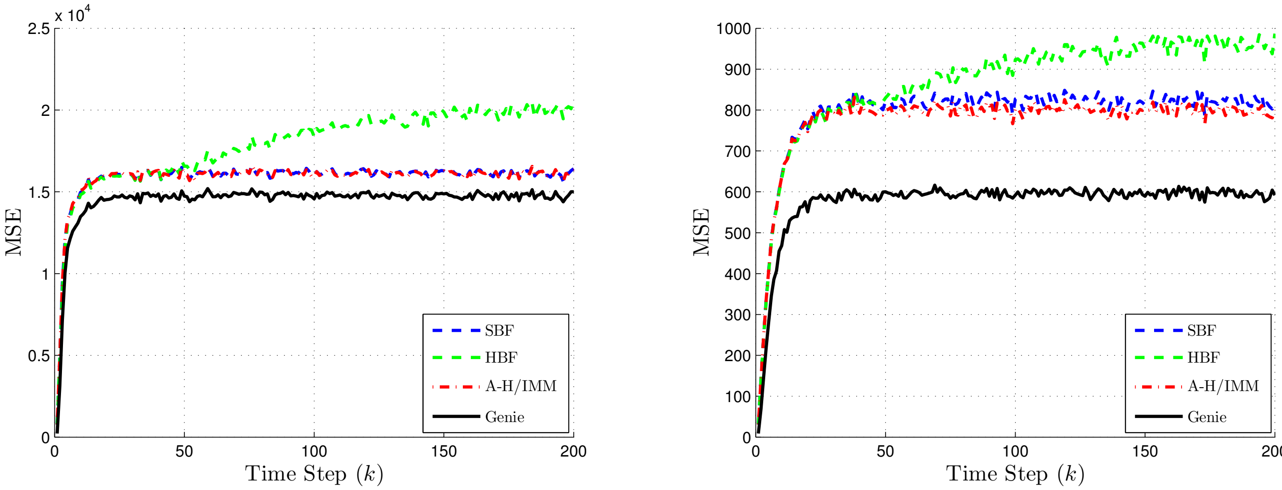

(a) Position **(b) Velocity**

Figure 4: Squared position and velocity estimation errors of Example 3.

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