The Big Bang, Mapping the Brain, and Geometry

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The big bang
Early Development of the Universe

Big Bang

Big Bang plus tiniest fraction of a second \((10^{-43})\)

Inflation

COBE Sky Map

Big Bang plus 300,000 years

Light from first galaxies

Big Bang plus 15 billion years
The Nobel Prize in Physics 2006

"for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation"

John C. Mather
NASA
b. 1946

George F. Smoot
Berkeley
b. 1945
DIRBE Solar Elongation 90° Maps: Mid-Infrared

4.9 µm

12 µm

25 µm

60 µm
DMR's Two Year CMB Anisotropy Result
Center for Astrophysics (CfA) survey

10,506 galaxies in the cone-shaped survey region, which extends out to 135 megaparsecs in the northern hemisphere, with the earth at the apex of the cone.
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Mapping the brain
Gross brain function

- Environment Mapping
- Object Recognition
- Voice Command Response
- Obistical Navigation
- Motion Control
- Input from Sensors
- Output to Controllers
Gender differences?
Gender differences?
Equipment: 1

Electroencephalogram (EEG)
Equipment: 1

Electroencephalogram (EEG)
Equipment: 2

Functional magnetic resonance imaging (fMRI)
Equipment: 2

Functional magnetic resonance imaging (fMRI)
Geometry
\[ \lambda_2 (\text{Tube}(M, \rho)) = \pi \rho^2 + \rho \times 4L + L^2 \]

\[ = \sum_{j=0}^{2} \omega_{2-j} \rho^{2-j} \mathcal{L}_j(M) \]

where

\[ \omega_j = \frac{\pi^{j/2}}{\Gamma\left(\frac{j}{2} + 1\right)} = \frac{s_j}{j} \]
\[ \lambda_3 \left( \text{Tube}(M, \rho) \right) \]

\[ = \frac{4}{3} \pi \rho^3 + 12 \cdot \frac{1}{4} \pi \rho^2 \cdot L + 6 \rho L^2 + L^3 \]

\[ = \sum_{j=0}^{3} \omega_{3-j} \rho^{3-j} L_j(M) \]
\[ \lambda_3 (\text{Tube}(M, \rho)) = \sum_{j=0}^{2} \omega_{3-j} \rho^{3-j} \mathcal{L}_j(M) \]
For nice (e.g. convex) $M \in \mathbb{R}^N$, and $N' \geq N$, the volume of

$$\text{Tube}(M, \rho) = \left\{ t \in \mathbb{R}^{N'} : d_{N'}(t, M) \leq \rho \right\}$$

is, for $\rho < \rho_c(M)$, given by,

$$\lambda_{N'}(\text{Tube}(M, \rho)) = \sum_{j=0}^{N} \omega_{N'-j} \rho^{N'-j} \mathcal{L}_j(M)$$

The $\mathcal{L}_j$ can be defined via the tube formula and are intrinsic.
\[ \lambda_{N'} (\text{Tube}(M, \rho)) = \sum_{j=0}^{N} \omega_{N'-j} \rho^{N'-j} L_j(M) \]
\[ \lambda_{N'}(\text{Tube}(M, \rho)) = \sum_{j=0}^{N} \omega_{N'-j} \rho^{N'-j} \mathcal{L}_j(M) \]

Wilhelm Killing  
Germany  
1847-1923

Rudolf Lipschitz  
Germany  
1832-1903
\[ \lambda_{N'} (\text{Tube}(M, \rho)) = \sum_{j=0}^{N} \omega_{N'-j} \rho^{N'-j} L_j(M) \]

Wilhelm Killing
Germany
1847-1923

Rudolf Lipschitz
Germany
1832-1903

Hermann Weyl
Germany/USA
1885-1951
Why convexity?
Sets with holes and negative contributions:

A bad example:
Sets with holes and negative contributions:

A bad example:

A good example:
\[ \lambda_3 \left( \text{Tube}(M, \rho) \right) \]

\[ = \frac{4}{3} \pi \rho^3 + 12 \cdot \frac{1}{4} \pi \rho^2 \cdot L + 6 \rho L^2 + L^3 \]

\[ = \sum_{j=0}^{3} \omega_{3-j} \rho^{3-j} \mathcal{L}_j(M) \]
Leonhard Euler
Switzerland
1707-1783

Jules Henri Poincaré
France
1854-1912
Dimension $N = 2$

4-5+2 = 1
5-8+4 = 1

12-20+8 = 0
16-35+18 = -1
Dimension $N = 3$

$5 - 9 + 7 - 2 = 1$
\( \mathcal{L}_0: \) The Euler-Poincaré characteristic

\[ M \subset \mathbb{R}^N \text{ is nice, of dimension } k, \text{ and } \text{“triangulisable”} \]
Ło: The Euler-Poincaré characteristic

$M \subset \mathbb{R}^N$ is nice, of dimension $k$, and “triangulisable”

$\alpha_0 = \text{number of vertices}$
\( \mathcal{L}_0: \) The Euler-Poincaré characteristic

\( M \subset \mathbb{R}^N \) is nice, of dimension \( k \), and “triangulisable”

\( \alpha_0 = \) number of vertices

\( \alpha_1 = \) number of lines
\( \mathcal{L}_0: \) The Euler-Poincaré characteristic

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\[ \cdots \]

\[ \cdots \]
$\mathcal{L}_0$: The Euler-Poincaré characteristic

$M \subset \mathbb{R}^N$ is nice, of dimension $k$, and “triangulisable”

$\alpha_0 = \text{number of vertices}$

$\alpha_1 = \text{number of lines}$

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$\alpha_k = \text{number of “full” simplices in the triangulation}$
$L_0$: The Euler-Poincaré characteristic

$M \subset \mathbb{R}^N$ is nice, of dimension $k$, and “triangulisable”

$\alpha_0 = \text{number of vertices}$
$\alpha_1 = \text{number of lines}$

$\alpha_k = \text{number of “full” simplices in the triangulation}$

$L_0(M) \equiv \text{Euler characteristic of } M \text{ is}$

$\varphi(M) = \alpha_0 - \alpha_1 + \cdots + (-1)^d \alpha_N$
3-d excursion sets

Meatball, EC=21
3-d excursion sets

Meatball, EC=21

Sponge, EC=-15
3-d excursion sets

Meatball, $EC = 21$

Sponge, $EC = -15$

Bubble, $EC = 1$
Random processes
The general structure

\[ f(t) = \sum_{k=1}^{\infty} \xi_k \varphi_k(t) \]

\( \xi_k \) are independent \( \mathcal{N}(0,1) \) \( \Rightarrow f \) is GAUSSIAN.
The general structure

\[ f(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t) \]

\( \xi_k \) are independent \( N(0, 1) \) \( \Rightarrow \) \( f \) is GAUSSIAN
Averaged geometry of excursion sets
Excursion sets

\[ \text{Excursion sets} \]

\[ A_u \equiv A_u(f, T) \Delta = \{ t \in M : f(t) \geq u \} \]
Excursion sets

\[ A_u \equiv A_u(f, T) \triangleq \{ t \in M : f(t) \geq u \} \]
Suppose \( f \) is Gaussian, mean zero, variance \( \sigma^2 \), stationary, and isotropic, with second spectral moment \( \lambda_2 \).
Then:

\[
E\{L_0(Au)\} = e^{-u^2/2\sigma^2}N\sum_{k=1}^{\infty} \left(\frac{Nk}{\lambda_k}\right)T_k \lambda_k^2 \left(\frac{2\pi}{k+1}\right)^{\frac{1}{2}} \sigma_k H_k - 1 \left(\frac{u\sigma}{\lambda_2}\right) + \Psi\left(\frac{u\sigma}{\lambda_2}\right).
\]
Suppose \( f \) is Gaussian, mean zero, variance \( \sigma^2 \), stationary, and isotropic, with second spectral moment \( \lambda_2 \). Then:

\[
\mathbb{E} \{ \mathcal{L}_0 (A_u) \} = e^{-u^2/2\sigma^2} \sum_{k=1}^{N} \frac{(N_k) T^k \lambda_2^k}{(2\pi)^{(k+1)/2} \sigma^k} H_{k-1} \left( \frac{u}{\sigma} \right) + \Psi \left( \frac{u}{\sigma} \right).
\]
Suppose $f$ is Gaussian, mean zero, variance $\sigma^2$, stationary, and isotropic, with second spectral moment $\lambda_2$. Then:

$$
\mathbb{E} \{ \mathcal{L}_0 (A_u) \} = e^{-u^2/2\sigma^2} \sum_{k=1}^{N} \frac{\binom{N}{k} T^k \lambda_2^{k/2}}{(2\pi)^{(k+1)/2} \sigma^k} H_{k-1} \left( \frac{u}{\sigma} \right) + \Psi \left( \frac{u}{\sigma} \right).
$$

where

$$
H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j! (n-2j)! 2^j}, \quad n \geq 0, \ x \in \mathbb{R}
$$

$$
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-x^2/2} \, dx
$$
One dimension: A line of length $T$

\[ \mathbb{E} \{ \mathcal{L}_0 (A_u(f, [0, T])) \} = \psi(u/\sigma) + \frac{T \lambda_2^{1/2}}{2\pi \sigma} e^{-u^2/2\sigma^2}, \]
Two dimensions: A square of side length $T$

\[
\left[ \frac{T^2 \lambda_2}{(2\pi)^{3/2}} u + \frac{2T \lambda_2^{1/2}}{2\pi} \right] e^{-u^2/2} + \Psi(u).
\]
Three dimensions: A cube of side length $T$

\[
\left[ \frac{T^3 \lambda_2^{3/2}}{(2\pi)^2} u^2 + \frac{3T^2 \lambda_2}{(2\pi)^{3/2}} u + \frac{3T \lambda_2^{1/2}}{2\pi} - \frac{T^3 \lambda_2^{3/2}}{(2\pi)^2} \right] e^{-u^2/2} + \Psi(u).
\]
About the proof

\[ f: M \rightarrow \mathbb{R} \]
About the proof

Marston Morse
1901-1977
U.S.A
A more general result

\[ A \equiv A \left( f, T \right) \Delta = \{ t \in M : f(t) \in D \} \]

\[ E \{ L_j(A_D) \} = N - j \sum_{l=0}^{\infty} \left[ j + l \left( 2\pi - j/2 \right) L_j + l \left( M_k \right) \right] \]
A more general result

\[ A_D \equiv A_D(f, T) \triangleq \{ t \in M : f(t) \in D \} \]
A more general result

\[ A_D \equiv A_D(f, T) \triangleq \{ t \in M : f(t) \in D \} \]

\[
\mathbb{E} \{ \mathcal{L}_j(A_D) \} = \sum_{l=0}^{N-j} \begin{bmatrix} j + l \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M) M^{(k)}_l(D)
\]
Gaussian tube formula

Gauss measure on $\mathbb{R}^k$

$$\gamma_k(A) \triangleq \frac{1}{(2\pi)^{k/2}} \int_A e^{-\|x\|^2/2} \, dx$$
Gaussian tube formula

Gauss measure on $\mathbb{R}^k$

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Gauss measure on $\mathbb{R}^k$

$$\gamma_k(A) \triangleq \frac{1}{(2\pi)^{k/2}} \int_A e^{-\|x\|^2/2} \, dx$$

Gaussian tube formula

$$\gamma_k(\text{Tube}(M, \rho)) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} M_j^\kappa(M)$$
Gaussian tube formula

Gauss measure on $\mathbb{R}^k$

\[ \gamma_k(A) \triangleq \frac{1}{(2\pi)^{k/2}} \int_A e^{-\|x\|^2/2} \, dx \]

Gaussian tube formula

\[ \gamma_k(\text{Tube}(M, \rho)) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^\gamma(M) \]

\[ \mathbb{E}\{\mathcal{L}_j(A_D)\} = \sum_{l=0}^{N-j} \binom{j + l}{l} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M) \mathcal{M}_l^{(k)}(D) \]
Kinematic fundamental formula
Kinematic fundamental formula

\[ \int_{L_i} (M_1 \cap g M_2) \, d\nu(g) = N - \sum_{j=0}^{i} [i+j] N_j - 1 \]

\[ L_{i+j}(M_1) \cap N_{i-j}(M_2) \]
Kinematic fundamental formula

\[ \int \mathcal{L}_i (M_1 \cap gM_2) \, dv(g) = \sum_{j=0}^{N-i} \begin{bmatrix} i + j \end{bmatrix} \begin{bmatrix} N \end{bmatrix}^{-1} \mathcal{L}_{i+j}(M_1) \mathcal{L}_{N-j}(M_2) \]
A more general result

\[ A_D \equiv A_D(f, T) \triangleq \{ t \in M : f(t) \in D \} \]

\[ \mathbb{E}\{\mathcal{L}_j(A_D)\} = \sum_{l=0}^{N-j} \binom{j + l}{l} (2\pi)^{-j/2} \mathcal{L}_{j+l}(M) \mathcal{M}_l^{(k)}(D) \]
The Big Bang
and
Mapping the Brain
Observed and expected EC for the PET data and the expected EC if there is no activation due to the linguistic task. In particular, at \( u = 3.3 \) we expect an EC of 1, but we observe 4. At the 5% critical value of \( u = 4.22 \), we expect 0.05 but we observe 2 components.
Center for Astrophysics (CfA) survey

10,506 galaxies in the cone-shaped survey region, which extends out to 135 megaparsecs in the northern hemisphere, with the earth at the apex of the cone.
The observed EC of the set of high-density regions of the CfA Galaxy survey. Also shown is the expected EC for randomly distributed galaxies with no structure; the CfA data has smaller EC than expected, indicating less “blobs” and more clumping of galaxies into clusters, strings, and “walls”.
DMR's Two Year CMB Anisotropy Result
In this case, the search region is the surface of a unit sphere, so the volume is zero, the surface area is $8\pi$, the caliper diameter is zero, and the EC of the search region is 2 (because it is a hollow sphere, like a tennis ball). We can use these results to calculate the expected EC and to plot it and the observed EC against threshold level.

![Plot of the observed EC of excursion sets of the anomalies in the cosmic microwave background radiation (jagged line), and the expected EC from the formula (smooth line) if there are no real anomalies.](image)

**Figure 12.** Plot of the observed EC of excursion sets of the anomalies in the cosmic microwave background radiation (jagged line), and the expected EC from the formula (smooth line) if there are no real anomalies. The observed microwave background radiation produces an EC curve similar in shape to that expected, but somewhat lower and spread more in the tails—evidence that some of the anomalies are real and not just due to random noise. This discrepancy points to a Gaussian random field model for the anomalies, with a larger standard deviation and a larger smoothness than the background noise.
Excursion probabilities

\[ \mathbb{P} \{ \sup_{t \in M} f(t) \geq u \} \]
Excursion probabilities

\[ \mathbb{P} \left\{ \sup_{t \in M} f(t) \geq u \right\} \sim \mathbb{E} \left\{ \mathcal{L}_0 (A_u(f, M)) \right\} \]
Excursion probabilities

\[ \mathbb{P} \left\{ \sup_{t \in M} f(t) \geq u \right\} \]

\[ \sim \quad \mathbb{E} \left\{ \mathcal{L}_0 \left( A_u(f, M) \right) \right\} \]

\[ \liminf_{u \to \infty} u^{-2} \log |\mathbb{P} - \mathbb{E}| \geq \frac{1}{2} + \frac{1}{2\sigma^2(f)} \]
This monograph is devoted to a completely new approach to geometric problems arising in the study of random fields. The groundbreaking material in Part III, for which the background is carefully prepared in Parts I and II, is of both theoretical and practical importance, and striking in the way in which problems arising in geometry and probability are beautifully intertwined.

The three parts to the monograph are quite distinct. Part I presents a user-friendly yet comprehensive background to the general theory of Gaussian random fields, treating classical topics such as continuity and boundedness, entropy and majorizing measures, Borell and Slepian inequalities. Part II gives a quick review of geometry, both integral and Riemannian, to provide the reader with the material needed for Part III, and to give some new results and new proofs of known results along the way. Topics such as Crofton formulae, curvature measures for stratified manifolds, critical point theory, and tube formulae are covered. In fact, this is the only concise, self-contained treatment of all of the above topics, which are necessary for the study of random fields. The new approach in Part III is devoted to the geometry of excursion sets of random fields and the related Euler characteristic approach to extremal probabilities.

Random Fields and Geometry will be useful for probabilists and statisticians, and for theoretical and applied mathematicians who wish to learn about new relationships between geometry and probability. It will be helpful for graduate students in a classroom setting, or for self-study. Finally, this text will serve as a basic reference for all those interested in the companion volume of the applications of the theory. These applications, to appear in a forthcoming volume, will cover areas as widespread as brain imaging, physical oceanography, and astrophysics.