Gaussian processes, kinematic formulae and Poincaré's limit

Jonathan E. Taylor
Stanford University & Université de Montréal

Robert J. Adler
Technion

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Abstract

We consider vector valued, unit variance Gaussian processes y defined over piecewise C^2 stratified manifolds M and consider the geometry of their (random) excursion sets $M \cap y^{-1}D$ for D a stratified subset of Euclidean space. In particular, we develop an explicit formula for the expectation of all the Lipschitz-Killing curvatures of these sets. This formula has an interpretation as a version of the classic kinematic fundamental formula of Integral Geometry, in which integration over the isometry group with respect to Haar measure is replaced by integration over a function space with respect to an appropriate Gaussian measure.

Particularly novel is the method of proof, based on approximating the Gaussian processes by processes on spheres, the orthonormal expansions of which have (random) coefficients on the n-sphere. The $n \to \infty$ limit is handled via recent extensions of the classic Poincaré limit theorem.

1 Introduction

Our problem lies somewhere between the realms of Probability Theory and Integral and Differential Geometry, although one of its primary motivations comes from Applied Statistics.

It lies in Probability Theory because the basic objects with which we shall work are $\mathbb{R}^k$ valued random processes f, defined over parameter spaces M.

The geometry enters in two ways. The first is via M, which we shall take to be a Whitney stratified manifold satisfying some side conditions. The second is that we shall primarily be concerned with the (random) *excursion sets*

$$A(f, M, D) \ \stackrel{\Delta}{=} \ \{t \in M : \ f(t) \in D\}, \tag{1.1}$$

where D is also a suitable Whitney stratified submanifold, this time in $\mathbb{R}^k$.

Our aim is to study the global geometry of excursion sets, as measured through their Lipschitz-Killing curvatures, $\mathcal{L}_j(A(f, M; D))$, $j = 0, \dots, \dim(M)$. In particular, since these curvatures are random variables, we shall be interested in computing their expectations.

We cannot do this for all f. For a start, both M and f will both have to be smooth enough (C^2 with some side conditions will suffice) for basic Differential Geometric techniques to be applicable. Furthermore, we shall need some distributional assumptions on f. As a first step, we take $f \equiv y = (y_1, \dots, y_k) : M \to \mathbb{R}^k$ to be a random process, the components of which are independent, identically distributed (hereafter i.i.d.) centered Gaussian processes of constant variance, which we take to be 1. For such a y we shall prove that

$$\mathbb{E}\left\{\mathcal{L}_i\left(A(y,M,D)\right)\right\} \;=\; \sum_{j=0}^{\dim M-i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M)\, \mathcal{M}_j^{\gamma}(D), \tag{1.2}$$

where the combinatorial flag coefficients are defined below at (6.8) and the $\mathcal{M}_j^{\gamma}$, defined in Section 9.3, are certain (Gaussian) Minkowski functionals that, to a certain extent, play the rôle of Lipschitz-Killing curvatures in Gauss space. The Lipschitz-Killing curvatures on both sides of (1.2) are computed with respect to a specific Riemannian metric induced on M by the component processes y_j. Note, however, that $\mathcal{L}_0(A)$ is the Euler-Poincaré characteristic of A, and so independent of any Riemannian structure. (cf. Theorem 5.1 for a formal statement of (1.2).)

The general structure of (1.2) has significant implications for a class of problems out of the purely Gaussian scenario. Taking $F : \mathbb{R}^k \to \mathbb{R}$ to be piecewise C^2, again along with appropriate side conditions, and defining a (now non-Gaussian) process

$$f(t) \;=\; F(y(t)) \;=\; F(y_1(t), \dots, y_k(t)), \tag{1.3}$$

with y Gaussian as above, it follows immediately from (1.2) that

$$\mathbb{E}\left\{\mathcal{L}_i\left(A_u(f,M)\right)\right\} \;=\; \sum_{j=0}^{\dim M-i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M)\, \mathcal{M}_j^{\gamma}(F^{-1}[u,+\infty)), \tag{1.4}$$

where, for a real valued f,

$$A_u(f,M) \;\stackrel{\Delta}{=}\; A(f,M,[u,\infty)) \;=\; \{t \in M : f(t) \geq u\}. \tag{1.5}$$

Non-Gaussian processes of the form (1.3) appear naturally in a wide variety of statistical applications of smooth random fields (e.g. [2, 3, 4, 23, 27, 28, 29] with an excellent introductory review in [30]) but in this paper we want to place a different, inherently geometric, interpretation on the central result (1.2).

Recall the Kinematic Fundamental Formula (henceforth KFF) of Integral Geometry, which, in its simplest form, states that for nice subsets M_1 and M_2 of $\mathbb{R}^n$,

$$\int_{G_n} \mathcal{L}_i\left(M_1 \cap g_n M_2\right)\, d\nu_n(g_n) \;\;=\;\; \sum_{j=0}^{n-i} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}^{-1} \mathcal{L}_{i+j}(M_1)\mathcal{L}_{n-j}(M_2), \tag{1.6}$$

where G_n is the isometry group of $\mathbb{R}^n$ with Haar measure ν_n normalised so that, for any $x \in \mathbb{R}^n$ and any Borel $A \subset \mathbb{R}^n$,

$$\nu_n\left(\{g_n \in G_n : g_n x \in A\}\right) \;=\; \mathcal{H}_n(A), \tag{1.7}$$

where $\mathcal{H}_n$ is n-dimensional Hausdorff measure. (See [16, 22] for M_j elements of the convex ring or similar, and [7] for more esoteric M_j in the spirit of this paper.)

Now reconsider (1.2). Taking $(\Omega, \mathcal{F}, \mathbb{P})$ as the probability space on which y lives, (1.2) can be rewritten as

$$\int_\Omega \mathcal{L}_i\left(M \cap (y(\omega))^{-1} D\right)\, d\mathbb{P}(\omega) \;=\; \sum_{j=0}^{\dim M - i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M)\, \mathcal{M}_j^\gamma(D). \tag{1.8}$$

Written this way, it is clear on comparing (1.6) and (1.8) that our main result can now be interpreted as a KFF over Gaussian function space, rather than over the isometry group on Euclidean space. We find this interpretation novel and intruiging, bridging as it does between a probabilistic problem with a geometric answer of classic form.

The result (1.2) has a long history. Indeed, if M is the simple interval $[0,1]$, y is real valued and stationary, and $D = [u, \infty)$, then (1.2) is essentially the famous Rice formula, which gives the mean number of upcrossings of the level u by f, and dates back to 1954 [20]. Since 1954 there have been tens, if not hundreds, of papers extending the original Rice formula in many ways, with the developments up until 1980 summarised in [1]. More recently, there was a series of papers by Worsley (eg. [27, 28, 29, 31]) that were important precursors to the general theory presented in this paper. However, the first precursor to (1.2), at the level of processes over manifolds, appeared only in 2002 in [25], where we considered only the first Lipschitz-Killing curvature $\mathcal{L}_0(A_u(f, M))$ and then only for real valued y. In [24] one of us (JET) extended this to vector valued y, which allowed for the derivation of the far more general, and far more beautiful, (1.2), although in [24] the manifold M was required to have a C^2 boundary and only the case of $\mathcal{L}_0$ was treated. The main technical difference between these two earlier papers and the current one, therefore, is the extension to all Lipschitz-Killing curvatures and to processes over more general geometric structures. Far more significant, however, is the difference in the method of proof. The proofs in the current paper are new, and far more geometric than the earlier ones. In particular, the proof in [24] progressed primarily by evaluating both sides of (1.2) and then showing that they were equivalent. The current proof starts on the left hand side and, eventually, yields the right hand side. The geometric nature of the current proof also explains *why* the two sides *should* be equal.

The proof of (1.2) is, unfortunately, not short and involves a wider collection of techniques than one might expect given the simplicity of the problem and the conciseness of the answer. Nevertheless, we found it rather interesting in its own right lying, as it does, somewhere between

Probability and Geometry. In particular, we found the Gaussian Crofton formula of Theorem 11.1, which is new and seems to have no predecesor beyond the much simpler classical Crofton formula, especially interesting, also outside of the context of the current paper.

2 A plan of action

This paper is, in essence, divided into three parts. We shall start the first, in Section 3, with setting the geometric basis for our results, by defining what class of manifolds M we shall be considering, and then defining the Lipschitz-Killing curvatures $\mathcal{L}_j$ and Gaussian Minkowski functionals $\mathcal{M}_j^\gamma$ in Section 4. In Section 5 we shall discuss Gaussian random processes, y, on manifolds and introduce a natural Riemannian metric g that y induces on its parameter space. Once this is done, then at very least all the terms in (1.2) will be clearly defined, and we shall be able to state the formal result as Theorem 5.1. A reader interested in the result but not the proof need read no further.

The second part of the paper covers Sections 6–9, in which we prove the main result of the paper for a specific rotationally invariant process restricted to submanifolds of the n-sphere. This is the most interesting part of the paper, and contains not only new results but, more importantly, a conceptually new way of looking at things.

However, even for this specific choice of process, chosen primarily for its simplicity, the computations are far from trivial and so we approach them in an indirect fashion. In particular, we shall start by looking at non-Gaussian processes with finite expansions and with coefficients coming from random variables distributed uniformly over high dimensional spheres. This will enable us to take expectations using an appropriate version of the KFF on spheres. These preliminary expectations are computed in Section 7, after we formulate the requisite KFF in Section 6.

The passage from this scenario to the Gaussian one will be via a limit theorem for projections of uniform random variables on the n-sphere as $n \to \infty$, historically associated with the name of Poincaré, although we shall need a slight extension of some more recent versions due to Diaconis and Freedman [9] and their generalisations to matrices in [10]. Poincaré's result is also given in Section 6 and applied to our setting in Section 9, after first seeing how it works in the case of the Euler-Poincaré characteristic, $\mathcal{L}_0$, and for real valued y, in Section 8. It is only in Section 9, however, that we shall finally see how the $\mathcal{M}_j^\gamma$ arise in (1.2) in arbitrary dimensions.

The third part of the paper commences in Section 10, where we shall claim that mean Lipschitz-

Killing curvatures of excursion sets must be of the form

$$\mathbb{E}\left\{\mathcal{L}_i\left(A(y,M,D)\right)\right\} \;=\; \sum_{j=0}^{\dim M-i} \mathcal{L}_{i+j}(M)\,\widetilde{\rho}\,(i,j,D)\,, \tag{2.1}$$

where $\widetilde{\rho}$ depends on all the parameters displayed, but not on the distribution of the underlying Gaussian processes y_j. (The distribution of y does, however, enter into this equation, since all the Lipschitz-Killing curvatures are computed with respect to the Riemannian metric g that it induces on M.) With (2.1) established we argue that the general result (1.2) then follows from a combination of (2.1) and the special case of Section 9.

In Section 11 we make the argument that it actually suffices to establish a result like (2.1) when $i = 0$; i.e. for the Euler-Poincaré characteristic $\mathcal{L}_0$. This claim is based on a new version of the classical Crofton formula, which shows how to derive information about Lipschitz-Killing curvatures from their 'cross-sections' with *random* manifolds. As noted above, we believe this result to be of interest well beyond our use of it. With this result in hand, we give a prove of (2.1) in Section 12. The proofs in Sections 10–12 are not as full as those in the second part of the paper, for two reasons. The main one is that many of the arguments here rely on perturbations of previous arguments in the study of Gaussian fields, and so, unlike those in the preceeding sections, are not conceptually new. The second is that to give them in full would probably double the length of the paper.

Finally, in Sections 13 and 14 we establish some technical lemmas required to complete some of the earlier proofs.

3 Stratified manifolds

We have two main aims in this section. The first is to set up notation for, and to be more specific about, the stratified manifolds M that appeared in the Introduction. The second aim is to say a few words about Morse theory.

3.1 Stratified manifolds

We start with what are known as Whitney stratified spaces, for which our basic references are Goresky and MacPherson [13] and Pflaum [19], and for which we need a C^k ambient manifold $\widetilde{M}$. Then $M \subset \widetilde{M}$ is called a C^l, $l \le k$, stratified manifold if there exists a partition $\mathcal{Z}$ of M such that each piece, or stratum, $S \in \mathcal{Z}$ is an embedded C^l submanifold of $\widetilde{M}$, without boundary, and for $R, S \in \mathcal{Z}$, if $R \cap \overline{S} \neq \emptyset$ then $R \subset \overline{S}$. In such a case R is said to be incident to S. There is a natural partial order on $\mathcal{Z}$, namely $R \preceq S$ if $R \subset \overline{S}$.

We shall usually collect all strata of dimension j together, and write their union as $\partial_j M$, $0 \leq j \leq \dim(M)$, so that we can write M as the disjoint union

$$M = \bigcup_{j=0}^{\dim M} \partial_j M. \tag{3.1}$$

In a chart (U, φ) on $\widetilde{M}$ containing t, a stratified space $(M, \mathcal{Z})$ is said to satisfy Whitney condition (B) at $t \in S$ if the following is satisfied for every $S \preceq \widetilde{S}$:

(B) Let $t_n \to t$ and $s_n \to t$ be such that $t_n \in S, s_n \in \widetilde{S}$, for all n. Furthermore, suppose that the sequence of line segments $\overline{\varphi(t_n)\varphi(s_n)}$ converges in projective space to a line ℓ and the sequence of tangent spaces $T_{s_n}\widetilde{S}$ converges in the natural Grassmannian to a subspace $\tau \subset T_t\widetilde{M}$. Then, $\varphi_*^{-1}(\ell) \subset \tau$. A limit of tangent spaces $T_{s_n}\widetilde{S}$ is called a *generalised tangent space*.

A stratified manifold $(M, \mathcal{Z})$ is called a *Whitney stratified manifold* if it satisfies Whitney condition (B) at every $t \in M$. These are the basic manifolds with which we shall work, although we shall also require a little more regularity, obtained by generalising somewhat the 'cone' structure of [19], who has the following definition for the case $m = 0$ (i.e. for homeomorphisms).

Definition 3.1 *Let $M \subset \widetilde{M}$ be a C^l stratified manifold with stratification $(\mathcal{Z}, S)$, with l a non-negative integer. Then M is said to be a cone manifold of class C^l and depth 0 if it is the topological sum of countably many connected C^l manifolds (without boundary), the strata S of which are the unions of connected components of equal dimension.*

M is said to be a cone manifold of class $C^{l,m}$ ($m \geq 0$) and depth $d+1$ ($d \geq 0$) if every $t \in S \subset M$ has a neighbourhood $U \subset \widetilde{M}$ such that $U \cap M$ is C^m diffeomorphic to $(U \cap S) \times \mathrm{Cone}(L_S)$, where L_S is a compact C^l cone manifold of depth d, and $\mathrm{Cone}(L_S)$ denotes the cone over L_S.

Our next step will be to assume that the ambient manifold $\widetilde{M}$ is equipped with a Riemannian metric $\widetilde{g}$, and that the $\partial_j M$ in the stratification (3.1) of M inherit the metrics induced by $\widetilde{g}$. We shall always write the induced metrics as g (although they should actually depend on j) and talk of the stratified Riemannian manifold (M, g).

3.2 Morse functions and indices

We shall need two indices, both related to the local geometry of a stratified Riemannian manifold M. The first is a measure of local change in the topology of the manifold, and is known as the *normal Morse index*. It is denoted by $\alpha : M \times T_t^{\perp} \to \mathbb{Z}$, where, for $t \in M$, $T_t^{\perp}$ is the orthogonal complement to $T_t M$ in $T_t\widetilde{M}$. In particular $\alpha(t, \nu_t)$, or simply $\alpha(\nu_t)$ measures change in the direction ν_t, $t \in M$. For a more information see either [13] or [3].

If the support cone, $\mathcal{S}_t M$, the local linearization of M at t, is convex for every $t \in M$ we call M *locally convex*. For locally convex M, the *normal cone* of M at t is defineds as

$$N_t M \;\stackrel{\Delta}{=}\; \left\{X_t \in T_t\widetilde{M} : \widetilde{g}(X_t, Y_t) \le 0 \;\text{ for all } Y_t \in \mathcal{S}_t M\right\}, \tag{3.2}$$

where $T_t\widetilde{M}$ is the tangent space to $\widetilde{M}$ at t. For locally convex stratified manifolds, $\alpha(\nu_t) = \mathbb{1}_{N_t M}(-\nu_t)$.

To define the second type of index, we need first to talk about critical points of functions on stratified Riemannian manifolds. We shall therefore now assume that $(\widetilde{M}, \widetilde{g})$ is a C^3 manifold, and that (M, g) is piecewise C^2. If $\widetilde{f} \in C^2(\widetilde{M}, \mathbb{R})$, a *critical point* of $\widetilde{f}$ is a point $t \in \widetilde{M}$ such that $\widetilde{\nabla} \widetilde{f}_t = 0$, where $\widetilde{\nabla}$ is the Riemannian gradient determined by $\widetilde{g}(\widetilde{\nabla}\widetilde{f}_t,\, X_t) = X_t f_{|_t}$ for all $t \in \widetilde{M}$.

If $f = \widetilde{f}_{|\partial_j M}$ is the restriction of $\widetilde{f}$ to a stratum of M, then it is easy to see that $t \in \partial_j M$ will be a critical point of the restricted function if, and only if, $\widetilde{\nabla}\widetilde{f}_t \in T_t^{\perp}\partial_j M$. We call the set

$$\bigcup_{j=0}^{\dim M} \left\{t \in \partial_j M : \widetilde{\nabla}\widetilde{f}_t \in T_t^{\perp}\partial_j M\right\}$$

the *set of critical points of* $f_{|M}$. All other points are known as *regular* points.

A critical point $t \in \partial_j M$ of $f_{|M}$ is called *non-degenerate* if the covariant Hessian, $\nabla^2 f\big|_{\partial_j M}$, of the restriction of f to $\partial_j M$ is non-degenerate, when considered as a linear mapping from $T_t\partial_j M$ to itself. A function $\widetilde{f} \in C^2(\widetilde{M})$ is said to be *non-degenerate* on M if the critical points of $\widetilde{f}_{|M}$ are all non-degenerate.

We can now define the second of our indices, the *tangential Morse index*

$$\iota_{f,\partial_j M}(t) \tag{3.3}$$

of a non-degenerate critical point $t \in \partial_j M$ of f as the dimension of the largest subspace L of $T_t\partial_j M$ such that $\nabla^2 f(t)\big|_L$ is negative definite. For more on Morse Theory, see Section 14.

3.3 Regular stratified manifolds

We have avoided one technicality in the above discussion, related the generalised tangent spaces appearing in the Whitney condition (B). We shall say that a vector ν *annihilates* a generalised tangent space of M at t if $\widetilde{g}(\nu, X) = 0$ for all X in the space. Vectors ν for which there exist such generalised tangent spaces are called *degenerate tangent vectors* while all others are *non-degenerate.*

Since there can be problems in defining the normal Morse index for degenerate vectors we adopt a condition which ensures that this can not happen. We call this tameness, somewhat abusing a similar, but slightly less restrictive terminology that already exists (e.g. ([7]).

Definition 3.2 *If C a positive integer, then a closed stratified manifold M embedded in an ambient manifold $\widetilde{M}$ is said to be 'C-tame', or simply 'tame' if it satisfies the Whitney condition (B) as well the following two conditions:*

(i) If S is a stratum of M, then the set $\{\lim_{t_n\to t} T_{t_n}S : t \in \partial S\}$ of all generalised tangent spaces coming from S has Hausdorff dimension less than $\dim(S)$ in the appropriate Grassmanian.

(ii) Wherever the normal Morse index is defined, we have $|\alpha(\nu_t; M)| \ \leq \ C$.

We now define the class manifolds with which we shall work from now on.

Definition 3.3 *Let M be an orientable C^2 Whitney stratified manifold, embedded in an ambient manifold $\widetilde{M}$. Assume that M is also a $C^{2,1}$ cone manifold of arbitrary depth and that M is C-tame for some finite C. Then M is called a regular stratified manifold.*

4 Curvature integrals and tubes

Perhaps the easiest way to meet Lipschitz-Killing curvatures is via Weyl's tube formula for embedded submanifolds of $\mathbb{R}^l$, which states that, for sufficiently small $\rho \geq 0$, there exist numbers $\mathcal{L}_j(M)$ for which

$$\mathcal{H}_l\left(\text{Tube}(M,\rho)\right) \ = \ \sum_{i=0}^{\dim M} \rho^{l-i}\omega_{l-i}\mathcal{L}_i(M), \tag{4.1}$$

where, for M belonging to a metric space (S,d), $\text{Tube}(M,\rho) \stackrel{\Delta}{=} \{t \in S : \inf_{s\in M} d(s,t) \leq \rho\}$, and ω_m is the volume of the unit ball in $\mathbb{R}^m$ (cf. (6.8)).

Weyl's formula actually defines the $\mathcal{L}_j(M)$ for this Euclidean case for all $0 \leq j \leq \dim(M)$, and we define $\mathcal{L}_j(M) \equiv 0$ for all $j > \dim(M)$. However, they can also be computed directly, with an intrinsic representation that defines them for more general stratified manifolds, as $\mathcal{L}_j(M) \equiv$

$\mathcal{L}_j(M,M)$, where the *Lipschitz-Killing measures* $\mathcal{L}_j(M,\cdot)$ are defined by

$$\begin{aligned}\mathcal{L}_i(M,A) &= \sum_{j=i}^{N}(2\pi)^{-(j-i)/2}C(l-j,j-i) \\ &\quad\times\int_{\partial_j M\cap A}\int_{S(\mathbb{R}^{l-j})}\frac{1}{(j-i)!}\mathrm{Tr}^{T_t\partial_j M}(S_\eta^{j-i})\alpha(\eta)\,\mathcal{H}_{l-j-1}(d\eta)\mathcal{H}_j(dt).\end{aligned} \tag{4.2}$$

Here S is the scalar second fundamental form of $\partial_j M$ as it sits in $\mathbb{R}^l$. The constants $C(m,i)$ are given by

$$C(m,i) \triangleq \begin{cases}\frac{(2\pi)^{i/2}}{s_{m+i}} & m+i>0,\\ 1 & m=0,\end{cases}$$

where s_n is the surface measure of $S(\mathbb{R}^n)$ (cf. (6.8)). For a proof see, for example, [3, 14, 26].

The form of (4.2) motivates the following definition of Lipschitz-Killing (signed) measures on an N-dimensional regular stratified manifold embedded in a Riemannian manifold $(\widetilde{M},\widetilde{g})$ as

$$\begin{aligned}\mathcal{L}_i(M,A) &= \sum_{j=i}^{N}(2\pi)^{-(j-i)/2}\sum_{m=0}^{\lfloor\frac{j-i}{2}\rfloor}C(N-j,j-i-2m)\frac{(-1)^m}{m!\,(j-i-2m)!} \\ &\quad\times\int_{\partial_j M\cap A}\int_{S(T_t\partial_j M^\perp)}\mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m\widetilde{S}_\eta^{j-i-2m}\right)\alpha(\eta)\,\mathcal{H}_{N-j-1}(d\eta)\mathcal{H}_j(dt),\end{aligned} \tag{4.3}$$

where $\widetilde{R}$ is the Riemannian curvature operator on $(\widetilde{M},\widetilde{g})$, $\widetilde{S}$ the corresponding scalar second fundamental form of $\partial_j M$ as it sits in $\widetilde{M}$, and $\mathrm{Tr}^{T_t\partial_j M}$ is the trace in $T_t\partial_j M$. The Hausdorff measures are now the volume measures induced by the Riemannian metric. Note that, on the innermost integral in (4.3), $S(T_t\partial_j M^\perp)$ is actually empty when $j=M$. In this case, as will always be the case when we evaluate α on vectors 'belonging to' the empty space, we take $\alpha\equiv 1$. We shall assume now and throughout the paper that Lipschitz-Killing measures are finite. For more details, such as a proof of the fact that $\mathcal{L}_j$ are independent of the stratification, see [6].

Finally, we note that the fact that $\mathcal{L}_0(M)$ is equivalent to the *Euler-Poincaré characteristic* of M, and so independent of any Riemannian structure, is the celebrated Chern-Gauss-Bonnet Theorem. The remaining Lipschitz-Killing curvatures also appear under a variety of other names, such as Quermassintegrales, Minkowski, Dehn and Steiner functionals, and invariant volumes, although in many of these cases the ordering and normalisations are different.

Having determined the meaning of the $\mathcal{L}_j(M)$ of the main result (1.2), we now turn to the $\mathcal{M}_j(D)$. Again, a tube formula approach is the easiest to take, although we shall need integral representations akin to (4.2) and (4.3) before we are done. For this, let $\gamma\equiv\gamma_{\mathbb{R}^l}$ be Gauss measure on $\mathbb{R}^l$, so that $X\sim\gamma_{\mathbb{R}^l}\iff X\sim N(0,I_{l\times l})$ where for a k vector μ and $k\times k$ positive definite, symmetric matrix Σ we denote by $N(\mu, \Sigma)$ the k-dimensional Gaussian distribution with mean μ and covariance matrix Σ.

Suppose $M \subset \mathbb{R}^l$ is a compact C^2, locally convex, regular stratified manifold, and that side conditions in Theorem 9.3 below hold. Then there exist numbers $\mathcal{M}_j^{\gamma}(M) = \mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(M)$, $j \geq 1$, so that, for ρ small enough,

$$\gamma_{\mathbb{R}^l}(\mathrm{Tube}(M, \rho)) \;=\; \gamma_{\mathbb{R}^l}(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(M). \tag{4.4}$$

For a proof of this result, see either [24] where, to the best of our knowledge, it first appeared for manifolds with C^2 boundary, or [3] where it is proven in the stratified case.

There are two quite significant differences between (4.4) and Weyl's tube formula (4.1) for Lebesgue measure. The first lies in the fact that while the expansion in (4.1) has only $\dim(M)+1$ terms, that in (4.4) has infinitely many. Fortunately, for our application in (1.2) we need only the first $\dim(M) + 1$ of the $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}$. The other, which has geometric significance but which will not be all that important for us, is that the $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}$ depend on l, the dimension of the ambient manifold for M, and, for $A \subset \mathbb{R}^l$, satisfy

$$\gamma_{\mathbb{R}^{l+k}}(A \times \mathbb{R}^k) = \gamma_{\mathbb{R}^l}(A).$$

In common with the $\mathcal{L}_j$ there is a representation of the $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}$ somewhat akin to (4.2) and (4.3). Since we shall not require the details of this for a while, we postpone it to Theorem 9.3.

5 Gaussian processes and the induced metric

The Lipschitz-Killing curvatures defined in Section 4 were computed with respect to the Riemannian metric g on M and, with the exception of $\mathcal{L}_0$, depend on the metric. This implies that our main result (1.2) is meaningless until we determine which metric is involved in the computation of the $\mathcal{L}_j(M)$ there. This we now do.

If $y : M \to \mathbb{R}$ is an almost surely C^1, centered Gaussian process, we define the *(Riemannian) metric induced by* y as

$$g(X, Y) \;\equiv\; g_t\left(X_t, Y_t\right) \;\stackrel{\Delta}{=}\; \mathbb{E}\left\{(X_t y) \cdot (Y_t y)\right\}, \tag{5.1}$$

where $X_t, Y_t \in T_t M$, and $X_t y$ denotes the derivative of y in the direction X_t. The fact that (5.1) actually gives a Riemannian metric follows immediately from the positive semi-definiteness of covariance functions. Without further mention, we will assume that g is truly a Riemannian metric, i.e. g_t is non-degenerate for each $t \in M$. If, as in the Introduction, y is a actually a vector valued process made up of i.i.d. components $y_1, \ldots, y_k$, then we define the metric induced by y to be that induced by any of the y_i. As an aside for Gaussian theorists, we note that if $\mathcal{H}$ is the natural L^2 space associated with y, then it has a natural geometric structure coming from the usual inner product given by covariances. Given this, it is easy to see that the Riemannian structure induced on M by y is simply the pull-back of this structure on $\mathcal{H}$.

With the induced metric defined, it worthwhile to look again at the structure of (1.2). We now see that as far as the right hand side of the equation goes, there is an elegant factoring of parameters. The covariance structure of y appears only in the $\mathcal{L}_j(M)$, as does the geometry of the parameter space, while the $\mathcal{M}_j^\gamma(D)$ are computed without any reference to either of these. This particularly useful when applied to (1.4), since it says that when working with f of the form $F(y)$ the computations related to M and the distributional properties of y separate out completely from the properties of F.

It is obvious that g is closely related to the covariance function $C_f(s,t) = \mathbb{E}(f_s f_t)$ of f. In particular, it follows from (5.1) that

$$g_t(X_t, Y_t) \;=\; X_s Y_t C_f(s,t)\big|_{s=t} \tag{5.2}$$

Consequently, it is also obvious that the tools of Riemannian manifolds – connections, curvatures, etc. – can be expressed in terms of covariances. In particular, it turns out that all of these tools also have interpretations in terms of conditional means and variances, a fact that is crucial to the Gaussian computations in the third part of the paper.

We now have all that is needed for understanding the result (1.2). Indeed, with just one more definition, we can state it properly.

Suppose y is a centered, real valued Gaussian process over $T \subseteq \mathbb{R}^N$. Then we say that it is *suitably regular* over T if, for each $t \in T$, the joint distributions of $(y, \partial y/\partial t_i,\ \partial^2 f/\partial t_j \partial t_k)_{i,j,k=1,\ldots,N}$ at t are non-degenerate, and if, for some finite K, and all $s,t \in T$,

$$\max_{i,j} \left| C_{f_{ij}}(t,t) + C_{f_{ij}}(s,s) - 2C_{f_{ij}}(s,t) \right| \;\le\; K \left| \ln|t-s| \right|^{-(1+\alpha)}, \tag{5.3}$$

where $C_{f_{ij}} = \partial^4 C_f / \partial^2 t_i \partial^2 t_j$ is the covariance function of $\partial^2 f/\partial t_i \partial t_j$. Suppose y is a now centered, real valued Gaussian process over a manifold M of dimension N. Let $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in I}$ be a countable atlas for M such that for every $\alpha \in I$ the Gaussian process $f_\alpha = f \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_a) \subset \mathbb{R}^N$ is suitably regular on $\varphi_\alpha(U_\alpha)$, $f = f_\alpha$. Then, again, we call f suitably regular. Finally, if M is a C^2 stratified manifold, we call a centered Gaussian $y : M \to \mathbb{R}$ suitably regular if the same can be said for each of the restrictions $y_{|\partial_j M} : \partial_j M \to \mathbb{R}$.

Suitable regularity has many consequences, among them the fact that (almost surely) y is C^2

and that it is a Morse function for M (cf. Section 14 for a definition of Morse functions and, for example, [3] for a proof of these facts.) It also enables us to formally state the main result of this paper.

Theorem 5.1 *Let M be a regular stratified manifold, and D a regular, stratified submanifold of $\mathbb{R}^k$. Let $y = (y_1, \dots, y_k) : M \to \mathbb{R}^k$ be a vector valued random process, the components of which are independent, identically distributed, real valued, suitably regular, centered, unit variance, Gaussian processes. Then*

$$\mathbb{E}\left\{\mathcal{L}_i\left(M \cap y^{-1}(D)\right)\right\} = \sum_{j=0}^{N-i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M) \mathcal{M}_j^\gamma(D),$$

where the $\mathcal{L}_j$, $j = 0, \dots, N$ are the Lipschitz-Killing measures on M with respect to the metric induced by the y_i, and the $\mathcal{M}_j^\gamma$ are the Gaussian Minkowski functionals on $\mathbb{R}^k$.

6 Two results about spheres

Before we start can start the proof of Theorem 5.1 we need some results about probability and geometry on spheres.

6.1 Poincaré's limit

A very old, and quite elementary result in Probability is known as the Poincaré limit theorem, or Poincaré's limit, although whether it really is due to Poincaré is not clear [9]. To state it, we need some notation which we introduce in slightly more generality than we need at the moment.

If H is a metric space, then we write $S_\lambda(H)$ and $B_\lambda(H)$ for the sphere and ball of radius λ, with $S(H) \triangleq S_1(H)$ and $B(H) \triangleq B_1(H)$.

In its simplest form, Poincaré's limit states that if $\eta_n = (\eta_{n1}, \dots, \eta_{nn})$ is uniformly distributed on $S_{\sqrt{n}}(\mathbb{R}^n)$, and $k \geq 1$ is fixed, then the joint distribution of $(\eta_{n1}, \dots, \eta_{nk})$ converges weakly to that of k independent standard Gaussians as $n \to \infty$. The proof is elementary. More formally, consider the random vector

$$X_{k,n} \triangleq \pi_{\sqrt{n},n,k}(\eta_n),$$

where $\pi_{\lambda,n,k} : S_\lambda(\mathbb{R}^n) \to B_{\mathbb{R}^k}(0, \lambda)$, defined by

$$\pi_{\lambda,n,k}(x_1, \dots, x_n) = (x_1, \dots, x_k), \tag{6.1}$$

is projection from $S_\lambda(\mathbb{R}^n)$ onto the first $k \leq n$ coordinates. Then the basic Poincaré limit result states that, for k fixed, and as $n \to \infty$

$$X_{k,n} \ \stackrel{\mathcal{L}}{\to}\ X \ \sim\ N(0, I_{k\times k}). \tag{6.2}$$

We shall need a little more, in particular covergence in total variation norm, from which (6.3) follows.

Theorem 6.1 *Fix $l, k \geq 1$ and suppose that $g_n \in O(n) \sim \mu_n$ is a Haar distributed random orthogonal matrix. Consider the random $l \times k$ matrix $X_{l,k,n}$ with (i,j)-th entry given by*

$$\left(X_{l,k,n}\right)_{i,j} \ =\ \left(\pi_{\sqrt{n},n,k}(\sqrt{n} g_n e_i)\right)_j, \qquad 1 \leq i \leq j,\ 1 \leq k,$$

where $\{e_1, \ldots, e_n\}$ is the usual orthonormal basis of $\mathbb{R}^n$. Then the matrix $X_{l,k,n}$ converges in total variation norm to $X_{l,k}$, an $l \times k$ matrix of i.i.d. $N(0,1)$ random variables. Furthermore, if F is a real valued function of matrices for which $\mathbb{E}\{|F(X_{lk})|\} < \infty$, then

$$\lim_{n\to\infty} \mathbb{E}\{F(X_{l,k,n})\} \ =\ \mathbb{E}\{F(X_{l,k})\}. \tag{6.3}$$

Proof. In [10] it was shown that that if $\mathbb{P}_{l,k}$ and $\mathbb{P}_{l,k,n}$ denote the distributions of $X_{l,k}$ and $X_{l,k,n}$ then the Radon-Nikodym derivative $d\mathbb{P}_{l,k,n}/d\mathbb{P}_{l,k}$ is bounded and converges uniformly to 1 as $n \to \infty$. This, together with the convergence in total variation norm proven there and a notational change from the uniform distribution on $S(\mathbb{R}^n)$ to the uniform distribution on $O(n)$ is essentially all that is needed to establish (6.3).

6.2 The Kinematic Fundamental Formula on $S_\lambda(\mathbb{R}^n)$

Although we met the KFF for Euclidean spaces back in the Introduction (cf. (1.6)), what we shall really need in this paper is a corresponding version for the sphere $S(\mathbb{R}^n)$, good references for which include [6, 7, 8, 12, 17, 21, 22]. To formulate the KFF on spheres we shall need to somewhat extend the Lipschitz-Killing curvatures of Section 4 to a one parameter family of measures defined, for $\kappa \geq 0$ and $0 \leq i \leq N = \dim(M)$, by

$$\begin{aligned}\mathcal{L}_i^\kappa(M, A) \ &\triangleq\ \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{(-1)^m C(N-j, j-i-2m)}{m!(j-i-2m)!} \\ &\quad \times \int_{\partial_j M \cap A} \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{T_t\partial_j M}\left(\left(\widetilde{R} + \frac{\kappa}{2} I^2\right)^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m}\right) \\ &\quad \times \alpha(\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j})\, \mathcal{H}_j(dt),\end{aligned} \tag{6.4}$$

where M is a C^2 Whitney stratified manifold embedded in $(\widetilde{M}, \widetilde{g})$ and all terms are as defined for (4.3). For $i > N$ we define $\mathcal{L}_i^\kappa \equiv 0$. We also define the one parameter family of curvatures integrals $\mathcal{L}_i^\kappa(M) \stackrel{\Delta}{=} \mathcal{L}_i^\kappa(M, M)$. For lack of a better terminology, we call these new curvature measures *extended* Lipschitz-Killing curvatures.

On comparing (6.4) with (4.3), it is immediate that $\mathcal{L}_i^0(M, \cdot) \equiv \mathcal{L}_i(M, \cdot)$. It takes some algebra (cf. [3]) but it is not too hard to show that there are simple equivalences between the $\mathcal{L}_j$ and $\mathcal{L}_j^\kappa$, given by

$$\mathcal{L}_i^\kappa = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(4\pi)^n} \frac{(i+2n)!}{n!i!} \mathcal{L}_{i+2n}, \qquad \mathcal{L}_i = \sum_{n=0}^{\infty} \frac{\kappa^n}{(4\pi)^n} \frac{(i+2n)!}{n!i!} \mathcal{L}_{i+2n}^\kappa. \tag{6.5}$$

(Note that the summations here are actually finite, since all curvatures are zero for $i > \dim(M)$.)

It turns out that in dealing with subsets of $S_\lambda(\mathbb{R}^n)$ the $\mathcal{L}_j^{\lambda^{-2}}$ are more natural to deal with than are the original Lipschitz-Killing curvatures. For example, there is a elegant version of Weyl's tube formula involving them and, more importantly for us, there is a nice KFF on on $S_\lambda(\mathbb{R}^l)$.

Let $G_{n,\lambda}$ denote the group of isometries (i.e. rotations) on $S_\lambda(\mathbb{R}^n)$, with Haar measure $\nu_{n,\lambda}$ normalised so that, for any $x \in S_\lambda(\mathbb{R}^n)$ and every Borel $A \subset S_\lambda(\mathbb{R}^n)$,

$$\nu_{n,\lambda}\left(\{g_n \in G_{n,\lambda} : g_n x \in A\}\right) = \mathcal{H}_{n-1}(A). \tag{6.6}$$

The KFF on $S_\lambda(\mathbb{R}^n)$ then reads as follows, where M_1 and M_2 are tame stratified manifolds in $S_\lambda(\mathbb{R}^n)$:

$$\begin{aligned} \int_{G_{n,\lambda}} \mathcal{L}_i^{\lambda^{-2}}(M_1 \cap g_n M_2)\, d\nu_{n,\lambda}(g_n) &= \sum_{j=0}^{n-1-i} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} n-1 \\ j \end{bmatrix}^{-1} \mathcal{L}_{i+j}^{\lambda^{-2}}(M_1) \mathcal{L}_{n-1-j}^{\lambda^{-2}}(M_2) \quad (6.7) \\ &= \sum_{j=0}^{n-1-i} \frac{s_{i+1} s_n}{s_{i+j+1} s_{n-j}} \mathcal{L}_{i+j}^{\lambda^{-2}}(M_1) \mathcal{L}_{n-1-j}^{\lambda^{-2}}(M_2), \end{aligned}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}, \quad [n]! = n!\omega_n, \quad \omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad s_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \tag{6.8}$$

7 A model process on the l-sphere

We now introduce a particularly simple process on the l-sphere $S(\mathbb{R}^l)$, for which we shall be able to compute all excursion set (and more) mean Lipschitz-Killing curvatures with calculations using no more than the KFF on the sphere. These processes are actually (distributional) approximations to the so-called *canonical isotropic process*, the components of which are independent, centred Gaussian with covariance function

$$\mathbb{E}\left\{y_i(s)y_i(t)\right\} \;=\; \langle s,t\rangle, \tag{7.1}$$

where $\langle\, ,\, \rangle$ is the usual Euclidean inner product. In Section 8 we shall exploit the results of this section to obtain analogous formulae for the canonical isotropic process.

The approximations are given by a sequence, $\{y^{(n)}\}_{n\geq l}$, of smooth $\mathbb{R}^k$-valued processes on stratified subsets M of $S(\mathbb{R}^l)$. To define them, for each $n\geq l$ we first embed $S(\mathbb{R}^l)$ in $S(\mathbb{R}^n)$ in the natural way, by setting $S(\mathbb{R}^l)=\{t=(t_1,\dots,t_n)\in S(\mathbb{R}^n) : t_{l+1}=\cdots=t_n=0\}$. Taking $O(n)$, equipped with its normalized Haar measure μ_n, as an underlying probability space, the n-th process $y^{(n)}$ is defined by

$$y^{(n)}(t,g_n) \;\stackrel{\Delta}{=}\; \pi_{\sqrt{n},n,k}\left(\sqrt{n}g_nt\right), \tag{7.2}$$

where $t\in S(\mathbb{R}^l)$, $g_n\in O(n)$ and $\pi_{\sqrt{n},n,k}$ is the projection from $S_{\sqrt{n}}(\mathbb{R}^n)$ to $\mathbb{R}^k$ given by (6.1).

From Theorem 6.1 it is clear that the processes $y^{(n)}$ converge in variation to a $\mathbb{R}^k$ valued Gaussian process all of whose independent components are versions of the canonical isotropic process on $\mathbb{R}^l$.

It is remarkably straightforward to compute the mean Lipschitz-Killing curvatures of many sets generated by the $y^{(n)}$, using only the KFF on $S_{\sqrt{n}}(\mathbb{R}^n)$. The key result, from which everything else follows, is the following.

Lemma 7.1 *Let $y^{(n)}$ be the model process* (7.2) *on a regular stratified manifold $M\subset S(\mathbb{R}^l)$, with $n\geq l$. Then, for any regular stratified manifold $D\subset\mathbb{R}^k$,*

$$\begin{aligned}\mathbb{E}\left\{\mathcal{L}_i^1(M\cap(y^{(n)})^{-1}D)\right\} \;&=\; \sum_{j=0}^{\dim M-i}\left(n^{j/2}\begin{bmatrix}n-1\\ j\end{bmatrix}^{-1}\right)\begin{bmatrix}i+j\\ j\end{bmatrix}\mathcal{L}_{j+i}^1(M)\,\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi_{\sqrt{n},n,k}^{-1}D\right)}{s_n n^{(n-1)/2}}\\ &=\; \sum_{j=0}^{\dim M-i}\frac{s_{i+1}}{s_{i+j+1}}\mathcal{L}_{j+i}^1(M)\,\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi_{\sqrt{n},n,k}^{-1}D\right)}{s_{n-j}n^{(n-1-j)/2}}.\end{aligned} \tag{7.3}$$

(It is important to understand the meaning of $\pi_{\sqrt{n},n,k}^{-1}D$ above, and in all that follows. The problem is that, for all $t\in S_{\sqrt{n}}(\mathbb{R}^n)$, $\pi_{\sqrt{n},n,k}(t)\in B_{\sqrt{n}}(\mathbb{R}^k)$, which may, or may not, cover D. Thus, since $\pi_{\sqrt{n},n,k}^{-1}D \;=\; \left\{t\in S_{\sqrt{n}}(\mathbb{R}^n) : \pi_{\sqrt{n},n,k}(t)\in D\right\}$, it follows that $\pi_{\sqrt{n},n,k}^{-1}D$ may be only the inverse image of a subset of D.)

Proof. Since $\pi_{\sqrt{n},n,k}^{-1}D$ is a regular stratified manifold in $S(\mathbb{R}^n)$, it follows from the construction of $y^{(n)}$ that

$$\begin{aligned}
\mathbb{E}\left\{\mathcal{L}_i^1(M\cap(y^{(n)})^{-1}D)\right\} &= \int_{O(n)} \mathcal{L}_i^1(M\cap(y^{(n)})^{-1}D)(g_n)d\mu_n(g_n) \\
&= \int_{O(n)} \mathcal{L}_i^1\left(M\cap n^{-1/2}g_n^{-1}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)\right)d\mu_n(g_n) \\
&= n^{-i/2}\int_{O(n)} \mathcal{L}_i^{n^{-1}}\left(\sqrt{n}M\cap g_n^{-1}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)\right)d\mu_n(g_n) \\
&= \frac{1}{s_n n^{(n-1+i)/2}}\int_{G_{n,n^{-1}}} \mathcal{L}_i^{n^{-1}}\left(\sqrt{n}\,M\cap g_n\left(\pi^{-1}_{\sqrt{n},n,k}D\right)\right)d\nu_{n,n^{-1}}(g_n),
\end{aligned}$$

where the second last line follows from the scaling properties of Lipschitz-Killing curvatures and the last is really no more than a notational change, using (6.6).

However, applying the KFF (6.7) to the last line above, we immediately have that it is equal to

$$\begin{aligned}
&\sum_{j=0}^{\dim M-i} n^{j/2}\begin{bmatrix} i+j \\ i\end{bmatrix}\begin{bmatrix} n-1 \\ j\end{bmatrix}^{-1}\frac{\mathcal{L}_{j+i}^{n^{-1}}(\sqrt{n}\,M)}{n^{(i+j)/2}}\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)}{s_n n^{(n-1)/2}} \\
&\qquad = \sum_{j=0}^{\dim M-i} n^{j/2}\begin{bmatrix} n-1 \\ j\end{bmatrix}^{-1}\begin{bmatrix} i+j \\ j\end{bmatrix}\mathcal{L}_{j+i}^{1}(M)\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)}{s_n n^{(n-1)/2}},
\end{aligned}$$

which proves the lemma. □

Suppose we send $n\to\infty$ in (7.3), which by Poincaré's limit is effectively equivalent to replacing the model process $y^{(n)}$ with a $\mathbb{R}^k$ valued canonical Gaussian y. Then, in order for $\mathbb{E}\{\mathcal{L}_j(M\cap y^{-1}D)$ to be finite for the limiting process y, we would like to have the following limits existing for each finite j:

$$\widetilde{\rho}_j(D) \;\triangleq\; \lim_{n\to\infty} n^{j/2}\begin{bmatrix} n-1 \\ j\end{bmatrix}^{-1}\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)}{s_n n^{(n-1)/2}}. \tag{7.4}$$

Note that a Stirling's formula computation shows that if the limit here does exist, then

$$\widetilde{\rho}_j(D) \;=\; (2\pi)^{-j/2}[j]!\lim_{n\to\infty}\frac{\mathcal{L}_{n-1-j}^{n^{-1}}\left(\pi^{-1}_{\sqrt{n},n,k}D\right)}{s_n n^{(n-1)/2}}. \tag{7.5}$$

Sending $n\to\infty$ in Lemma 7.1 and applying Poincaré's limit of Theorem 6.1, we see that if $\mathbb{E}\left\{|\mathcal{L}_i^1(M\cap y^{-1}D)|\right\} < \infty$ then

$$\begin{aligned}
\mathbb{E}\left\{\mathcal{L}_i^1(M\cap y^{-1}D)\right\} &= \lim_{n\to\infty}\mathbb{E}\left\{\mathcal{L}_i^1(M\cap(y^{(n)})^{-1}D)\right\} \qquad (7.6)\\
&= \sum_{j=0}^{\dim M-i}\begin{bmatrix} i+j \\ i\end{bmatrix}\mathcal{L}_{j+i}^1(M)\widetilde{\rho}_j(D).
\end{aligned}$$

This is starting to take the form of the (1.2). The combinatorial flag coefficients are in place, but both sides of the equation are based on the $\mathcal{L}^1_{j+i}$ curvatures rather than the $\mathcal{L}_{j+i}$, and we have yet to identify the functions $\widetilde{\rho}_j$. Note the important fact, however, that on the right hand side of the equation we have already managed to split into product form, each factor of which depends on the underlying manifold M or the set D, but not both. Moving to the desired curvatures, under the assumption that the limits (7.4) exist, is our next step.

Theorem 7.2 *Let $M \subset S(\mathbb{R}^l)$ be a regular stratified manifold, and assume that, for $0 \le j \le \dim(M)$, the $\widetilde{\rho}_j(D)$ of (7.5) are well defined and finite and*

$$\mathbb{E}\left\{|\mathcal{L}_i(M \cap y^{-1}D)|\right\} \ < \ \infty, \tag{7.7}$$

where y is the $\mathbb{R}^l$ valued canonical isotropic Gaussian process on $S(\mathbb{R}^l)$. Then

$$\mathbb{E}\left\{\mathcal{L}_i(M \cap y^{-1}D)\right\} \ = \ \sum_{l=0}^{\dim M - i} \begin{bmatrix} i+l \\ l \end{bmatrix} \mathcal{L}_{i+l}(M)\, \rho_l(D), \tag{7.8}$$

where

$$\rho_j(D) \ = \ \begin{cases} (j-1)! \sum_{l=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^l}{(4\pi)^l l! (j-1-2l)!} \widetilde{\rho}_{j-2l}(D) & j \ge 1, \\ \gamma_{\mathbb{R}^k}(D) & j = 0. \end{cases} \tag{7.9}$$

Proof. As usual, set $N = \dim(M)$. Combining (6.5) and (7.6) we have

$$\begin{aligned} \mathbb{E}\left\{\mathcal{L}_i(M \cap y^{-1}D)\right\} &= \mathbb{E}\left\{ \sum_{m=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{1}{(4\pi)^m m!} \frac{(i+2m)!}{i!} \mathcal{L}^1_{i+2m}(M \cap y^{-1}D) \right\} \\ &= \sum_{m=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{1}{(4\pi)^m m!} \frac{(i+2m)!}{i!} \sum_{j=0}^{N-i-2m} \begin{bmatrix} i+2m+j \\ j \end{bmatrix} \mathcal{L}^1_{i+2m+j}(M) \widetilde{\rho}_j(D) \\ &= \sum_{m=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{1}{(4\pi)^m m!} \frac{(i+2m)!}{i!} \sum_{j=0}^{N-i-2m} \begin{bmatrix} i+2m+j \\ j \end{bmatrix} \widetilde{\rho}_j(D) \\ &\qquad \times \sum_{l=0}^{\lfloor \frac{N-i-2m-j}{2} \rfloor} \frac{(-1)^l}{(4\pi)^l l!} \frac{(i+2m+j+2l)!}{(i+2m+j)!} \mathcal{L}_{i+2m+j+2l}(M). \end{aligned}$$

To save space, suppress the dependence of $\widetilde{\rho}$ and $\mathcal{L}$ on D and M, respectively. Proceeding,

$\mathbb{E}\left\{\mathcal{L}_i(M \cap y^{-1}D)\right\}$ can be written as

$$
\begin{aligned}
&\sum_{n=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{1}{(4\pi)^m m!} \frac{(i+2m)!}{i!} \sum_{j=0}^{N-i-2m} \left[\begin{matrix} i+2m+j \\ j \end{matrix}\right] \tilde{\rho}_j \sum_{l=0}^{\lfloor \frac{N-i-2m-j}{2} \rfloor} \frac{(-1)^l}{(4\pi)^l l!} \frac{(i+2m+j+2l)!}{(i+2m+j)!} \mathcal{L}_{i+2m+j+2l} \\
&= \sum_{j=0}^{N-i} \tilde{\rho}_j \sum_{m=0}^{\lfloor \frac{N-i-j}{2} \rfloor} \frac{1}{(4\pi)^m m!} \frac{(i+2m)!}{i!} \left[\begin{matrix} i+2m+j \\ j \end{matrix}\right] \sum_{l=0}^{\lfloor \frac{N-i-2m-j}{2} \rfloor} \frac{(-1)^l}{(4\pi)^l l!} \frac{(i+2m+j+2l)!}{(i+2m+j)!} \mathcal{L}_{i+2m+j+2l} \\
&= \sum_{j=0}^{N-i} \tilde{\rho}_j \sum_{\alpha=0}^{\lfloor \frac{N-i-j}{2} \rfloor} \sum_{\beta=0}^{\alpha} \frac{1}{(4\pi)^{\alpha-\beta}(\alpha-\beta)!} \frac{(i+2(\alpha-\beta))!}{i!} \\
&\qquad\qquad \times \left[\begin{matrix} i+2(\alpha-\beta)+j \\ j \end{matrix}\right] \frac{(-1)^\beta}{(4\pi)^\beta \beta!} \frac{(i+j+2\alpha)!}{(i+2(\alpha-\beta)+j)!} \mathcal{L}_{i+j+2\alpha},
\end{aligned}
$$

where the first and third equalities come from a change of order of summation and the second from the transformation $(\alpha, \beta) = (n+l, l)$. Again changing the order of summation we find that this is the same as

$$
\begin{aligned}
&\sum_{\alpha=0}^{\lfloor \frac{N-i}{2} \rfloor} \sum_{j=0}^{N-i-2\alpha} \sum_{\beta=0}^{\alpha} \tilde{\rho}_j \frac{1}{(4\pi)^\alpha (\alpha-\beta)!} \frac{(i+2(\alpha-\beta))!}{i!} \\
&\qquad\qquad \times \left[\begin{matrix} i+2(\alpha-\beta)+j \\ j \end{matrix}\right] \frac{(-1)^\beta}{\beta!} \frac{(i+j+2\alpha)!}{(i+2(\alpha-\beta)+j)!} \mathcal{L}_{i+j+2\alpha}.
\end{aligned}
$$

Making now the further change of variables $(m, k) = (\alpha, j+2\alpha)$, the above is equivalent to

$$
\begin{aligned}
&\sum_{k=0}^{N-i} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{\beta=0}^{m} \tilde{\rho}_{k-2m} \frac{1}{(4\pi)^m (m-\beta)!} \frac{(i+2(m-\beta))!}{i!} \left[\begin{matrix} i+k-2\beta \\ k-2m \end{matrix}\right] \frac{(-1)^\beta}{\beta!} \frac{(i+k)!}{(i+k-2\beta)!} \mathcal{L}_{i+k} \\
&\quad = \sum_{k=0}^{N-i} \mathcal{L}_{i+k} \frac{(i+k)!}{i!} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{\rho}_{k-2m} \sum_{\beta=0}^{m} \frac{1}{(4\pi)^m (m-\beta)!} (i+2(m-\beta))! \\
&\qquad\qquad \times \left[\begin{matrix} i+k-2\beta \\ k-2m \end{matrix}\right] \frac{(-1)^\beta}{\beta!} \frac{1}{(i+k-2\beta)!},
\end{aligned}
$$

the last line being just a minor reorganisation of the preceeding one.

We must therefore show that

$$
\begin{aligned}
&\sum_{k=0}^{N-i} \mathcal{L}_{i+k} \frac{(i+k)!}{i!} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{\rho}_{k-2m} \sum_{\beta=0}^{m} \frac{1}{(4\pi)^m (m-\beta)!} (i+2(m-\beta))! \\
&\qquad\qquad \times \left[\begin{matrix} i+k-2\beta \\ k-2m \end{matrix}\right] \frac{(-1)^\beta}{\beta!} \frac{1}{(i+k-2\beta)!} \\
&= \sum_{k=0}^{N-i} \mathcal{L}_{i+k}(M) \left[\begin{matrix} i+k \\ k \end{matrix}\right] \rho_k,
\end{aligned}
$$

where the functionals ρ_k are those of (7.9). Equivalently, we must show that

$$\begin{aligned}\rho_k &= \begin{bmatrix} i+k \\ k \end{bmatrix}^{-1} \frac{(i+k)!}{i!} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{\rho}_{k-2m} \sum_{\beta=0}^{m} \frac{1}{(4\pi)^m (m-\beta)!} (i+2(m-\beta))! \\ &\qquad \times \begin{bmatrix} i+k-2\beta \\ k-2m \end{bmatrix} \frac{(-1)^\beta}{\beta!} \frac{1}{(i+k-2\beta)!} \\ &= \frac{k!\omega_k \omega_i}{\omega_{i+k}} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{\rho}_{k-2m} \sum_{\beta=0}^{m} \frac{(-1)^\beta}{(4\pi)^m (m-\beta)!\beta!(k-2m)!} \frac{\omega_{i+k-2\beta}}{\omega_{k-2m}\omega_{i+2m-2\beta}}.\end{aligned}$$

This is equivalent, by (7.9), to proving that the following identity holds for all non-negative integers $k \geq 1, i \geq 0$

$$\frac{(-1)^m}{m!} \frac{\omega_{i+k}}{k\omega_i\omega_k/2} = \sum_{\beta=0}^{m} \frac{(-1)^\beta}{(m-\beta)!\beta!} \frac{\omega_{i+k-2\beta}}{\omega_{i+2m-2\beta} \cdot (k-2m)\omega_{k-2m}/2}.$$

Finally, after some further simple manipulations, this is equivalent to the identity

$$B\left(\frac{i}{2}+1, \frac{k}{2}\right) = \sum_{\beta=0}^{m} (-1)^{m-\beta} \binom{m}{\beta} B\left(\frac{i+2m-2\beta}{2}+1, \frac{k-2m}{2}\right),$$

where B is the Beta function

$$B(\gamma, \delta) = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} = \int_0^1 p^{\gamma-1}(1-p)^{\delta-1}\, dp.$$

That this identity holds is verified in the following lemma, and so the proof is complete. □

Lemma 7.3 *For every $\gamma, \delta > 0$ and every integer $0 \leq m \leq \gamma$*

$$B(\gamma, \delta) = \sum_{\beta=0}^{m} (-1)^{m-\beta} \binom{m}{\beta} B(\gamma - m, \delta + m - \beta).$$

Proof. The identity is clearly equivalent to

$$1 = \frac{1}{B(\gamma,\delta)} \sum_{\beta=0}^{m} (-1)^{m-\beta} \binom{m}{\beta} \int_0^1 p^{\gamma-m-1}(1-p)^{\delta+m-\beta-1}\, dp.$$

Suppose, then, that P is a random variable with distribution Beta(γ, δ), so that it has density

$$f_P(p) = \frac{1}{B(\gamma,\delta)}\, p^{\gamma-1}(1-p)^{\delta-1}, \qquad 0 \leq p \leq 1.$$

Then,

$$1 \;=\; P^{-m}(1-(1-P))^m \;=\; \sum_{\beta=0}^{m} \binom{m}{\beta} (-1)^{m-\beta}(1-P)^{m-\beta} P^{-m}$$

Since $m < \gamma$ we can take expectations of both sides above to obtain

$$1 \;=\; \mathbb{E}\{P^{-m}(1-(1-P))^m\} \;=\; \frac{1}{B(\gamma,\delta)} \sum_{\beta=0}^{m} \binom{m}{\beta} (-1)^{m-\beta} \int_0^1 p^{\gamma-1-m}(1-p)^{\delta+m-\beta-1}\, dp,$$

and we are done. □

What remains now to do is to check that the limits (7.5) are indeed well-defined, and to evaluate them. Before treating the general case, however, we shall work out one special one, for which the computations will be more transparent and for which the result, which will not explicitly involve the Gaussian Minkowski curvatures, will be somewhat simpler than in the general case.

8 The canonical process on $S(\mathbb{R}^l)$ and in half-spaces

In this section we shall prove Theorem 5.1 for the special case in which y is the canonical isotropic Gaussian process on $S(\mathbb{R}^l)$ and D is the half-space

$$D \;=\; \left\{ y \in \mathbb{R}^k : \langle y, \eta \rangle \geq u \right\}, \tag{8.10}$$

for some unique unit vector $\eta \in S(\mathbb{R}^k)$ and $u \in \mathbb{R}_+$. In this case the Minkowski curvatures do not appear explicitly, but are hidden in the Hermite polynomials of the result.

Theorem 8.1 *Let $M \subset S(\mathbb{R}^l)$ be a regular stratified manifold, y the real valued, canonical isotropic Gaussian process on $S(\mathbb{R}^l)$ and D the half-space in $\mathbb{R}^k$ defined above. Then*

$$\mathbb{E}\left\{ \mathcal{L}_i(M \cap y^{-1}D) \right\} \;=\; \sum_{l=0}^{\dim M - i} \begin{bmatrix} i+l \\ l \end{bmatrix} \mathcal{L}_{i+l}(M)\, \rho_l(D), \tag{8.11}$$

where, for $j \geq 1$,

$$\rho_j(D) \;=\; \begin{cases} (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2} & 1 \leq j \leq \dim(M), \\ \gamma_{\mathbb{R}^k}(D), & j = 0, \end{cases}$$

and H_n is the n-th Hermite polynomial defined by

$$H_n(x) \;=\; n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j!\,(n-2j)!\,2^j}, \qquad n \geq 0,\; x \in \mathbb{R}. \tag{8.12}$$

The proof is based on the following lemma.

Lemma 8.2 *Under the assumptions of Theorem 8.1, the $\widetilde{\rho}_l(D)$ of (7.5) are well defined and are given by*

$$\widetilde{\rho}_j(D) \;=\; \begin{cases} (2\pi)^{-(j+1)/2}u^{j-1}e^{-u^2/2} & 1 \le j \le \dim(M), \\ 1-\Phi(u) & j=0, \end{cases}$$

where $\Phi(u) = (2\pi)^{-1/2}\int_u^\infty e^{-x^2/2}\,dx$ is the tail of a standard Gaussian distribution.

Proof. Changing notation a little for this proof, write $B_M(x,r)$ for a ball in M of radius r, centered at $x \in M$. Otherwise adopting the notation of the preceeding section, we begin with the observation that, as long as $n > u$,

$$\pi^{-1}_{\sqrt{n},n,k}D \;=\; B_{S_{\sqrt{n}}(\mathbb{R}^n)}\left(\sqrt{n}\eta, \cos^{-1}(u/\sqrt{n})\right), \tag{8.13}$$

a geodesic ball, or spherical cap, in $S_{\sqrt{n}}(\mathbb{R}^n)$ of radius $\cos^{-1}(u/\sqrt{n})$. Consequently, it follows from the definition (7.5) of the $\widetilde{\rho}_l(D)$ that they are given by

$$\widetilde{\rho}_l(D) \quad=\quad (2\pi)^{-j/2}[j]!\lim_{n\to\infty}\frac{\mathcal{L}^{n^{-1}}_{n-1-j}\left(\sqrt{n}\,B_{S(\mathbb{R}^n)}(\eta,\cos^{-1}(u/\sqrt{n}))\right)}{s_n n^{(n-1)/2}},$$

for all $j > 0$ and all $u > 0$. In order to compute this limit, we move from the spherical Lipschitz-Killing curvatures to Euclidean ones, with the claim that

$$\mathcal{L}^{n^{-1}}_{n-1-j}\left(\sqrt{n}\,B_{S(\mathbb{R}^n)}(\eta,\cos^{-1}(u/\sqrt{n}))\right) \;=\; \left(\frac{u}{\sqrt{n}}\right)^{j-1}\mathcal{L}_{n-1-j}\left(B_{\mathbb{R}^{n-1}}(0,\sqrt{n-u^2})\right). \tag{8.14}$$

To prove this, we first need to replace the intrinsic definitions (4.3) and (6.4) of the $\mathcal{L}_j$ and $\mathcal{L}^\kappa_j$ by extrinsic ones. It is not too hard to show (cf. Chapter 10 of [3]) that if M is embedded in $\mathbb{R}^l$ with the canonical Riemannian structure on $\mathbb{R}^l$, then it follows from the flatness of $\mathbb{R}^l$ that

$$\begin{aligned}\mathcal{L}_i(M,A) \quad=\quad \sum_{j=i}^{N}\frac{C(l-j,j-i)}{(2\pi)^{-(j-i)/2}}\int_{\partial_j M\cap A}\int_{S(\mathbb{R}^{l-j})}\frac{1}{(j-i)!}\mathrm{Tr}^{T_t\partial_j M}(S^{j-i}_\eta) \\ \times\alpha(\eta)\mathcal{H}_{l-j-1}(d\eta)\mathcal{H}_j(dt),\end{aligned} \tag{8.15}$$

where S is the scalar second fundamental form of $\partial_j M$ as it sits in $\mathbb{R}^l$, $T_t\partial_j M^\perp$ is the orthogonal complement of $T_t\partial_j M$ in $T_t\mathbb{R}^l$, and $\widehat{N_t M}$ is the normal cone at $t \in \partial_j M$ as it sits in $T_t\mathbb{R}^l$.

Furthermore, if $\kappa > 0$ and $M \subset S_{\kappa^{-1/2}}(\mathbb{R}^l)$, then

$$\begin{aligned}\mathcal{L}^\kappa_i(M,A) \quad=\quad \sum_{j=i}^{N}\frac{C(l-1-j,j-i)}{(2\pi)^{-(j-i)/2}}\int_{\partial_j M\cap A}\int_{S(T_t\partial_j M^\perp)}\frac{1}{(j-i)!}\mathrm{Tr}^{T_t\partial_j M}(S^{j-i}_\eta) \\ \times\alpha(\eta)\mathcal{H}_{l-j-2}(d\eta),\end{aligned} \tag{8.16}$$

where now S is the scalar second fundamental form of $\partial_j M$ as it sits in $S_{\kappa^{-1/2}}(\mathbb{R}^l)$, $T_t\partial_j M^\perp$ is the orthogonal complement of $T_t\partial_j M$ in $T_t S_{\kappa^{-1/2}}(\mathbb{R}^l)$, and $\widehat{N_t M}$ is the normal cone at $t \in \partial_j M$ as it sits in $T_t S_{\kappa^{-1/2}}(\mathbb{R}^l)$.

We now return to the proof of (8.14), which we start by writing

$$H \;\stackrel{\Delta}{=}\; \sqrt{n}\, B_{S(\mathbb{R}^n)}\left(\eta, \cos^{-1}(u/\sqrt{n})\right)$$

and compute $\mathcal{L}^{n^{-1}}_{n-1-j}(H)$ using the extrinsic formula (8.16). Note that, for $j > 0$, the only contributions to $\mathcal{L}^{n^{-1}}_{n-1-j}(H)$ come from ∂H, which is a sphere of radius $(n-u^2)^{1/2}$. Consequently, up to a constant, $\mathcal{L}^{n^{-1}}_{n-1-j}(H)$ is the integral of $\mathrm{Tr}\left(S_\eta^{j-1}\right)$ over ∂H, where S is the second fundamental form of ∂H in $S_{\sqrt{n}}(\mathbb{R}^n)$.

However, although up to this point we have been considering ∂H as it sits in $S_{\sqrt{n}}(\mathbb{R}^n)$, it can also be treated as a subset of the hyperplane

$$L \;\stackrel{\Delta}{=}\; \{y \in \mathbb{R}^n : \langle y, \eta\rangle = u\}\,.$$

Consequently, we can also compute the Euclidean Lipschitz-Killing curvatures $\mathcal{L}_{n-i-j}(H)$, this time using the extrinsic representation (8.15). While these will not be the same as the $\mathcal{L}^{n^{-1}}_{n-1-j}(H)$, they too will be given, up to constants, by the integral of the trace of a scalar second fundamental form over ∂H. In this case, however, the scalar form comes from considering ∂H as a subset of L, and we write it as $\tilde{S}$. The two fundamental forms are related by

$$S \;=\; \frac{u}{\sqrt{n}}\tilde{S},$$

in the sense that if η is the unit outward pointing normal of ∂H in $S_{\sqrt{n}}(\mathbb{R}^n)$ at $t \in \partial H$ and $\tilde{\eta}$ is the unit outward pointing normal of ∂H in L at $t \in \partial H$ then, for any $X_t, Y_t \in T_t\partial H$,

$$S_\eta(X_t, Y_t) \;=\; \frac{u}{\sqrt{n}}\tilde{S}_{\tilde{\eta}}(X_t, Y_t).$$

Using this equivalence and substituting into (8.16) and (8.15) to get the constants right, (8.14) now follows on noting that

$$\mathcal{L}_{n-1-j}(B_L(u\eta, \sqrt{n-u^2})) \;=\; \mathcal{L}_{n-1-j}(B_{\mathbb{R}^{n-1}}(0, \sqrt{n-u^2})).$$

However, the Lipschitz-Killing curvatures of balls (and spheres) are easy to compute, either from any of the integral definitions or from Weyl's tube formula. In particular,

$$\mathcal{L}_j\left(B_r(\mathbb{R}^n)\right) \;=\; r^j \binom{n}{j} \frac{\omega_n}{\omega_{n-j}}.$$

The right hand side of (8.14) is now also easy to compute, and so, for $j > 0$ and $n > u$ we have

$$\mathcal{L}^{n^{-1}}_{n-1-j}\left(\sqrt{n}\, B_{S(\mathbb{R}^n)}(\eta, \cos^{-1}(u/\sqrt{n}))\right) = \left(\frac{u}{\sqrt{n}}\right)^{j-1} \begin{bmatrix} n-1 \\ n-1-j \end{bmatrix} \omega_{n-1-j}(n-u^2)^{(n-1-j)/2}. \tag{8.17}$$

The rest of the proof for $j > 0$ is a straightforward application of Stirling's formula, specifically the fact that for any fixed α

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \overset{n\to\infty}{\sim} \left(\frac{n}{2}\right)^{\alpha/2}.$$

Applying this asymptotic equivalence we easily check that

$$\begin{bmatrix} n-1 \\ n-1-j \end{bmatrix} \sim \frac{(2\pi)^{j/2} n^{j/2}}{[j]!}$$

and

$$\frac{\omega_{n-1-j} n^{(n-1-j)/2}}{s_n n^{(n-1)/2}} \sim (2\pi)^{-(j+1)/2} n^{-1/2},$$

so that

$$\frac{[j]! u^{j-1} n^{-(j-1)/2}}{(2\pi)^{j/2}} \begin{bmatrix} n-1 \\ n-1-j \end{bmatrix} \frac{\omega_{n-1-j} n^{(n-1-j)/2}}{s_n n^{(n-1)/2}} \left(1 - \frac{u^2}{n}\right)^{(n-1-j)/2} \sim (2\pi)^{-(j+1)/2} u^{j-1} e^{-u^2/2}.$$

Substituting this into (8.17) gives that, for $j > 0$ and $u > 0$,

$$\widetilde{\rho}_l(D) = (2\pi)^{-(j+1)/2} u^{j-1} e^{-u^2/2},$$

as required.

It remains only to treat the case $j = 0$. But since it is an immediate implication of Poincaré's limit that, for all $u > 0$,

$$\lim_{n\to\infty} \frac{\mathcal{L}_{n-1}^{n^{-1}}\left(\sqrt{n}\, B_{S(\mathbb{R}^n)}(\eta, \cos^{-1}(u/\sqrt{n}))\right)}{s_n n^{(n-1)/2}} = 1 - \Phi(u),$$

this case is simple, and we are done. □

Proof of Theorem 8.1. The case $j = 0$ is an immediate consequence of Theorem 7.2 and Lemma 8.2. Relying on the same results, but now taking $j \geq 1$, we have

$$\begin{aligned}
\rho_j(D) &= (j-1)! \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^k}{(4\pi)^k k! (j-1-2k)!} \widetilde{\rho}_{j-2k}(D) \\
&= (j-1)! \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^k (2\pi)^{-(j-1-2k)/2}}{(4\pi)^k k! (j-1-2k)!} u^{j-1-2k} e^{-u^2/2} \\
&= (2\pi)^{-(j+1)/2} (j-1)! \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{(-1)^k}{2^k k! (j-1-2k)!} u^{j-1-2k} e^{-u^2/2} \\
&= (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2},
\end{aligned}$$

on noting the definition (8.12) of the Hermite polynomials. □

9 The canonical Gaussian process on $S(\mathbb{R}^l)$

In this section, we again want to prove Theorem 5.1 for the canonical Gaussian process on $S(\mathbb{R}^l)$, but now for general D. As in the preceeding section, this involves understanding the Lipschitz-Killing curvatures of $\pi^{-1}_{\sqrt{n},n,k}D$ as $n \to \infty$, a computation that turns out to be part geometric and part asymptotics. We start with the geometry.

9.1 Warped products

Our first observation is that $\pi^{-1}_{\sqrt{n},n,k}D \in S_{\sqrt{n}}(\mathbb{R}^n)$ is, topologically, a disjoint union

$$\pi^{-1}_{\sqrt{n},n,k}D \;\simeq\; \big(D \cap S_{\sqrt{n}}(\mathbb{R}^k)\big) \sqcup \Big(D \cap \big(B_{\mathbb{R}^k}(0,\sqrt{n})\big)^{\circ} \times S(\mathbb{R}^{n-k})\Big)\,. \tag{9.1}$$

Since we are assuming that D itself is a tame stratified manifold, the same is true of $\pi^{-1}_{\sqrt{n},n,k}D$ and of each of the two components above. Consequently, their Lipschitz-Killing curvatures are well defined. One is easy to compute. Since $D \cap S_{\sqrt{n}}(\mathbb{R}^k)$ is a tame stratified subset of $S_{\sqrt{n}}(\mathbb{R}^k)$ its Lipschitz-Killing curvatures can be computed using (6.4).

The second set in the union is, however, somewhat more complex, since we have written it as a product set. Furthermore, what we have written in (9.1) is a topological equivalence, whereas we shall need precise Lipschitz-Killing curvatures, which, with the exception of $\mathcal{L}_0$, are not topological invariants.

The way to handle these problems is two-fold. First of all, we need to break the Riemannian structure of products into a product of structures, at least along each stratum of a stratified manifold. Secondly, we need to keep track of the fact that while (9.1) is topologically precise, at each point in $D \cap (B_{\mathbb{R}^k}(0,\sqrt{n}))^{\circ}$ the corresponding $S(\mathbb{R}^{n-k})$ is likely to have a different radius.

In fact, the rightmost part of (9.1) is a subset of a *Riemannian warped product*, and each stratum of $D \cap (B_{\mathbb{R}^k}(0,\sqrt{n}))^{\circ} \times S(\mathbb{R}^{n-k})$ inherits this warped product structure. Once we choose the appropriate warp we can, and shall, relate to (9.1) no longer as a topologically correct relationship, but as if it were correct without qualifiers.

Recall that the Riemannian warped product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) with a smooth warp function $\sigma^2 : M_1 \to [0, +\infty)$ is the Riemannian manifold

$$(M_1, M_2, \sigma) \;\stackrel{\Delta}{=}\; \big(M_1 \times M_2, g_1 + \sigma^2 g_2\big)\,. \tag{9.2}$$

Usually, as in our case, M_2 is a sphere. As an example of a warped product consider

$$\widetilde{M}_{\sigma} \;=\; \big(B_{\mathbb{R}^k}(0,\sqrt{n})\big)^{\circ} \times S(\mathbb{R}^{n-k}),$$

where the Riemannian metric on the open ball is given by

$$g_\sigma \;=\; g_{\mathbb{R}^k} + \nabla\sigma \otimes \nabla\sigma, \tag{9.3}$$

for $\sigma^2(t) \;=\; n - \|t\|^2_{\mathbb{R}^k}$, and the Riemannian metric on $S(\mathbb{R}^{n-k})$ is the canonical one inherited from $\mathbb{R}^{n-k}$. The importance of this example for us is that each $(n-k-1+j)$-dimensional stratum $\widetilde{D}_{n-k-1+j}$ of $\left(D \cap \left(S(\mathbb{R}^k)\right)^c\right) \times S(\mathbb{R}^{n-k})$ is isometrically embedded in this warped product and has the form $D_j \times S(\mathbb{R}^{n-k})$ for some j-dimensional submanifold D_j of the open ball $B_{\mathbb{R}^k}(0,\sqrt{n})^\circ$. Using this embedding we can compute

$$\mathcal{L}^{1/n}_{n-1-i}\left(\pi^{-1}_{\sqrt{n},n,k}D, \widetilde{D}_{n-k-1+j}\right),$$

the contribution of these strata to the Lipschitz-Killing curvatures of $\pi^{-1}_{\sqrt{n},n,k}D$.

The first step to computing these contributions is to determine the form of the Levi-Civita connection $\widetilde{\nabla}^\sigma$ of $\widetilde{M}_\sigma$, as this is needed to in order to compute the second fundamental form of $D_j \times S(\mathbb{R}^{n-k})$ in $\widetilde{M}_\sigma$. Consider, therefore, a general warped product (M_1, M_2, σ) and denote the Levi-Civita connection on each M_j by ∇^j. Use E, or E_j, to denote vector fields on M_1, identified with their natural extensions on $M_1 \times M_2$. Similarly, F, or F_j, denote vector fields on M_2 extended to $M_1 \times M_2$. Then, from the definition of Levi-Civita connections, the product structure of M and Koszul's formula it is not too hard to check that

$$\widetilde{\nabla}^\sigma_{E_1}E_2 \;=\; \nabla^1_{E_1}E_2, \qquad \widetilde{\nabla}^\sigma_{F_1}F_2 \;=\; \nabla^2_{F_1}F_2, \qquad \widetilde{\nabla}^\sigma_{E}F \;=\; \widetilde{\nabla}^\sigma_{F}E \;=\; E(\log\sigma)F. \tag{9.4}$$

9.2 A second fundamental form

With the Levi-Civita connection $\widetilde{\nabla}^\sigma$ determined, the next step towards computing Lipschitz-Killing curvatures lies in determining the second fundamental form over the sets $D_j \times S(\mathbb{R}^k)$ as they sit in $\widetilde{M}_\sigma$, as well as traces of their powers. For this we need to describe the normal spaces $T_{(t,\eta)}\widetilde{M}_\sigma^\perp$ for $(t,\eta) \in D_j \times S(\mathbb{R}^{n-k})$. A simple argument shows that, at these points,

$$(T_{(t,\eta)}\widetilde{M}_\sigma)^\perp \;=\; \left(T_tD_j \oplus T_\eta S(\mathbb{R}^{n-k})\right)^\perp \;\simeq\; T_tD_j^\perp, \tag{9.5}$$

where $T_tD_j^\perp$ is the orthogonal (with respect to g_σ) complement of T_tD_j in $T_tB_{\mathbb{R}^k}(0,\sqrt{n})$. From this, it is now not hard to see that the normal Morse index $D_j \times S(\mathbb{R}^k)$ as it sits in $\widetilde{M}^\sigma$ is actually the same as the Morse index of D_j as it sits in $B_{\mathbb{R}^k}(0,\sqrt{n})$. We can therefore state

Lemma 9.1 *Retaining the above notation, for $0 \le l \le n-1$, take*

$$(t,\eta) \in D_j \times S(\mathbb{R}^{n-k}), \qquad \nu_{k-j} \in (T_{(t,\eta)}D_j \times S(\mathbb{R}^{n-k}))^\perp.$$

Then

$$\frac{1}{l!}\mathrm{Tr}(S^l_{\nu_{k-j}}) \;=\; \sum_{r=0}^{l}\binom{n-k-1}{l-r}(-1)^{l-r}\left(\nu_{k-j}(\log\sigma_t)\right)^{l-r}\mathrm{Tr}(S^r_{\sigma,\nu_{k-j}}),$$

where S is the shape operator of $D_j \times S(\mathbb{R}^{n-k})$ in $\widetilde{M}_\sigma$ and S_σ is the shape operator of D_j in $(B_{\mathbb{R}^k}(0,\sqrt{n}), g_\sigma)$.

Proof. Fix an orthonormal (under g) basis $(E_1,\dots,E_k,F_1,\dots,F_{n-k-1})$ of $T_{(t,\eta)}\widetilde{M}_\sigma$ such that $(E_1,\dots,E_j)$ forms an orthonormal basis of T_tD_j. Observation (9.5) implies that any ν_{k-j} can be expressed as

$$\nu_{k-j} \;=\; \sum_{r=1}^{j} a_r E_r,$$

for some constants a_r. Applying (9.4) and the Weingarten equation we find

$$\begin{aligned}
S_{\nu_{k-j}}(E_r,E_s) &= -g(\nabla^\sigma_{E_r}\nu_{k-j},E_s) \;=\; -g_\sigma(\nabla^\sigma_{E_r}\nu_{k-j},E_s) \;=\; S_{\sigma,\nu_{k-j}}(E_r,E_s),\\
S_{\nu_{k-j}}(F_r,F_s) &= -g(\nabla^\sigma_{F_r}\nu_{k-j},F_s) \;=\; -\nu_{k-j}(\log\sigma_t)g(F_r,F_s) \;=\; -\nu_{k-j}(\log\sigma_t)\delta_{rs},\\
S_{\nu_{k-j}}(E_r,F_s) &= 0.
\end{aligned}$$

Therefore, for each ν_{k-j}, the matrix of the shape operator $S_{\nu_{k-j}}$ in our chosen orthonormal basis is block diagonal with one block, of size j, being $\left\{S_{\sigma,\nu_{k-j}}(E_r,E_s)\right\}_{1\le r,s\le j}$ and the other, of size $n-k-1$, being $-\nu_{k-j}(\log\sigma_t)I_{(n-k-1)\times(n-k-1)}$. Therefore, applying basic combinatorial properties of the trace operator we have that, for $l \le n-k-1$,

$$\frac{1}{l!}\mathrm{Tr}(S^l_{\nu_{k-j}}) \;=\; \sum_{r=0}^{l}\binom{n-k-1}{l-r}(-1)^{l-r}\left(\nu_{k-j}(\log\sigma_t)\right)^{l-r}\frac{1}{r!}\mathrm{Tr}(S^r_{\sigma,\nu_{k-j}}),$$

which completes the proof. □

Since Lipschitz-Killing curvatures are no more than integrals of powers of traces of second fundamental forms, we now have all nearly all we need to begin the final computation. All that remains between us and the final stage is a deeper understanding of Gaussian Minkowski functionals.

9.3 Gaussian Minkowski functionals

To complete our evaluation of the illusive limits (7.5), and via them the all-important functions $\rho_j(D)$ of (7.9), we want to express the $\rho_j(D)$ in terms of Gaussian Minkowski functionals. For this we need a representation of them beyond the tube formula of (4.4). As a first step, we define a family of generalised Lipschitz-Killing curvature measures $\widetilde{\mathcal{L}}_j$ as follows.

Definition 9.2 *Let $(M, \mathcal{Z}) \subset \mathbb{R}^l$ be a tame stratified C^2 manifold embedded in $\mathbb{R}^l$. The generalised Lipschitz-Killing curvature measures of M, defined on Borel sets $A \subset \mathbb{R}^l, B \subset S(\mathbb{R}^l)$ and supported on $M \times S(\mathbb{R}^l)$, are defined, for $0 \leq i \leq l-1$, by*

$$\begin{aligned} \widetilde{\mathcal{L}}_i(M, A \times B) \triangleq\ & \sum_{j=i}^{l} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{(-1)^m C(l-j, j-i-2m)}{m!\,(j-i-2m)!} \\ & \times \int_{\partial_j M \cap A} \int_{S(T^\perp \partial_j M) \cap B} \mathrm{Tr}^{T_t \partial_j M} \left(\widetilde{R}^m \widetilde{S}^{j-i-2m}_{\nu_{l-j}} \right) \alpha(\nu_{l-j})\, \mathcal{H}_{l-j-1}(d\nu_{l-j}) \mathcal{H}_j(dt). \end{aligned} \tag{9.6}$$

For any Borel function $f : \mathbb{R}^l \times S(\mathbb{R}^l) \to \mathbb{R}$ we use the standard notation $\widetilde{\mathcal{L}}_i(M, f)$ to denote the integral of f with respect to $\widetilde{\mathcal{L}}_i(M, \cdot)$. If $i = l$, we define $\widetilde{\mathcal{L}}_l(M, \cdot)$ only on Borel functions on $\mathbb{R}^l \times S(\mathbb{R}^l)$ that are constant over $S(\mathbb{R}^l)$. Specifically,

$$\widetilde{\mathcal{L}}_l(M, f) \triangleq \begin{cases} \int_M f(t, \nu)\, \mathcal{H}_l(dt) & f \in \mathcal{B}(\mathbb{R}^l) \times S(\mathbb{R}^l), \\ 0 & \textit{otherwise}, \end{cases} \tag{9.7}$$

for some fixed but arbitrary $\nu \in S(\mathbb{R}^l)$ where $\mathcal{B}(\mathbb{R}^l) \times S(\mathbb{R}^l) = \left\{ A \times S(\mathbb{R}^l) : A \in \mathcal{B}(\mathbb{R}^l) \right\}$ is the sub-sigma algebra of $\mathcal{B}(\mathbb{R}^l) \otimes \mathcal{B}(S(\mathbb{R}^l))$ generated by functions that are constant over $S(\mathbb{R}^l)$.

The Minkowski curvature measures and generalised Minkowski curvature measures are then

$$\mathcal{M}_j(M, A) \triangleq (j!\,\omega_j)\, \mathcal{L}_{l-j}(M, A), \qquad \widetilde{\mathcal{M}}_j(M, A \times B) \triangleq (j!\,\omega_j)\, \widetilde{\mathcal{L}}_{l-j}(M, A \times B). \tag{9.8}$$

With this definition we can now give the formal statement of the tube formula (4.4). For a full proof see [24] for manifolds with C^2 boundary or [3] in the full generality of the following.

Lemma 9.3 *Suppose $(M, \mathcal{Z}) \subset \mathbb{R}^l$ is a C^2, locally convex, tame, Whitney stratified manifold, and that for every $\varepsilon > 0$ there exists $K(\varepsilon) \subset \mathbb{R}^l$ compact such that*

$$\int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!} |\widetilde{\mathcal{M}}_{j+1}| \left(M, \mathbb{1}_{K(\varepsilon)^c} \circ F_{-r} \left| \sum_{m=0}^{n} \frac{r^m}{m!} H_m\left(\langle \eta, t \rangle\right) \mathrm{e}^{-|t|^2/2} \right| \right) dr \ < \varepsilon,$$

uniformly in n, where H_m is the m-th Hermite polynomial (8.12). Then

$$\gamma(\mathrm{Tube}(M, \rho)) = \gamma(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^\gamma(M), \tag{9.9}$$

where

$$\mathcal{M}_j^\gamma(M) \triangleq (2\pi)^{-l/2} \sum_{m=0}^{j-1} \binom{j-1}{m} \widetilde{\mathcal{M}}_{m+1} \left(M, H_{j-1-m}\left(\langle \eta, t \rangle\right) \mathrm{e}^{-|t|^2/2} \right). \tag{9.10}$$

Note that while the tube formula here is claimed to hold only under some side conditions on M, the definition (9.10) of the $\mathcal{M}_j^\gamma$ makes sense without these. In many cases, the $\mathcal{M}_j^\gamma$ are quite easy to compute. For example, if $M = [u,\infty) \subset \mathbb{R}$, then, since $\mathrm{Tube}([u,\infty),\rho) = [u-\rho,\infty)$, comparing a Taylor series expansion of $\gamma(\mathrm{Tube}(M,\rho))$ with (9.9) easily gives $\mathcal{M}_j^\gamma([u,\infty)) = (\sqrt{2\pi})^{-1}H_{j-1}(u)\mathrm{e}^{-u^2/2}$. Additional examples, also accessible from simple calculus, can be found in [24].

9.4 Some volume computations

With the geometry behind us, we now turn to the asymptotics required for computing the limits (7.5), for which the following lemma is a crucial step. The final move to the Gaussian Minkowski functionals appearing in Theorem 5.1 above will then involve no more than some careful asymptotics.

Lemma 9.4 *As usual, D is a tame stratified subset of $\mathbb{R}^k$ with j-dimensional strata D_j. Suppose that $\widetilde{D}_{n-k-1+j} = D_j \times S(\mathbb{R}^{n-k})$ is an $(n-k-1+j)$-dimensional stratum of $\pi^{-1}_{\sqrt{n},n,k}D$ such that*

$$\widetilde{D}_{n-k-1+j} \cap S(\mathbb{R}^k) \;=\; \emptyset.$$

Then, for all $i \geq k-j \geq 0$,

$$\begin{aligned}
&\mathcal{L}^{1/n}_{n-1-i}\left(\pi^{-1}_{\sqrt{n},n,k}D, \widetilde{D}_{n-k-1+j}\right) \qquad (9.11)\\
&\quad = \; s_{n-k}\sum_{l=0}^{i+j-k}\frac{s_{k+l-j}}{s_i}\binom{n-k-1}{i+j-k-l}\widetilde{\mathcal{L}}_{j-l}\left(D^\sigma,\, \sigma^{n+k-2i-2j+2l-1}(2\pi)^{-k/2}h_{i+j-k-l}\mathbb{1}_{D_j}\right)
\end{aligned}$$

where $D^\sigma = D \cap B_{\mathbb{R}^k}(0,\sqrt{n})$ is the regular stratified manifold obtained from the intersection of the embedding of D in $\mathbb{R}^k$ and the open ball of radius $\sqrt{n}$, endowed with the metric g_σ given by (9.3) *and*

$$h_l(t,\nu) \;\stackrel{\Delta}{=}\; \langle \nu, t\rangle^l_{\mathbb{R}^k}.$$

Furthermore, suppose that, for all $0 \leq r,s \leq i+k-j$,

$$\widetilde{\mathcal{L}}_r\left(D,\, |h_s|\,\varphi_k\mathbb{1}_{D_j}\right) \;<\; \infty, \qquad (9.12)$$

where φ_k denotes the k-dimensional Gaussian density $(2\pi)^{-k/2}e^{-|t|^2/2}$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{s_n\, n^{(n-1)/2}}\mathcal{L}^{1/n}_{n-1-i}\left(\pi^{-1}_{\sqrt{n},n,k}D, \widetilde{D}_{n-k-1+j}\right) \qquad (9.13)\\
&\quad = \sum_{l=0}^{i+j-k}\frac{[k+l-j]!}{[i]!}\binom{i-1}{k+l-j-1}\widetilde{\mathcal{L}}_{j-l}\left(D, \varphi_k(t)h_{i+j-k-l}\mathbb{1}_{D_j}\right).
\end{aligned}$$

Proof. From (8.16)

$$\mathcal{L}^{1/n}_{n-1-i}\left(\pi^{-1}_{\sqrt{n},n,k}D, \widetilde{D}_{n-k-1+j}\right) \;=\; \frac{C(k-j,i+j-k)}{(2\pi)^{(i+j-k)/2}(i+j-k)!}\int_{D_j\times S(\mathbb{R}^{n-k})}\int_{S(T_{(t,\eta)}D_j\times S(\mathbb{R}^{n-k})^{\perp})} \times \mathrm{Tr}(S^{i+j-k}_{\nu_{k-j}})\,\alpha(\nu_{k-j})\;\mathcal{H}_{k-j}(d\nu_{k-j})\;\mathcal{H}_{n-1-j+k}(dt,d\eta),$$

where

$$\mathcal{H}_{n-1-j+k}(dt,d\eta) \;=\; \sigma_t^{n-1-k}\;\mathcal{H}_{n-1-k}(d\eta)\mathcal{H}_j(dt)$$

is the Hausdorff measure that $D_j \times S(\mathbb{R}^{n-k})$ inherits from $\widetilde{D}^{\sigma}$, the warped product of (D^{σ}, g_{σ}) and $S(\mathbb{R}^k)$ with its usual metric and warp function σ^2 as in (9.3).

By Lemma 9.1, equation (9.5) and the subsequent remarks about the normal Morse index α,

$$\begin{aligned}
&\mathcal{L}^{1/n}_{n-1-i}\left(\pi^{-1}_{\sqrt{n},n,k}D, \widetilde{D}_{n-k-1+j}\right)\\
&\quad= C(k-j,i+j-k)(2\pi)^{-(i+j-k)/2}\int_{D_j}\int_{S(\mathbb{R}^{n-k})}\int_{S(T_tD_j^{\perp})}\\
&\quad\times \sum_{l=0}^{i+j-k}\binom{n-1-k}{i+j-k-l}\sigma_t^{n-1-k}(-1)^{i+j-k-l}\,(\nu_{k-j}(\log\sigma_t))^{i+j-k-l}\\
&\quad\times \;\frac{1}{l!}\mathrm{Tr}(S^{l}_{\sigma,\nu_{k-j}})\,\alpha(\nu_{k-j})\;\mathcal{H}_{k-j}(d\nu_{k-j})\;\mathcal{H}_{n-1-k}(d\eta)\;\mathcal{H}_j(dt).
\end{aligned}$$

Equation (9.11) now follows from the fact that

$$\frac{C(k-j,i+j-k)\,(2\pi)^{-(i+j-k)/2}}{C(k-j,l)\,(2\pi)^{-l/2}} \;=\; \frac{s_{k+l-j}}{s_i},$$

followed by integrating over $S(\mathbb{R}^{n-k})$ and noting that

$$\nu_{k-j}(\log\sigma_t) \;=\; -\frac{\langle \nu_{k-j}, t\rangle_{\mathbb{R}^k}}{\sigma_t^2}.$$

As for the second conclusion of the lemma, (9.13), note that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{s_{n-k}}{s_n n^{(n-1)/2}}\binom{n-k-1}{i+j-k-l}\sigma_t^{n+k-2i-2j+2l-1}\\
&\quad= \lim_{n\to\infty}\frac{s_{n-k}}{s_n n^{(n-1)/2}}\binom{n-k-1}{i+j-k-l}n^{(n+k-2i-2j-2l-1)/2}\left(1-\frac{\|t\|^2}{n}\right)^{(n+k-2i-2j-2l-1)/2}\\
&\quad= \frac{(2\pi)^{-k/2}}{(i+j-k-l)!}e^{-\|t^2\|/2}.
\end{aligned}$$

Also,

$$\frac{s_{k+l-j}}{s_i\,(i+j-k-l)!} \;=\; \frac{s_{k+l-j}\,(k+l-j-1)!}{s_i\,(i-1)!}\binom{i-1}{k+l-j} \;=\; \frac{[k+l-j]!}{[i]!}\binom{i-1}{k+l-j}.$$

Finally, since it is not hard to see that there exists a finite K such that, for all n large enough, $\left(1-|t|^2/n\right)^{n/2} \le Ke^{-|t|^2/2}$ for all $t \in B_{\mathbb{R}^k}(0,\sqrt{n})^\circ$, dominated convergence yields (9.13) and we are done. □

In Lemma 9.4 we computed the contribution of the sets $\widetilde{D}_{n-k-1+j} = D_j \times S(\mathbb{R}^{n-k})$ to the curvatures $\mathcal{L}^{n^{-1}}_{n-1-i}(\pi^{-1}_{\sqrt{n},n,k}D)$ under the assumption that $\widetilde{D}_{n-k-1+j} \cap S(\mathbb{R}^k) = \emptyset$.

Our task now is to show that if, in fact $\widetilde{D}_{n-k-1+j} \cap S(\mathbb{R}^k) \ne \emptyset$, then there is actually no contribution to $\mathcal{L}^{n^{-1}}_{n-1-i}(\pi^{-1}_{\sqrt{n},n,k}D)$ for n large enough. We write this as

Lemma 9.5 *Suppose that D satisfies the conditions of Lemma 9.4 and that $D_j \subset S_{\sqrt{n}}(\mathbb{R}^k)$ is a stratum of $D \cap B_{\mathbb{R}^k}(0,\sqrt{n})$ for some $n > 0$. Then, for $n > j+i+1$,*

$$\mathcal{L}^{n^{-1}}_{n-1-i}(\pi^{-1}_{\sqrt{n},n,k}D, \pi^{-1}_{\sqrt{n},n,k}D_j) \;=\; 0,$$

and so, for all sufficiently large n,

$$\mathcal{L}^{n^{-1}}_{n-1-i}(\pi^{-1}_{\sqrt{n},n,k}D, h\mathbb{1}_{\pi^{-1}_{\sqrt{n},n,k}D_j}) \;\equiv\; 0$$

for all h.

Proof. Since $D_j \subset S_{\sqrt{n}}(\mathbb{R}^k)$ it is obvious that $\pi^{-1}_{\sqrt{n},n,k}D_j = D_j$, and so $\pi^{-1}_{\sqrt{n},n,k}D_j$ is a j-dimensional stratum of $\pi^{-1}_{\sqrt{n},n,k}D$. From Definition 9.2, we see that such strata only contribute to the intrinsic volumes of order 0 to j, as required. □

9.5 Completing the proof of Theorem 5.1 on $S(\mathbb{R}^l)$

Theorem 5.1, for the case of the canonical process on $S(\mathbb{R}^l)$, will now follow immediately from Theorem 7.2 and the following result.

Theorem 9.6 *Suppose D satisfies the conditions of Lemma 9.4. Then, in the notation of that lemma,*

$$\rho_i(D) \;=\; (2\pi)^{-i}\mathcal{M}^\gamma_i(D), \tag{9.14}$$

where the functionals $\mathcal{M}^\gamma_i(D)$ are defined by (9.10)

Proof. We start by computing the $\widetilde{\rho}_i$. By Lemmas 9.4 and 9.5,

$$\begin{aligned}
\widetilde{\rho}_i(D) &= (2\pi)^{-i/2}[i]! \sum_{j=k-i}^{k-1} \sum_{l=0}^{i+j-k} \frac{[k+l-j]!}{[i]!} \binom{i-1}{k+l-j-1} \widetilde{\mathcal{L}}_{j-l}\left(D, \varphi_k h_{i+j-k-l} \mathbb{1}_{D_j}\right) \\
&= (2\pi)^{-i/2} \sum_{j=k-i}^{k-1} \sum_{l=0}^{i+j-k} \binom{i-1}{k+l-j-1} \widetilde{\mathcal{M}}_{k+l-j}\left(D, \varphi_k h_{i+j-k-l} \mathbb{1}_{D_j}\right) \\
&= (2\pi)^{-i/2} \sum_{m=0}^{i-1} \binom{i-1}{m} \widetilde{\mathcal{M}}_{m+1}\left(D, \varphi_k h_{i-1-m}\right),
\end{aligned}$$

where the $\widetilde{M}$ are the generalised Minkowski curvature measures defined at (9.8).

With the $\widetilde{\rho}_i$ determined, we can now turn to the ρ_j. By (7.9) these are given by

$$\begin{aligned}
\rho_i(D) &= (i-1)! \sum_{l=0}^{\lfloor \frac{i-1}{2} \rfloor} \frac{(-1)^l}{(4\pi)^l l!(i-1-2l)!} \widetilde{\rho}_{i-2l}(D) \\
&= (i-1)! \sum_{l=0}^{\lfloor \frac{i-1}{2} \rfloor} \sum_{m=0}^{i-2l-1} \frac{(-1)^l}{(4\pi)^l l!(i-1-2l)!} (2\pi)^{-(i-2l)/2} \\
&\qquad\qquad \times \binom{i-2l-1}{m} \widetilde{\mathcal{M}}_{m+1}\left(D, \varphi_k h_{i-2l-1-m}\right) \\
&= (2\pi)^{-(i+k)/2} \sum_{m=0}^{i-1} \binom{i-1}{m} \widetilde{\mathcal{M}}_{m+1}\left(D, H_{i-m-1}(\langle \eta, t\rangle) e^{-|t|^2/2}\right) \\
&= (2\pi)^{-i} \mathcal{M}_i^{\gamma}(D),
\end{aligned}$$

the second last line coming from the definition (8.12) of the Hermite polynomials, and the last from the definition (9.10) of the $\mathcal{M}_i^{\gamma}(D)$. □

10 A general representation for $\mathbb{E}\{\mathcal{L}_j(A_u(f,m))\}$.

Our aim over the next few sections will be to prove the following result, which is the formal version of (2.1).

Theorem 10.1 *Let M, D and y be as in Theorem 5.1. Then, retaining the notation of that theorem, there exist functions $\widetilde{\rho}(i,j,D)$ dependent on all the parameters displayed, but not on the distribution of y, such that*

$$\mathbb{E}\left\{\mathcal{L}_i(M \cap y^{-1}(D)\right\} = \sum_{j=0}^{\dim M - i} \mathcal{L}_{i+j}(M)\, \widetilde{\rho}(i,j,D). \tag{10.1}$$

As described in Section 2, the main result of the paper, Theorem 5.1, follows immediately from Theorems 7.2, 9.6, and the above result.

The proof of Theorem 10.1 is not short and so we shall break it up into a series of smaller calculations with some necessary digressions along the way. The basic idea is to write $M \cap y^{-1}(D)$ as

$$M \cap y^{-1}(D) \;=\; \bigcup_{j=0}^{\dim M} \bigcup_{l=0}^{j} \partial_j M \cap y^{-1}\left(\partial_{k-l} D\right) \;\stackrel{\Delta}{=}\; \bigcup_{j=0}^{\dim M} M_{jl}. \tag{10.2}$$

If we also assume, without loss of generality, that $k \geq \dim(\widetilde{M})$, then, with probability one, each M_{jl} will be a (random) manifold of dimension $j-l$ in $\widetilde{M}$. Since Lipschitz-Killing curvatures are additive, it follows that

$$\mathcal{L}_i(M \cap y^{-1} D) \;=\; \sum_{j=i}^{\dim M} \sum_{l=0}^{j-i} \mathcal{L}_i\left(M \cap y^{-1} D, M_{jl}\right). \tag{10.3}$$

Thus all that we need show, in order to prove Theorem 10.1, is that

$$\mathbb{E}\left\{\mathcal{L}_i\left(M, M_{jl}\right)\right\} \;=\; \sum_{m=0}^{j-i} \mathcal{L}_{i+m}(M, \partial_j M) \cdot \widehat{\rho}(i, m, \partial_l D), \tag{10.4}$$

a result that we shall finally prove in Section 12. Firstly, however, we want to claim that it will actually suffice to establish (10.4) for the case $i = 0$; i.e. for the Euler-Poincaré characteristic. That this is in fact the case follows from the following section.

11 Crofton formulae

The classic Crofton formula works with nice subsets M of $\mathbb{R}^N$ and states that

$$\int_{\mathrm{Graff}(N,N-k)} \mathcal{L}_j(M \cap V)\, d\lambda^N_{N-k}(V) \;=\; \begin{bmatrix} k+j \\ j \end{bmatrix} \mathcal{L}_{k+j}(M), \tag{11.1}$$

where $\mathrm{Graff}(N,k)$ is the affine Grassmanian of k-dimensional flats in $\mathbb{R}^N$ with invariant measure ν^N_k normalised so that $\nu^N_k(\mathrm{Gr}(N,k)) = \begin{bmatrix} N \\ k \end{bmatrix}$. In its original form, (11.1) was proven for M in the convex ring (cf. [15, 17, 21]) and was extended to the setting of Whitney stratified manifolds in [7]. The special case $k = 0$ of Crofton's formula is generally known as *Hadwiger's formula* and is given by

$$\mathcal{L}_k(M) \;=\; \int_{\mathrm{Graff}(N,N-k)} \mathcal{L}_0(M \cap V)\, d\lambda^N_{N-k}(V). \tag{11.2}$$

The importance of Hadwiger's formula is gives all the Lipschitz-Killing curvatures of M in terms of the Euler-Poincaré characteristic of its cross-sections, and it is this aspect of it that we wish to exploit later.

Note, however, that (11.1) and (11.2) are Euclidean results and that the $\mathcal{L}_j$ appearing there are all computed with respect to the standard (Euclidean) Riemannian metric on $\mathbb{R}^N$. This is important and somewhat limiting. To avoid this limitation we shall develop a different, but analogous, result, the basic idea of which is to replace the cross-sections $M \cap V$ by something more appropriate for the manifold setting. In particular, we shall introduce an auxiliary set of *random* manifolds D for which the union of the $M \cap D$ over all D again gives M and, more importantly, for which $\mathcal{L}_j(M)$ can be computed by averaging the Euler-Poincaré characteristics $\mathcal{L}_0(M \cap D)$ over D. The result, which would seem to be of significant interest beyond the application that we have in mind, is as follows.

Theorem 11.1 *Let $\widetilde{M}$ be a C^2, n-dimensional manifold (without boundary) embedded in a C^3 manifold. Let $y^1, \dots, y^k$ be centred, unit variance, i.i.d. Gaussian processes on $\widetilde{M}$ satisfying the conditions of the y_j of Theorem 5.1. For $u \in \mathbb{R}^k$ define the (random) submanifold*

$$D_u = \left\{ t \in \widetilde{M} : \, y^1_t = u_1, \dots, y^k_t = u_k \right\}$$

of $\widetilde{M}$, and suppose $Z_k = (Z^1_k, \dots, Z^k_k) \sim \gamma_{\mathbb{R}^k}$ independently of the process $y = (y^1, \dots, y^k)$. Then, for $0 \le j \le \dim(\widetilde{M}) - k$,

$$\mathbb{E}\left\{ \mathcal{L}_j(\widetilde{M} \cap D_{Z_k}) \right\} \;=\; (2\pi)^{-k/2} \frac{[k+j]!}{[j]!} \mathcal{L}_{k+j}(\widetilde{M}), \tag{11.3}$$

where the $\mathcal{L}_j$ are computed with respect to the Riemannian metric (5.1) induced on $\widetilde{M}$ by the y^i.

Proof. We start by noting that it suffices to prove (11.3) for the Euler-Poincaré characteristic $\mathcal{L}_0$. That is, we need only prove that

$$\mathbb{E}\left\{ \mathcal{L}_0(\widetilde{M} \cap D_{Z_k}) \right\} \;=\; (2\pi)^{-k/2} [k]! \mathcal{L}_k(\widetilde{M}). \tag{11.4}$$

This follows from the following observation: For $j > 0$, take $Z_j \sim \gamma_{\mathbb{R}^j}$ independent of everything else, and write Z_{k+j} for the concatenation of Z_k and Z_j. Note that, in distribution,

$$\widetilde{M} \cap D_{Z_{k+j}} \;=\; \left(\widetilde{M} \cap D_{Z_j} \right) \cap D_{Z_k}.$$

Conditioning first on Z_k, apply (11.4) to the manifold $\widetilde{M} \cap D_{Z_k}$ to see that

$$\mathbb{E}\left\{ \mathcal{L}_j \left(\widetilde{M} \cap D_{Z_k} \right) \right\} \;=\; \frac{(2\pi)^{j/2}}{[j]!} \mathbb{E}\left\{ \mathcal{L}_0 \left(\widetilde{M} \cap D_{Z_{k+j}} \right) \right\} \;=\; (2\pi)^{-k/2} \frac{[k+j]!}{[j]!} \mathcal{L}_{k+j}(\widetilde{M}),$$

which establishes the general case.

We now turn to establishing (11.4). To do this we shall adopt an approach based on Morse theory (e.g. [18]) which allows us to write the Euler-Poincaré characteristics in terms of the critical points of an appropriate Morse function. To provide such a Morse function, we take an independent realisation of the y_j, which we denote by $\widetilde{y}$.

There is a problem however, since the set $\widetilde{M} \cap D_{Z_k}$ is random, and classical Morse theory is designed to handle deterministic sets and deterministic Morse functions. We can somewhat simplify things by first conditioning on both $Z_k = u \in \mathbb{R}^k$ and a fixed realisation of $\widetilde{y}$. However, even then, D_u remains random, since it depends on y. Fortunately, there is a long history of tackling expectations of Euler-Poincaré characteristics of such random manifolds, which gives the following representation for fixed Z_k and $\widetilde{y}$:

$$\begin{aligned}
&\mathbb{E}\left\{\mathcal{L}_0\left(\widetilde{M} \cap D_u\right)\right\} && (11.5)\\
&\quad = \frac{1}{(n-k)!}\int_{\widetilde{M}} \mathbb{E}\left\{\mathrm{Tr}^{L_t^\perp}\left((-\nabla^2 \widetilde{y}_{|D})^{n-k}\right)\, J_t \Big| y = u, P_{L_t}^\perp \nabla\widetilde{y} = 0\right\} p_{y, P_{L_t}^\perp \nabla \widetilde{y}}(u, 0)\, \mathcal{H}_n(dt).
\end{aligned}$$

There are many terms here requiring definition or explanation. Firstly, give $\widetilde{M}$ the Riemannian metric induced by y. Then ∇ is the associated Riemannian gradient, so that $\nabla \widetilde{y}_t$ is in

$$L_t \;\stackrel{\Delta}{=}\; \mathrm{span}\left(\nabla y_t^i,\ 1 \le i \le k\right).$$

The Riemannian Hessian is denoted by ∇^2, and with G_t the $k \times k$ matrix with elements $g_{ij,t} = \langle \nabla y_t^i, \nabla y_t^j\rangle$ and g_t^{ij} the elements of G_t^{-1}, a few calculations show that

$$\nabla^2 \widetilde{y}_{|D,t} \;=\; \nabla^2 \widetilde{y}_t - \sum_{i,j=1}^{k} \langle \nabla \widetilde{y}_t, \nabla y_t^i \rangle g_t^{ij} \nabla^2 y_t^j \;=\; \nabla^2 \widetilde{y}_t - \sum_{i,j=1}^{k} \langle P_{L_t} \nabla \widetilde{y}_t, \nabla y_t^i \rangle g_t^{ij} \nabla^2 y_t^j. \qquad (11.6)$$

Finally, $p_{y, P_{L_t \nabla \widetilde{y}}^\perp}$ is the joint probability density of y and $P_{L_t \nabla \widetilde{y}}^\perp$, and $J_t = (\det(G_t))^{1/2}$.

Equation (11.5) is not 'easy to see' (although its general form will be familiar to afficiandos) and comes from a long history of results computing expectations of functionals of point processes. A full proof would also be quite long and, although (11.5) is crucial to the proof of Theorem 11.1 it is unrelated to all other arguments in this paper. We therefore refer you to [3] for full details.

The remainder of the proof now hinges on evaluating the integrand in (11.5) and taking its expectation over u. It is standard Gaussian fare that, since the y_j have constant variance, y_t and ∇y_t are independent. Consequently the same is true of y_t and L_t, and y_t and J_t, and so we can actually integrate out u first in (11.5) to obtain

$$\mathbb{E}\left\{\mathcal{L}_0\left(\widetilde{M} \cap D_{Z_k}\right)\right\} \;=\; \frac{1}{(n-k)!}\int_{\widetilde{M}} \mathbb{E}\left\{\mathrm{Tr}^{L_t^\perp}\left((-\nabla^2 \widetilde{y}_{|D})^{n-k}\right)\, J_t \Big| P_{L_t}^\perp \nabla\widetilde{y} = 0\right\} p_{P_{L_t}^\perp \nabla \widetilde{y}}(0)\, \mathcal{H}_n(dt).$$

Now note that, given the subspace L_t, the pair $(\nabla^2 \widetilde{y}_{|D,t},\ J_t)$ is conditionally independent of $P_{L_t}^\perp \nabla \widetilde{y}$, which then has a standard Gaussian distribution on L_t. From this it is immediate that the integrand above is given by

$$(2\pi)^{-(n-k)/2}\mathbb{E}\left\{\mathrm{Tr}^{L_t^\perp}\left((-\nabla^2 \widetilde{y}_{|D})^{n-k}\right)\, J_t\right\}.$$

We now condition on $(\nabla^2\widetilde{y}_{|D,t}, J_t)$, so that the next step is to compute the conditional expectation

$$\mathbb{E}\left\{\mathrm{Tr}^{L_t^\perp}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right) J_t \middle| \nabla^2\widetilde{y}_{|D,t}, J_t\right\}.$$

Because of the conditioning, this is just the expected value of the trace of a double form α on $T_t\widetilde{M}$, restricted to a random subspace of dimension $n-k$.. Lemma 13.1 below shows how to evaluate such expectations, and in our case it follows from the lemma that

$$\mathbb{E}\left\{\mathrm{Tr}^{L_t^\perp}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right) J_t \middle| \nabla^2\widetilde{y}_{|D,t}, J_t\right\} \;=\; \binom{n}{k}^{-1} \mathrm{Tr}^{T_t\widetilde{M}}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right) J_t. \tag{11.7}$$

To complete the computation, we now need to evaluate

$$\mathbb{E}\left\{\mathrm{Tr}^{T_t\widetilde{M}}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right) J_t\right\}.$$

Recall (11.6), which gives a different way of writing $\nabla^2\widetilde{y}_{|D}$. In particular, consider the second term in the last expression there, which involves the term

$$\sum_{i,j=1}^k \langle P_{L_t}\nabla\widetilde{y}_t, \nabla y_t^i\rangle g_t^{ij}\nabla^2 y_t^j.$$

We want a more user-friendly version of this. To obtain it, for the moment we drop the dependence on t and define

$$V_i \;\triangleq\; \sum_{j=1}^k g_{ij}^{-1/2}\langle P_L\nabla\widetilde{y}, \nabla y^i\rangle,$$

where the $g_{ij}^{-1/2}$ are the elements of the matrix $G^{-1/2}$ $(= G_t^{-1/2})$. Then, conditional on $\nabla y^1, \dots, \nabla y^k, \nabla^2 y^1, \dots, \nabla^2 y^k$, we have $V \sim N(0, I_{k\times k})$. Since the conditional distribution does not depend on the conditioning variables, the V_i are actually independent of them. Furthermore, since the ∇y^i and $\nabla^2 y^j$ are all independent of one another, we have that, for each t,

$$\sum_{i,j=1}^k \langle P_{L_t}\nabla\widetilde{y}_t, \nabla y_t^i\rangle g_t^{ij}\nabla^2 y_t^j \quad\stackrel{\mathcal{L}}{=}\quad \sum_{i,j=1}^k V_i W_{ij}^{-1/2}\nabla^2 y^j,$$

where, $W \sim \mathrm{Wishart}(n, I_{k\times k})$, independently of everything else and $\stackrel{\mathcal{L}}{=}$ denotes equivalence in law. (cf. Section 13 for details on Wishart distributions.) Consequently, by (11.6),

$$\mathrm{Tr}^{T_t\widetilde{M}}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right) J_t \;\stackrel{\mathcal{L}}{=}\; \mathrm{Tr}^{T_t\widetilde{M}}\left(\left(\nabla^2\widetilde{y} - \sum_{i,j=1}^k V_i W_{ij}^{-1/2}\nabla^2 y^i\right)^{n-k}\right)\sqrt{\det(W)},$$

where, as above, $W \sim \mathrm{Wishart}(n, I_{k\times k})$ and the random matrices $\nabla^2\widetilde{y}_t$, $\nabla^2 y_t^1, \dots, \nabla^2 y_t^k$ are independent and identically distributed.

From Lemma 13.2 below it then follows that

$$\mathbb{E}\left\{\mathrm{Tr}^{T_t\widetilde{M}}\left((-\nabla^2\widetilde{y}_{|D})^{n-k}\right)\, J_t\right\} \;=\; (2\pi)^{-k/2}\frac{n!}{(n-k)!}\omega_k\mathbb{E}\left\{\mathrm{Tr}^{T_t\widetilde{M}}\left((-\nabla^2\widetilde{y})^{n-k}\right)\right\}. \qquad (11.8)$$

This expectation is, however, something that has been computed before (cf. [24, 25]), as well as its integral over $\widetilde{M}$, giving

$$\int_{\widetilde{M}}\mathbb{E}\left\{\mathrm{Tr}^{T_t\widetilde{M}}((-\nabla^2\widetilde{y}_t)^{n-k})\right\}\mathcal{H}_n(dt) \;=\; (2\pi)^{(n-k)/2}(n-k)!\,\mathcal{L}_k(\widetilde{M}).$$

Putting all the pieces together we have enough to prove (11.3) and so we are done, modulo proving Lemmas 13.1 and 13.2, which we defer to Section 13. □

We shall also need an extension of Theorem 11.1 to regular stratified manifolds. The proof is similar, although somewhat more involved due to the stratification (cf. the proof of Theorem 10.1 in Section 12) and details can be found in [3].

Theorem 11.2 *Let $\widetilde{M}$, $y^1,\dots,y^k$, Z_k and D_u be as in Theorem 11.1, and let M be a regular stratified submanifold of $\widetilde{M}$. Then, for all $j\le \dim(M)-k$,*

$$\mathbb{E}\left\{\mathcal{L}_j(M\cap D_{Z_k})\right\} \;=\; (2\pi)^{-k/2}\frac{[k+j]!}{[j]!}\mathcal{L}_{k+j}(M).$$

12 Proof of Theorem 10.1

In the notation of the previous two sections, we shall prove that, under the conditions of Theorem 10.1,

$$\mathbb{E}\left\{\mathcal{L}_0\left(M, M_{jl}\right)\right\} \;=\; \sum_{i=0}^{j}\mathcal{L}_i^1(M,\partial_j M)\cdot\widehat{\rho}(i,\partial_{k-l}D), \qquad (12.1)$$

for some functions $\widehat{\rho}$ depending only on the parameters displayed. Once this is established, we can apply the linear relationships (6.5) between the $\mathcal{L}_i^1$ and the $\mathcal{L}_i$ to move from a representation involving the $\mathcal{L}_i$ rather than the $\mathcal{L}_i^1$. (While the functionals $\widehat{\rho}$ will then change, this is not important for us.) The proof of Theorem 10.1 then follows from the addivity of the curvature measures $\mathcal{L}_i^1(M,\cdot)$ and the Crofton formula of Theorem 11.2. The argument goes as follows.

Fix $0\le j\le\dim(M)$ and introduce an auxiliary set of Gaussian processes $y^1,\dots,y^j$ and a Gaussian random variable Z_j as in Theorem 11.1, satisfying the conditions required there. As before, write $D_u = \{t \in M : y_t^1 = u_1, \ldots, y_t^j = u_j\}$. Then, by Theorem 11.2,

$$\begin{aligned}\mathbb{E}\left\{\mathcal{L}_j(M \cap y^{-1}(D))\right\} &= \frac{(2\pi)^{j/2}}{[j]!}\mathbb{E}\left\{\mathcal{L}_0(M \cap y^{-1}(D)) \cap D_{Z_j})\right\} \\ &= \frac{(2\pi)^{j/2}}{[j]!}\mathbb{E}\left\{\mathbb{E}\left\{\mathcal{L}_0(M \cap y^{-1}(D)) \cap D_{Z_j})\big|D_{Z_j}\right\}\right\}.\end{aligned}$$

We now note that, with probability one, the sets $M \cap y^{-1}(D) \cap D_{Z_j}$ are regular stratified manifolds, and so we can apply (12.1) and additivity to compute the inner expectation above, giving

$$\begin{aligned}\mathbb{E}\left\{\mathcal{L}_j(M \cap y^{-1}(D))\right\} &= \sum_{l=0}^{\dim M - j} \widehat{\rho}(j,l,D)\,\mathbb{E}\left\{\mathcal{L}_l^1(M \cap D_{Z_k})\right\} \\ &= \sum_{l=0}^{\dim M - j} \widehat{\rho}(j,l,D)\,\mathbb{E}\left\{\mathcal{L}_l(M \cap D_{Z_k})\right\} \\ &= \sum_{l=0}^{\dim M - j} \widehat{\rho}(j,l,D)\mathcal{L}_{j+l}^1(M),\end{aligned}$$

again by (12.1), and allowing the $\widehat{\rho}(j,l,D)$ to change from line to line, but always remaining functions only of j, l and D. Converting again from the $\mathcal{L}_j^1$ to $\mathcal{L}_j$ this is (10.1) and so Theorem 10.1 is established.

All that remains, therefore, is to establish (12.1). As in the proof of the Gaussian Crofton Theorem 11.1 we introduce an independent process $\widetilde{y}$ with the same distribution as the y_j to act as a (random) Morse function and use it to find an integral expression analogous to (11.5) for $\mathbb{E}\{\mathcal{L}_0(M; M_{jl})\}$. Unfortunately, however, we shall first require a little more notation and shall need to write the strata of D in a form that will simplify computations.

In particular, suppose that $F_l : \mathbb{R}^k \to \mathbb{R}^l$ is a C^2 function with $F_l^{-1}(0)$ a $(k-l)$-dimensional C^2 submanifold containing of $\partial_{k-l}D$. The stratum $\partial_{k-l}D$ will generally have to be broken up into smaller pieces to achieve this, in which case we can think of $\partial_{k-l}D$ as a generic stratum of dimension $k-l$ rather than the union of all $(k-l)$-dimensional strata. With this in mind, from now on we shall assume that F_l has been chosen so that

$$\partial_{k-l}D \;\subset\; \left\{x \in \mathbb{R}^k : F_{l,r}(x) = 0,\, 1 \le r \le l\right\}.$$

We now define $f_t = F_l(y_t)$, a $\mathbb{R}^l$ valued random process on M with components $f_r(t) = F_{l,r}(y_t)$, $r = 1, \ldots, l$, dropping the unnecessary dependence on l, which will be fixed for the remainder of the proof. Given f, we introduce the 'Jacobians' $\widetilde{J}f(t)$ with entries

$$\widetilde{J}f(t)_{rs} \;=\; \left\langle P_{T_t\partial_j M}\nabla f_r(t), P_{T_t\partial_j M}\nabla f_s(t)\right\rangle.$$

Note, for later use, that given y_t the Jacobians have conditional distributions

$$\widetilde{J}f(t)\,\big|\, y_t \quad\sim\quad \text{Wishart}\left(j, \widetilde{J}F(y_t)\right) \;\stackrel{\Delta}{=}\; \text{Wishart}\left(j, \langle \nabla F_r(y_t), \nabla F_s(y_t)\rangle\right).$$

With notation now set, we can start our computation. Appealing again to Morse theory for a characterisation of $\mathcal{L}_0\,(M, M_{jl})$ via the critical points of $\widetilde{y}$, and to point process theory to compute expectations, some work shows that

$$\begin{aligned}\mathbb{E}\left\{\mathcal{L}_0\,(M, M_{jl})\right\} \;=\; \frac{1}{(j-l)!}\int_{\partial_j M} \mathbb{E}\Big\{&\alpha(\eta_t; M\cap y^{-1}D)\mathrm{Tr}^{T_tM_{jl}}\left((-\nabla^2\widetilde{y}_{|M_{jl}})^{j-l}\right)|\widetilde{J}f(t)|\mathbb{1}_t \\ &\Big| f(t)=0,\; P_{T_tM_{jl}}\nabla\widetilde{y}=0\Big\}p_{f,P_{jl}}(0,0)\;\mathcal{H}_j(dt),\end{aligned} \tag{12.2}$$

where, in order to make the formulae a little more manageable, we write $\mathbb{1}_t$ for $\mathbb{1}_{\partial_{k-l}D}(y(t))$, $|\widetilde{J}f(t)|$ for $(\det(\widetilde{J}f(t)))^{1/2}$, $p_{f,P_{jl}}$ for the joint density of $f(t)$ and $P_{T_tM_{jl}}\nabla\widetilde{y}$, and

$$\eta_t \;\stackrel{\Delta}{=}\; P^{\perp}_{T_tM_{jl}}\nabla\widetilde{y}_t. \tag{12.3}$$

Note that is actually technically incorrect to call $p_{f,P_{jl}}$ a joint density as $P_{T_tM_{jl}}\nabla\widetilde{y}_t$ lives in $T_t\partial_jM$, a j-dimensional space and $f(y_t)$ lives in a l dimensional space. However, conditional on $y_t, y_{*,t}$ we can find a subspace of $T_t\partial_jM$ that is perpendicular to $y_t^*(T_{y_t}\partial_{k-l}D)^{\perp}$, the pull-back of the orthogonal complement of $T_{y_t}\partial_{k-l}D$ in $T_{y_t}\mathbb{R}^k$, as well as an orthonormal basis $X_{1,t},\dots,X_{j-l,t}$ for this subspace. The vector $P_{T_tM_{jl}}$ can thus be written as a linear combination of the $X_{i,t}$'s and so we think of the density $p_{f,P_{jl}}$ as the joint density of f and the coefficients of $P_{T_tM_{jl}}$ in this basis. This being the case, this density is easily seen to be

$$(2\pi)^{-(j-l)/2}p_f(0). \tag{12.4}$$

Our goal now is to show that the expectation in (12.2) factors, in an appropriate sense, into one factor related only to the Lipschitz-Killing curvature measures of M and another that is related only to $\partial_{k-l}D$. (Since the density term $p_{f,P_{jl}}(0,0)$ is independent of t, we can ignore it.) Thus, we focus on evaluating the conditional expectation in the integrand of (12.2).

To be more specific, we shall consider a finer conditional expectation, looking at

$$\mathbb{E}\Big\{\alpha(\eta_t; M\cap y^{-1}D)\mathrm{Tr}^{T_tM_{jl}}(-\nabla^2\widetilde{y}_{|M_{jl}})^{j-l}|\widetilde{J}f(t)|\mathbb{1}_t\Big|\; y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t}\Big\}, \tag{12.5}$$

where $y_{*,t}$ and $\widetilde{y}_{*,t}$ are the push-forwards of y and $\widetilde{y}$. We shall see that this factors as required and, after a little further manipulation of expectations, will achieve our goal.

We deal with the normal Morse index first. Theorem 14.3 implies that this Morse index factors into a product of two modified Morse indices, one for M and one for $y^{-1}D$. To set this up properly for the current scenario, note first that the support cone of $M \cap y^{-1}D$ at $t \in M_{jl}$ is the intersection of two support cones, *viz.* $\mathcal{S}_t M_{jl} = \mathcal{S}_t M \cap \mathcal{S}_t y^{-1} D \ \subset \ T_t\widetilde{M}$. Let

$$V_t \ \stackrel{\Delta}{=} \ \mathrm{span}\left\{\nabla f_i(t), 1 \le i \le l\right\}$$

be the normal to $T_t y^{-1}(\partial_{k-l}D)$ in all of $T_t\widetilde{M}$. By Theorem 14.3, for almost every $\nu_t \in T_t M_{jl}^{\perp}$,

$$\begin{aligned}
\alpha(\nu_t; M\cap y^{-1}D) &= \alpha(\nu_t; \mathcal{S}_t(M\cap y^{-1}D)) \\
&= \alpha\left(\mathcal{T}_{V_t}\nu_t; \mathcal{S}_t M\right)\cdot \alpha\left(\nu_t; P_{T_t\partial_j M}\left(\mathcal{S}_t\left(y^{-1}D\right)\right)\right) \\
&\stackrel{\Delta}{=} \alpha_1(\nu_t)\cdot\alpha_2(\nu_t),
\end{aligned} \tag{12.6}$$

where $\mathcal{T}_{V_t}$ is as defined in Theorem 14.3. Specifically,

$$\mathcal{T}_{V_t}\nu_t \ = \ P^{\perp}_{T_t\partial_j M}\nu_t - \sum_{r,s=0}^{l}\langle \nu_t, P_{T_t\partial_j M}\nabla f_r(t)\rangle \widetilde{J}f(t)^{rs} P^{\perp}_{T_t\partial_j M}\nabla f_s(t). \tag{12.7}$$

In particular, for the η_t of (12.3), we have

$$\alpha(\eta_t; M\cap y^{-1}D) \ = \ \alpha_1(\eta_t)\cdot\alpha_2(\eta_t).$$

With regard to the second of these indices, note that the vectors in the cone $\mathcal{S}_t\left(y^{-1}D\right)$ are push-forwards under y_t^{-1} of vectors in a cone in $T_{y_t}\mathbb{R}^k$. Consequently,

$$\alpha_2\left(\eta_t\right) \ = \ \alpha\left(\eta_t; P_{T_t\partial_j M}\left(\mathcal{S}_t\left(y^{-1}D\right)\right)\right) \ = \ \alpha\left(y_*(\eta_t); \mathcal{S}_{y_t}\left(D\right)\cap y_*(T_t\partial_j M)\right),$$

where, with some abuse of notation, we have written $y_*(T_t\partial_j M)$ to denote the collection of push-forwards, by y, of vectors in $T_t\partial_j M$. It is clear that, as a random variable, $\alpha_2\left(\eta_t\right)$ is dependent only on the collection $(y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t})$ of four random vectors. Hence, its conditional distribution in (12.5) does not depend on t.

In view of the above observations, and appealing to Lemma 13.1, we have that the conditional expectation (12.5) can be written as

$$\begin{aligned}
&\mathbb{E}\left\{\alpha(\eta_t; M\cap y^{-1}D)\mathrm{Tr}^{T_t M_{jl}}(-\nabla^2\widetilde{y}_{|M_{jl}})^{j-l}|\widetilde{J}f(t)|\mathbb{1}_t \,\middle|\, y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t}\right\} \\
&\quad = \ \frac{l!(j-l)!}{j!}\mathbb{E}\left\{\alpha(\eta_t; M\cap y^{-1}D)\mathrm{Tr}^{T_t\partial_j M}\left((-\nabla^2\widetilde{y}_{|M_{jl}})\right)^{j-l}|\widetilde{J}f(t)|\mathbb{1}_t \,\middle|\, y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t}\right\} \\
&\quad = \ \frac{l!(j-l)!}{j!}\alpha_2(\eta_t)|\widetilde{J}f(t)|\mathbb{1}_t\mathbb{E}\left\{\alpha_1(\eta_t)\mathrm{Tr}^{T_t\partial_j M}\left((-\nabla^2\widetilde{y}_{|M_{jl}})\right)^{j-l} \,\middle|\, y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t}\right\}.
\end{aligned}$$

To evaluate the expectation here, we need to look a little more closely at the structure of the Hessian $\nabla^2\widetilde{y}_{|M_{jl}}$. Applying the Weingarten equation gives us, after some computation, that

$$\nabla^2\widetilde{y}_{|M_{jl},t} \ = \ \nabla^2\widetilde{y}_t - \sum_{r,s=1}^{l}\langle\nabla\widetilde{y}_t, \nabla f_{r|\partial_j M}(t)\rangle\left(\widetilde{J}f(t)\right)^{rs}\nabla^2 f_{s|\partial_j M}(t).$$

This depends on the curvature of $F_l^{-1}(0)$. With some perversity (note how terms cancel) but with the future in mind, we acknowledge this dependence by writing

$$\nabla^2 \widetilde{y}_{|M_{jl},t} \;=\; \Xi_t \,+\, \Theta_t,$$

where

$$\begin{aligned}
\Xi_t &\triangleq \left(\nabla^2 \widetilde{y}_t \,+\, \widetilde{y}_t \cdot I\right) \,-\, S_{\eta_t} \\
&\quad - \sum_{r,s=1}^{l} \sum_{u=1}^{k} \langle \nabla \widetilde{y}_t, P_{T_t \partial_j M} \nabla f_r(t) \rangle \widetilde{J} f(t)^{rs} \frac{\partial F_s}{\partial y_u} \cdot \left(\nabla^2 y_u(t) + y_{u,t} \cdot I\right) \\
\Theta_t &\triangleq \Theta_t^1 \,+\, \Theta_t^2 \\
\Theta_t^1 &\triangleq \sum_{r,s=1}^{l} \sum_{u,v=1}^{k} \langle \nabla \widetilde{y}_t, P_{T_t \partial_j M} \nabla f_r(t) \rangle \widetilde{J} f(t)^{rs} \left.\frac{\partial^2 F_s}{\partial y_u \partial y_v}\right|_{y_t} \\
&\qquad\qquad \times \langle X_t, \nabla y_u(t) \rangle \cdot \langle Y_t, \nabla y_v(t) \rangle \\
\Theta_t^2 &\triangleq -\widetilde{y}_t \cdot I \,+\, \sum_{r,s=1}^{k-l} \sum_{u=1}^{k} \langle \nabla \widetilde{y}_t, P_{T_t \partial_j M} \nabla f_r(t) \rangle \widetilde{J} f(t)^{rs} \frac{\partial F_s}{\partial y_u} y_u, t \cdot I.
\end{aligned}$$

Written this way, the curvature of $F_l^{-1}(0)$ is contained in the double form Θ_t^1. Continuing, Lemma 2.1 of [24] implies that (12.5) is given by

$$\begin{aligned}
&\mathbb{E}\left\{ \frac{l!}{j!} \alpha_1(\eta_t) \alpha_2(\eta_t) \mathrm{Tr}^{T_t \partial_j M} \left((-\nabla^2 \widetilde{y}_{|M_{jl}}) \right)^{j-l} |\widetilde{J} f(t)| \mathbb{1}_t \,\middle|\, y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t} \right\} \\
&\quad = \sum_{m=l}^{j} \frac{l!(j-l)!}{(m-l)!(j-m)!j!} \mathrm{Tr}^{T_t \partial_j M} \left(\mathbb{E}\left\{ \alpha_1(\eta_t) \alpha_2(\eta_t) \cdot \Xi_t^{j-m} \Theta_t^{m-l} |\widetilde{J} f(t)| \mathbb{1}_t \,\middle|\, y_t, y_{*,t}, \widetilde{y}_t, \widetilde{y}_{*,t} \right\} \right).
\end{aligned}$$

We shall apply Lemma 13.2, conditional on y_t to each term in the sum above. To do so, we first identify variables and functions above with the variables and functions in Lemma 13.2. With this in mind, set

$$\begin{aligned}
Z_i &= \sum_{r=0}^{l} \langle \nabla \widetilde{y}_t, P_{T_t \partial_j M} \nabla f_r(t) \rangle \widetilde{J} f(t)_{ri}^{-1/2} \\
W &= \left(\widetilde{J} F(y_t)\right)^{-1/2} \widetilde{J} f(t) \left(\widetilde{J} F(y_t)^t\right)^{-1/2} \\
U_j &= \left(\nabla^2 y_j + y_j \cdot I, S_{P^{\perp}_{T_t \partial_j M} \nabla y_j}\right) \qquad 1 \le j \le l \\
U_{l+1} &= \left(\nabla^2 \widetilde{y}, S_{P^{\perp}_{T_t \partial_j M} \nabla \widetilde{y}}\right) \\
G_{j-m} &= \alpha_1 \cdot \Xi_t^{j-m} \\
\widetilde{G}_{m-l} &= \alpha_2 \cdot \Theta_t^{m-l}.
\end{aligned}$$

It is straightforward to verify that, conditional on y_t, the above random variables have the required conditional distributions. The result after application of the lemma with $\tilde{l} = j - m, \tilde{r} =$

$m-l, \tilde{n}=j, \tilde{k}=l$ is

$$(2\pi)^{-m/2}\frac{j!}{(j-l)!}\frac{\omega_m}{\omega_0}\mathbb{E}\left\{G_{m-l}\right\}\mathbb{E}\left\{\widetilde{G}_{j-m}\big|y_t\right\}. \tag{12.8}$$

There is only one conditional expectation above because the U_j's are in fact independent of y_t.

Applying Lemma 13.2 repeatedly, along with the fairly obvious fact that $\mathbb{E}\left\{\Theta_t^m\right\} = C_m I^m$ for any m, we find constants, $\widehat{\rho}(m, \partial_{k-l}D)$, which may change from line to line, such that the expected value of the above (after multiplication by the factor of 2π in (12.4)) is equivalent to

$$\begin{aligned}
&\sum_{m=l}^{j}\frac{l!}{(m-l)!(j-m)!}(2\pi)^{-(j-m)/2}\widehat{\rho}(m,\partial_{k-l}D)\\
&\qquad\times \mathrm{Tr}^{T_t\partial_j M}\left(\mathbb{E}\left\{\alpha(P^{\perp}_{T_t\partial_j M}\nabla\widetilde{y}_t; M)\cdot(\nabla^2\widetilde{y}_t+\widetilde{y}_t\cdot I)^{j-m}\right\}I^{m-l}\right)\\
&\quad=\sum_{m=l}^{j}\frac{l!}{(m-l)!(j-m)!}(2\pi)^{-(j-m)/2}\widehat{\rho}(m,\partial_{k-l}D)\\
&\qquad\quad\times \mathrm{Tr}^{T_t\partial_j M}\left(\mathbb{E}\left\{\alpha(P^{\perp}_{T_t\partial_j M}\nabla\widetilde{y}_t; M)\cdot(\nabla^2\widetilde{y}_t+\widetilde{y}_t)^{j-m}\right\}\right),
\end{aligned}$$

the last equality following from a trace formula of Federer ([11], p425).

Above, $\widehat{\rho}(m, \partial_{k-l}D)$ is a complicated expression involving the expected value of products of the α_2 of (12.6), powers of Θ_t^1 and Θ_t^2 and factors of 2π and ω_m's.

Finally, arguing as in the proof of Theorem 11.1, one can show that

$$\int_{\partial_j M}\mathrm{Tr}^{T_t\partial_j M}\left(\mathbb{E}\left\{\alpha(P^{\perp}_{T_t\partial_j M}\nabla\widetilde{y}_t; M)\cdot(\nabla^2\widetilde{y}_t+\widetilde{y}_t\cdot I)^{j-m}\right\}\right)\ \propto\ \mathcal{L}^1_m(M;\partial_j M),$$

with the proportionality constant independent of $\partial_l D$. The reason that $\mathcal{L}^1$ appears instead of $\mathcal{L}=\mathcal{L}^0$ is because of the appearance of $\widetilde{y}_t\cdot I$ above. This completes the proof as all that remains under the expectation are quantities whose distribution does not depend on t. □

13 Two Gaussian lemmas

We shall now prove two technical lemmas that we used earlier. They are stated in somewhat more generality than needed, but it turns out to be not much harder to prove the more general results and as currently formulated they may also turn out to have broader application. Both are exercises in (statistical) Multivariate Analysis, for which the standard reference is Anderson [5]. In particular, both proofs exploit some basic facts of *Wishart distributions* which we now summarise and proofs of which can be found in Section 13.3 of [5].

The Wishart(n, Σ) distribution is the distribution of a matrix W with elements of the form $W_{ij} = \sum_{m=1}^{n} X_{mi} X_{mj}$, where the k n-dimensional vectors $X_m = (X_{m1}, \ldots, X_{mk})$ are independent, each distributed as $N(0, \Sigma)$. Alternatively, writing X for the $n \times k$ matrix with X_m being the m-th row, we have $W \stackrel{\mathcal{L}}{=} X'X$.

We shall need two properties of the Wishart$(n, I_{k\times k})$ distribution. The first is the rather immediate fact that, for any $W \sim$ Wishart$(n, I_{k\times k})$ and any orthonormal matrix A,

$$AWA' \stackrel{\mathcal{L}}{=} W. \tag{13.1}$$

The second is that, for $W \sim$ Wishart$(n, I_{k\times k})$,

$$\det(W) \sim \prod_{j=1}^{k} \chi^2_{n+1-j}, \tag{13.2}$$

where we read the right hand side as "the product of k independent χ^2 variables, with degrees of freedom running from $n+1-k$ to n". An immediate consequence of this is that

$$\mathbb{E}\left\{\det(W)\right\} = \prod_{j=1}^{k}(n+1-j) = \frac{n!}{(n-k)!}. \tag{13.3}$$

Here is the first of our two lemmas.

Lemma 13.1 *Let $(V, \langle\,,\rangle)$ be an inner product space with $\dim(V) = n$ and associated wedge product $\wedge$. Suppose that $L_k \subset V$ is a uniformly distributed random subspace of V of dimension k, in the sense that $gL_k \stackrel{\mathcal{L}}{=} L_k$ for all orthonormal transformations g of V. Then, for all alternating double forms α of order j,*

$$\mathbb{E}\left\{\mathrm{Tr}^{L_k}(\alpha)\right\} = \begin{cases} \frac{\binom{k}{j}}{\binom{n}{j}}\mathrm{Tr}^V(\alpha) & j \le k, \\ 0 & \textit{otherwise,} \end{cases}$$

where by $\mathrm{Tr}^V(\alpha)$ and $\mathrm{Tr}^{L_k}(\alpha)$ we denote the trace of α on V and L_k, respectively.

Proof. Firstly, note that the case $j > k$ is trivial since then $\alpha \equiv 0$ by definition. Secondly, by the linearity of double forms note that it suffices to consider the case

$$\alpha = (\omega_1 \wedge \cdots \wedge \omega_j) \cdot (\omega_1 \wedge \cdots \wedge \omega_j) \tag{13.4}$$

for some orthonormal set $\{\omega_1, \ldots, \omega_j\}$. Note that from the definition of the trace

$$\mathrm{Tr}^V(\alpha) = \mathrm{Tr}^V\left((\omega_1 \wedge \cdots \wedge \omega_j) \cdot (\omega_1 \wedge \cdots \wedge \omega_j)\right) = 1. \tag{13.5}$$

Finally, we claim that it also suffices to consider only the case $j = k$. To see this, note that, for an α of the above type,

$$\mathrm{Tr}^{L_k}(\alpha) \;=\; \sum_{\{i_1,\dots,i_j\}\subset\{1,\dots,k\}} \alpha\left(\left(\theta_{i_1},\dots,\theta_{i_j}\right),\left(\theta_{i_1},\dots,\theta_{i_j}\right)\right),$$

where $\{\theta_1,\dots,\theta_k\}$ is a (random) orthonormal basis for L_k. Note that the individual terms in the sum all have an identical expectation, due to the assumed distributional invariance of L_k (and its subspaces) under orthonormal transformation. Hence

$$\mathbb{E}\left\{\mathrm{Tr}^{L_k}(\alpha)\right\} \;=\; \binom{k}{j}\mathbb{E}\left\{\alpha\left((\theta_1,\dots,\theta_j),(\theta_1,\dots,\theta_j)\right)\right\} \;=\; \binom{k}{j}\mathbb{E}\left\{\mathrm{Tr}^{L_j}(\alpha)\right\}.$$

Thus, to compute the expected value of the trace of a α on a uniform subspace of dimension k, it suffices to compute it on a uniform subspace of dimension j. Consequently, we shall now concentrate on the form (13.4) for the case $j = k$.

To this end we introduce an auxiliary set of random vectors, $X_1,\dots,X_k$, taken to be independent, identicially distributed with standard Gaussian distribution γ_V. (i.e. For any $x \in V^*$, the dual of V, the real valued random variable $x(X)$ is distributed as $N(0,\|x\|)$.) Then, for α of the form (13.4), it follows from the definition of the trace operator that

$$\alpha\left((X_1,\dots,X_k),(X_1,\dots,X_k)\right) \;=\; \det(g)\,\mathrm{Tr}^{L_k}(\alpha),$$

where $g_{ij} \;=\; \langle X_i, X_j\rangle \;\sim\;$ Wishart$(n, I_{k\times k})$ is independent of L_k.

Let $\widetilde{V}^* = \mathrm{span}\{\omega_1,\dots,\omega_k\}$ and, using the usual identification of V and its dual V^*, let $\widetilde{V}$ be the corresponding subspace of V. Define $Y_i = P_{\widetilde{V}}X_i \sim \gamma_{\widetilde{V}}$, and note that

$$\alpha\left((X_1,\dots,X_k),(X_1,\dots,X_k)\right) \;=\; \alpha\left((Y_1,\dots,Y_k),(Y_1,\dots,Y_k)\right).$$

However, the right hand side here has a particularly simple distribution, since $\alpha((Y_1,\dots,Y_k),$ $(Y_1,\dots,Y_k)) = \det(\widetilde{g})$, where $\widetilde{g}_{ij} = \langle Y_i, Y_j\rangle \sim$ Wishart$(k, I_{k\times k})$. Collecting equivalences, we find that

$$\det(g)\,\mathrm{Tr}^{L_k}(\alpha) \;=\; \det(\widetilde{g}).$$

Taking expectations over L_k, g and $\widetilde{g}$, recalling the independence of L_k and g, and applying (13.3) we find that

$$\mathbb{E}\left\{\mathrm{Tr}^{L_k}(\alpha)\right\} \;=\; \binom{n}{k}^{-1} \;=\; \binom{n}{k}^{-1}\mathrm{Tr}^{V}(\alpha),$$

the last equality following from (13.5). Combining this with (13.6) completes the proof. □

The second, unrelated, lemma is the following.

Lemma 13.2 *Consider the following three sets of random variables: a vector Z, a matrix W and a sequence $U_1, \ldots, U_{k+1}$ of vectors. All are independent, with distributions*

$$Z \sim N(0, I_{k\times k}), \qquad W \sim \text{Wishart}(n, I_{k\times k}), \qquad U_j \sim N(0, \Sigma),$$

where Σ is an arbitrary covariance matrix.

Then, for any homogeneous functions G_l of degree l and $\widetilde{G}_r$ of degree r,

$$\mathbb{E}\Big\{G_l\Big(U_{k+1} + \sum_{i,j=1}^{k} Z_i W_{ij}^{-1/2} U_j\Big)\widetilde{G}_r\Big(\sum_{i,j=1}^{k} Z_i W_{ij}^{-1/2}\Big)\sqrt{\det(W)}\Big\}$$
$$= (2\pi)^{-(k+r)/2}\frac{n!}{(n-k)!}\frac{\omega_{n-l}}{\omega_{n-l-k-r}}\mathbb{E}\left\{G_l(U_1)\right\}\mathbb{E}\{\widetilde{G}_r(Z)\}$$

In particular,

$$\mathbb{E}\Big\{\Big(U_{k+1} + \sum_{i,j=1}^{k} Z_i W_{ij}^{-1/2} U_j\Big)^{2l}\sqrt{\det(W)}\Big\} = \frac{(2l)!}{l!\,2^l}(2\pi)^{-k/2}\frac{n!}{(n-k)!}\frac{\omega_{n-2l}}{\omega_{n-2l-k}}\mathbb{E}\left\{U_1^{2l}\right\}.$$

Proof. For ease of notation, we first consider the case $\widetilde{G}_r \equiv 1$.

Let $\mathbb{P}_W$ denote the distribution of W on $\text{Sym}_{k\times k}$, the space of $k \times k$ symmetric matrices. Then

$$\mathbb{E}\Big\{G_l\Big(U_{k+1} + \sum_{i,j=1}^{k} Z_i W_{ij}^{-1/2} U_j\Big)\sqrt{\det(W)}\Big\} \tag{13.6}$$
$$= \int_{\text{Sym}_{k\times k}}\int_{\mathbb{R}^k}\sqrt{\det(w)}\mathbb{E}\Big\{G_l\Big(U_{k+1} + \sum_{i,j=1}^{k} z_i w_{ij}^{-1/2} U_j\Big)\Big\}\frac{e^{-|z|^2/2}}{(2\pi)^{k/2}}\,dz\mathbb{P}_W(dw)$$
$$= \int_{\text{Sym}_{k\times k}}\int_{\mathbb{R}^k}\det(w)\mathbb{E}\Big\{G_l\Big(U_{k+1} + \sum_{j=1}^{k} z_j U_j\Big)\Big\}\frac{e^{-zwz'/2}}{(2\pi)^{k/2}}\,dz\mathbb{P}_W(dw).$$

In order to simplify this expression, fix Z and consider the expectation over W. Take an orthornormal matrix, say O_Z, with first row equal to $Z/|Z|$ and the remaining rows chosen arbitrarily (but consistently with the first). Then, as we noted at (13.1), $O_Z W O_Z' \stackrel{\mathcal{L}}{=} W$ and we can write $\det(O_Z W O_Z')$ as the product of χ^2 random variables (cf. (13.2)) the first of which is actually $ZWZ'/|Z|^2$.

Taking the expectation over these terms, we have $k-1$ expectations of of χ^2 variables, along with

$$\mathbb{E}\Big\{\frac{ZWZ^t}{|Z|^2}e^{-ZWZ^t/2}\Big|Z\Big\} = \mathbb{E}\Big\{X_n e^{-|Z|^2 X_n/2}\Big|Z\Big\} = \frac{d}{d\lambda}M_{X_n}(\lambda)\Big|_{\lambda=-|Z|^2/2}$$

where $X_n \sim \chi^2_n$ and M_{X_n} is its moment generating function. However, this is equal to

$$\frac{d}{d\lambda}\left(\frac{1}{1-2\lambda}\right)^{n/2}\Bigg|_{\lambda=-|Z|^2/2} = n\left(\frac{1}{1+|Z|^2}\right)^{n/2-1}.$$

Substituting this into the last line of (13.6) gives that the expectation there is equivalent to

$$\begin{aligned}(2\pi)^{-k/2}\prod_{j=1}^{k-1}\mathbb{E}\left\{\chi^2_{n-j}\right\}\int_{\mathbb{R}^k} n(1+|z|^2)^{l/2-n/2-1}\mathbb{E}\left\{G_l\left(\frac{U_{k+1}+\sum_{i=1}^k z_iU_i}{\sqrt{1+|z|^2}}\right)\right\}dz \qquad (13.7)\\ = (2\pi)^{-k/2}\frac{n!}{(n-k)!}\mathbb{E}\left\{G_l\left(U_1\right)\right\}\int_{\mathbb{R}^k}(1+|z|^2)^{l/2-n/2-1}\,dz.\end{aligned}$$

Finally, as the integral here is, effectively, that of a multivariate t density with $n-l-k+2$ degrees of freedom and covariance parameter $(n-l-k+2)^{-1}I_{k\times k}$ (cf. [5]) we have

$$\int_{\mathbb{R}^k}(1+|z|^2)^{l/2-n/2-1}\,dz = \pi^{k/2}\frac{\Gamma\left(\frac{n-l-k}{2}+1\right)}{\Gamma\left(\frac{n-l}{2}+1\right)} = \frac{\omega_{n-l}}{\omega_{n-l-k}}$$

and, after putting all the pieces together, the proof is done for the case $\widetilde{G}_r = 1$.

Now, suppose that $\widetilde{G}_r$ is not identically 1. Then, the proof proceeds exactly as above up to the point (13.7), keeping in mind that we had made the substitution

$$z_i \mapsto \widetilde{z}_i = \sum_{i,j=1}^{k} W_{ij}^{-1/2} z_j,$$

and the integral

$$\int_{\mathbb{R}^k}(1+|z|^2)^{l/2-n/2-1}\,dz$$

is replaced with

$$\int_{\mathbb{R}^k}\widetilde{G}_r(z)\,(1+|z|^2)^{l/2-n/2-1}\,dz.$$

Finally, exploit the homogeneity of $\widetilde{G}_r$ to see that

$$\begin{aligned}\mathbb{E}\left\{\widetilde{G}_r(T_{n-l-k+2})\right\} &= \mathbb{E}\left\{\left(\chi^2_{n-l-k+2}\right)^{-r/2}\right\}\mathbb{E}\left\{\widetilde{G}_r(Z)\right\}\\ &= 2^{-r/2}\cdot\pi^{-(n-l-k)/2}\omega_{n-l-k}\frac{\Gamma\left(\frac{n-l-k+2-r}{2}\right)}{\Gamma\left(\frac{n-l-k+2}{2}\right)}\mathbb{E}\left\{\widetilde{G}_r(Z)\right\}.\end{aligned}$$

□

14 Two Morse theoretic results

We present here, without proofs, two Morse theoretic results that are used in the paper. Recall first that a function $f \in C^2(\widetilde{M})$, where M is a C^2 Whitney stratified manifold embedded in a C^3 ambient manifold $\widetilde{M}$, is called a *Morse function* on M if, for all $0 \le j \le \dim(M)$, the covariant Hessians $\nabla^2 f_{|T_t\partial_j M}$ are non-degererate at all critical points of $f_{|\partial_j M}$ in $\partial_j M$ and the restriction of f to $\overline{\partial_k M} = \bigcup_{j=0}^{k} \partial_j M$ has no critical points on $\bigcup_{j=0}^{k-1} \partial_j M$.

The classical Morse Theorem, in this settings, can be stated as follows (cf. [13]).

Theorem 14.1 (Morse's Theorem) *Let $(M, \mathcal{Z})$ be a compact C^2 Whitney stratified manifold embedded in a C^3 Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $\widetilde{f} \in C^2(\widetilde{M})$ be a Morse function on M. Then, setting $f = \widetilde{f}_{|M}$,*

$$\mathcal{L}_0(M) \;=\; \sum_{j=0}^{N} \sum_{\{t\in\partial_j M : \nabla f_t \in T_t^{\perp}\partial_j M\}} (-1)^{\iota_{f,\partial_j M}(t)} \alpha(P^{\perp}_{T_t\partial_j M} \nabla f_t; M), \tag{14.1}$$

where $P^{\perp}_{T_t\partial_j M}$ is orthogonal projection onto $(T_t\partial_j M)^{\perp}$, $\mathcal{L}_0(M)$ is the Euler-Poincaré characteristic of M, and the $\iota_{f,\partial_j M}(t)$ are the tangential Morse indices of (3.3).

More important for us is the following corollary of the standard Morse Theorem, for this is what lies behind the the point set representations leading to (11.5) and (12.2).

Corollary 14.2 *Let M be a regular stratified manifold embedded in a C^3 manifold $\widetilde{M}$. Let $\widetilde{f} \in C^2(\widetilde{M})$ be a Morse function on M, and let $u \in \mathbb{R}$ be a regular value of $\widetilde{f}_{|\partial_j M}$ for all $j = 0, \dots, N$. Then, writing $f = \widetilde{f}_{|M}$,*

$$\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right) \;=\; \sum_{j=0}^{N} \sum_{\{t\in\partial_j M : f_t \ge u, \nabla f_t \in T_t^{\perp}\partial_j M\}} (-1)^{\iota_{-f,\partial_j M}(t)} \alpha(-P^{\perp}_{T_t\partial_j M} \nabla f_t; M). \tag{14.2}$$

The proof of the Corollary is not hard (cf. [3] for details) but depends, in part, on the following lemma, a proof of which can also be found in Section 9.2 of [3] and which, to the best of our knowledge, does not appear elsewhere in the literature. Although hidden away in the details of the proof of Theorem 10.1 of Section 12, this lemma is actually crucial to the fact that in the expression (1.2) for the expected Lipschitz-Killing curvature of an excursion set the geometries of the parameter and Gaussian spaces separate out to product form.

Lemma 14.3 *Let M_1 and M_2 be regular stratified manifolds with stratifications*

$$M_1 \;=\; \bigcup_{j=0}^{\dim M_1} M_{1j}, \qquad M_2 \;=\; \bigcup_{k=0}^{\dim M_2} M_{2k}.$$

Suppose that, for each j and k, M_{1j} and M_{2k} intersect transversally and

$$\mathrm{Codim}\,(M_{1j}\cap M_{2k}) \;=\; \mathrm{Codim}\,(M_{1j})+\mathrm{Codim}\,(M_{2k})\,. \tag{14.3}$$

Fix a $t\in M_{1j}\cap M_{2k}$. Then, for every such t and almost every $\nu_t\in T_t(M_{1j}\cap M_{2k})^{\perp}$,

$$\begin{aligned}\alpha\left(\nu_t;\mathcal{S}_t(M_1\cap M_2)\right) &= \alpha\left(\nu_t;\mathcal{S}_tM_1\cap\mathcal{S}_tM_2\right) \qquad (14.4)\\ &= \alpha\left(\nu_t;P_{T_tM_{2k}}\mathcal{S}_tM_1\right)\cdot\alpha\left(\nu_t;P_{T_tM_{1j}}\mathcal{S}_tM_2\right)\\ &= \alpha\left(\mathcal{T}_{jk}\nu_t;\mathcal{S}_tM_1\right)\cdot\alpha\left(\nu_t;P_{T_tM_{1j}}\mathcal{S}_tM_2\right),\end{aligned}$$

where $\mathcal{T}_{jk}:\,T_t(M_{1j}\cap M_{2k})^{\perp}\to T_tM_{1j}^{\perp}$ is defined by

$$\mathcal{T}_{jk}\nu \;=\; P^{\perp}_{T_tM_{1j}}\nu-\sum_{r,s=1}^{\mathrm{Codim}(M_{2k})}\langle P_{T_tM_{1j}}\nu,P_{T_tM_{1j}}w_r\rangle\,\widetilde{g}^{rs}P^{\perp}_{T_tM_{1j}}w_s,$$

where $\{w_1,\dots,w_{\mathrm{Codim}(M_{2k})}\}$ is a collection of vectors spanning $T_tM_{2k}^{\perp}$ and the $\widetilde{g}^{rs}$ are the entries of the inverse of the matrix with elements

$$\widetilde{g}_{rs} \;=\; \langle P_{T_tM_{1j}}w_r,P_{T_tM_{1j}}w_s\rangle,\qquad 1\le r,s\le\mathrm{Codim}(M_{2k}).$$

Furthermore, if $\mathcal{S}_t(M_2)$ is a half-space then, for almost every $\nu_t\in T_t(M_{1,j}\cap M_{2,k})^{\perp}$,

$$\alpha\left(\nu_t;\mathcal{S}_t(M_1\cap M_2)\right) \;=\; \alpha\left(\nu_t;P_{T_tM_{2,k}}\mathcal{S}_tM_1\right) \;=\; \alpha\left(\mathcal{T}_{T_{jk}}\nu_t;\mathcal{S}_tM_1\right). \tag{14.5}$$

Although we are not going to prove this lemma here, a couple of words as to how the proof proceeds are in order. There are essentially two steps in the proof. The first is to show that the lemma holds when M is a complex of simplicial cones. The second is to extend this result to stratified manifolds via an approximation. That is, the manifolds, their tangent bundles and, importantly, their normal and tangential indices need to be approximated by those of simplicial cone complexes. At the this stage, various smoothness demands enter into consideration, in particular that M be a $C^{2,1}$ cone manifold.

References

[1] R.J. Adler. *The Geometry of Random Fields*. John Wiley & Sons Ltd., Chichester, 1981. Wiley Series in Probability and Mathematical Statistics.

[2] R.J. Adler. On excursion sets, tube formulae, and maxima of random fields. *Annals of Applied Prob.*, 10:1–74, 2000.

[3] R.J. Adler and J.E Taylor. *Random Fields and Geometry*. Springer, 2007. Currently available at http://iew3.technion.ac.il/-∼radler/publications.html.

[4] R.J. Adler, J.E. Taylor, and K.J. Worsley. *Random Fields and Geometry: Applications.* In preparation.

[5] T.W. Anderson. *An Introduction to Multivariate Statistical Analysis.* Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2003.

[6] A. Bernig and L. Bröcker. Lipschitz-Killing invariants. *Math. Nachr.*, 245:5–25, 2002.

[7] L. Bröcker and M. Kuppe. Integral geometry of tame sets. *Geom. Dedicata*, 82(1-3):285–323, 2000.

[8] S.S. Chern. On the kinematic formula in integral geometry. *J. Math. Mech.*, 16:101–118, 1966.

[9] P. Diaconis and D. Freedman. A dozen de Finetti-style results in search of a theory. *Ann. Inst. H. Poincaré Probab. Statist.*, 23(2, suppl.):397–423, 1987.

[10] P.W. Diaconis, M.L. Eaton, and S.L. Lauritzen. Finite de Finetti theorems in linear models and multivariate analysis. *Scand. J. Statist.*, 19(4):289–315, 1992.

[11] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[12] H. Federer. *Geometric Measure Theory.* Springer-Verlag, New York, 1969.

[13] M. Goresky and R. MacPherson. *Stratified Morse Theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1988.

[14] A. Gray. *Tubes.* Addison-Wesley, Redwood City, 1990.

[15] H. Hadwiger. Normale Körper im euklidischen Raum und ihre topologischen und metrischen Eigenschaften. *Math. Zeitschrift*, 71:124–140, 1959.

[16] D. Hug and R. Schneider. Kinematic and Crofton formulae of integral geometry: recent variants and extensions. In C. Barceló i Vidal, editor, *Homenatge al Professor Lus Santaló i Sors*, pages 51–80. Universitat de Girona, 2002.

[17] D.A. Klain and G-C. Rota. *Introduction to Geometric Probability.* Cambridge University Press, Cambridge, 1997.

[18] M. Morse and S. Cairns. *Critical Point Theory in Global Analysis and Differential Topology. An Introduction.* Academic Press, New York, 1969.

[19] M.J. Pflaum. *Analytic and Geometric Study of Stratified Spaces*, volume 1768 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2001.

[20] S.O. Rice. Mathematical analysis of random noise. *Bell System Tech. J.*, 24:46–156, 1945. Also in Wax, N. (Ed.) (1954), *Selected Papers on Noise and Stochastic Processes*, Dover, New York.

[21] L.A. Santalo. *Integral Geometry and Geometric Probability.* Encyclopedia of Mathematics and its Applications. Addison-Wesley, Reading, 1976.

[22] R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[23] D.O. Siegmund and K.J. Worsley. Testing for a signal with unknown location and scale in a stationary Gaussian random field. *Ann. Statist*, 23:608–639, 1995.

[24] J.E. Taylor. A Gaussian kinematic formula. *Ann. Probab*, 32, 2004. To appear.

[25] J.E. Taylor and R.J. Adler. Euler characteristics for Gaussian fields on manifolds. *Ann. Prob.*, 30:533–563, 2002.

[26] H. Weyl. On the volume of tubes. *Amer. J. Math.*, 61:461–472, 1939.

[27] K.J. Worsley. Local maxima and the expected Euler characteristic of excursion sets of χ^2, F and t fields. *Adv. in Appl. Probab.*, 26(1):13–42, 1994.

[28] K.J. Worsley. Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. Appl. Probab.*, 27:943–959, 1995.

[29] K.J. Worsley. Estimating the number of peaks in a random field using the Hadwiger characteristic of excursion sets, with applications to medical images. *Ann. Statist.*, 23:640–669, 1995.

[30] K.J. Worsley. The geometry of random images. *Chance*, 9:27–40, 1997.

[31] K.J. Worsley. Testing for signals with unknown location and scale in a χ^2 random field, with an application to fMRI. *Adv. in Appl. Probab.*, 33(4):773–793, 2001.

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Jonathan E. Taylor: Department of Statistics, Stanford University, Stanford, USA.
jtaylor@stat.stanford.edu, http://www-stat.stanford.edu/∼jtaylo

Robert J. Adler: Faculty of Industrial Engineering and Management, Technion - Israel Institute of Technology, Haifa, Israel
robert@ieadler.technion.ac.il, http://ie.technion.ac.il/Adler.phtml