

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

# Gauss curvature of random metrics

Y. Canzani (McGill), canzani@math.mcgill.ca

D. Jakobson (McGill), jakobson@math.mcgill.ca

I. Wigman (Cardiff), WigmanI@cardiff.ac.uk

ERA, Vol. 17 (2010)

arXiv:1002.0030

October 25, 2011

- $(M, g)$  is a compact surface with a Riemannian metric  $g$ .
- Goal: study Gauss curvature  $K$  of *random* Riemannian metrics on  $M$ .
- *Gauss curvature*: Geometric meaning: as  $r \rightarrow 0$ ,

$$\text{vol}(B_M(x_0, r)) = \pi r^2 \left[ 1 - \frac{K(x_0)r^2}{12} + O(r^4) \right].$$

$K > 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *convex*; volume grows *slower* than in  $\mathbf{R}^2$ .

$K < 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *concave*; volume grows *faster* than in  $\mathbf{R}^2$ .

- $(M, g)$  is a compact surface with a Riemannian metric  $g$ .
- Goal: study Gauss curvature  $K$  of *random* Riemannian metrics on  $M$ .
- *Gauss curvature*: Geometric meaning: as  $r \rightarrow 0$ ,

$$\text{vol}(B_M(x_0, r)) = \pi r^2 \left[ 1 - \frac{K(x_0)r^2}{12} + O(r^4) \right].$$

$K > 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *convex*; volume grows *slower* than in  $\mathbf{R}^2$ .

$K < 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *concave*; volume grows *faster* than in  $\mathbf{R}^2$ .

- $(M, g)$  is a compact surface with a Riemannian metric  $g$ .
- Goal: study Gauss curvature  $K$  of *random* Riemannian metrics on  $M$ .
- *Gauss curvature*: Geometric meaning: as  $r \rightarrow 0$ ,

$$\text{vol}(B_M(x_0, r)) = \pi r^2 \left[ 1 - \frac{K(x_0)r^2}{12} + O(r^4) \right].$$

$K > 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *convex*; volume grows *slower* than in  $\mathbf{R}^2$ .

$K < 0 \Rightarrow$  surface in  $\mathbf{R}^3$  is *concave*; volume grows *faster* than in  $\mathbf{R}^2$ .

- **Conformal class:** Metric  $g_1$  is *conformally equivalent* to  $g_0$  if for all  $x \in M$  and  $U, V \in T_x M$ ,

$$g_1(x)(U, V) = F(x) \cdot g_0(x)(U, V), \quad F(x) > 0.$$

The set of all such metrics is called a *conformal class*  $[g_0]$  of  $g_0$ .

- **Uniformization theorem:** in every conformal class, there exists a unique metric of constant Gauss curvature  $K_0$ .  $K_0 > 0$  for  $M = S^2$ ,  $K_0 = 0$  for  $M = \mathbf{T}^2$ , and  $K_0 < 0$  for surfaces of genus  $\gamma \geq 2$ .
- **Gauss-Bonnet theorem:**  $\int_M K dA = 2\pi\chi(M)$ , where  $\chi$  is the *Euler characteristic*,  
 $\chi(\text{sphere with } \gamma \text{ handles}) = 2 - 2\gamma$ , e.g.  
 $\chi(S^2) = 2, \chi(\mathbf{T}^2) = 0$  etc.

- **Conformal class:** Metric  $g_1$  is *conformally equivalent* to  $g_0$  if for all  $x \in M$  and  $U, V \in T_x M$ ,

$$g_1(x)(U, V) = F(x) \cdot g_0(x)(U, V), \quad F(x) > 0.$$

The set of all such metrics is called a *conformal class*  $[g_0]$  of  $g_0$ .

- **Uniformization theorem:** in every conformal class, there exists a unique metric of constant Gauss curvature  $K_0$ .  $K_0 > 0$  for  $M = S^2$ ,  $K_0 = 0$  for  $M = \mathbf{T}^2$ , and  $K_0 < 0$  for surfaces of genus  $\gamma \geq 2$ .
- **Gauss-Bonnet theorem:**  $\int_M K dA = 2\pi\chi(M)$ , where  $\chi$  is the *Euler characteristic*,  
 $\chi(\text{sphere with } \gamma \text{ handles}) = 2 - 2\gamma$ , e.g.  
 $\chi(S^2) = 2, \chi(\mathbf{T}^2) = 0$  etc.

- **Conformal class:** Metric  $g_1$  is *conformally equivalent* to  $g_0$  if for all  $x \in M$  and  $U, V \in T_x M$ ,

$$g_1(x)(U, V) = F(x) \cdot g_0(x)(U, V), \quad F(x) > 0.$$

The set of all such metrics is called a *conformal class*  $[g_0]$  of  $g_0$ .

- **Uniformization theorem:** in every conformal class, there exists a unique metric of constant Gauss curvature  $K_0$ .  $K_0 > 0$  for  $M = S^2$ ,  $K_0 = 0$  for  $M = T^2$ , and  $K_0 < 0$  for surfaces of genus  $\gamma \geq 2$ .
- **Gauss-Bonnet theorem:**  $\int_M K dA = 2\pi\chi(M)$ , where  $\chi$  is the *Euler characteristic*,  
 $\chi(\text{sphere with } \gamma \text{ handles}) = 2 - 2\gamma$ , e.g.  
 $\chi(S^2) = 2, \chi(T^2) = 0$  etc.

- **Questions:** Assume  $M \neq \mathbf{T}^2$ , and  $g_0$  has *non-vanishing* curvature  $K_0$ . What is the *probability* that a random metric  $g_1$  in the conformal class  $[g_0]$  also has non-vanishing curvature  $K_1$ ?
  - Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
  - Techniques: differential geometry; spectral theory of Laplacian; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

- **Questions:** Assume  $M \neq \mathbf{T}^2$ , and  $g_0$  has *non-vanishing* curvature  $K_0$ . What is the *probability* that a random metric  $g_1$  in the conformal class  $[g_0]$  also has non-vanishing curvature  $K_1$ ?
- Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
- Techniques: differential geometry; spectral theory of Laplacian; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

- **Questions:** Assume  $M \neq \mathbf{T}^2$ , and  $g_0$  has *non-vanishing* curvature  $K_0$ . What is the *probability* that a random metric  $g_1$  in the conformal class  $[g_0]$  also has non-vanishing curvature  $K_1$ ?
- Use *Laplacian* to define random metrics in a *conformal class* and to estimate that probability.
- Techniques: differential geometry; spectral theory of Laplacian; Gaussian random fields on manifolds (Borell, Tsirelson-Ibragimov-Sudakov, Adler-Taylor).

- $g_0$  - reference metric on  $M$ . Conformal class of  $g_0$ :  
 $\{g_1 = e^f \cdot g_0\}$ ;  $f$  is a random (suitably regular) function on  $M$ .
- $\Delta_0$  - Laplacian of  $g_0$ . Spectrum:  
 $\Delta_0 \phi_j + \lambda_j \phi_j = 0$ ,  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ . Define  $f$  by

$$f(x) = - \sum_{j=1}^{\infty} a_j c_j \phi_j(x), \quad (1)$$

where  $a_j \sim \mathcal{N}(0, 1)$  are i.i.d standard Gaussians,  
 $c_j = F(\lambda_j) \rightarrow 0$  are *decreasing*.

- $g_0$  - reference metric on  $M$ . Conformal class of  $g_0$ :  
 $\{g_1 = e^f \cdot g_0\}$ ;  $f$  is a random (suitably regular) function on  $M$ .
- $\Delta_0$  - Laplacian of  $g_0$ . Spectrum:  
 $\Delta_0 \phi_j + \lambda_j \phi_j = 0$ ,  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ . Define  $f$  by

$$f(x) = - \sum_{j=1}^{\infty} a_j c_j \phi_j(x), \quad (1)$$

where  $a_j \sim \mathcal{N}(0, 1)$  are i.i.d standard Gaussians,  
 $c_j = F(\lambda_j) \rightarrow 0$  are *decreasing*.

Gauss curvature

Question

Random metrics

$R_1$  changes sign

Using Borell-TIS

Real-analytic metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Functions on  $\mathbf{T}^2$ :  $h(x) = \sum_m c_m e^{i(x,m)}$ ,  $m \in \mathbf{Z}^2$ . Sobolev norm:  $(\|f\|_{H^s})^2 = \sum_m |c_m|^2 (1 + \|m\|^2)^s$ .

**General surface:**  $f(x) = \sum_j c_j \phi_j(x)$ .

$$\|f\|_{H^s}^2 = \sum_j c_j^2 (1 + \lambda_j)^s.$$

**Sobolev embedding theorem:** If  $s > k + 1$ , and

$\|f\|_{H^s} < \infty$ , then  $f \in C^k(M)$ .

**Weyl's law:**  $\lambda_j \asymp \text{const} \cdot j$ .

- *Random functions:*  $f$  as in (1), then

$$\mathbb{E}(\|f\|_{H^s}^2) = \sum_j c_j^2 (1 + \lambda_j)^s.$$

- **Proposition 1:** If  $c_j < C/\lambda_j^s$ ,  $s > 1$ , then  $f \in C^0(M)$  a.s.; if  $c_j < C/\lambda_j^s$ ,  $s > 2$ , then  $f \in C^2(M)$  a.s.

- Functions on  $\mathbf{T}^2$ :  $h(x) = \sum_m c_m e^{i(x,m)}$ ,  $m \in \mathbf{Z}^2$ . Sobolev norm:  $(\|f\|_{H^s})^2 = \sum_m |c_m|^2 (1 + \|m\|^2)^s$ .

**General surface:**  $f(x) = \sum_j c_j \phi_j(x)$ .

$$\|f\|_{H^s}^2 = \sum_j c_j^2 (1 + \lambda_j)^s.$$

**Sobolev embedding theorem:** If  $s > k + 1$ , and

$$\|f\|_{H^s} < \infty, \text{ then } f \in C^k(M).$$

**Weyl's law:**  $\lambda_j \asymp \text{const} \cdot j$ .

- *Random functions:*  $f$  as in (1), then

$$\mathbb{E}(\|f\|_{H^s}^2) = \sum_j c_j^2 (1 + \lambda_j)^s.$$

- **Proposition 1:** If  $c_j < C/\lambda_j^s$ ,  $s > 1$ , then  $f \in C^0(M)$  a.s.;  
if  $c_j < C/\lambda_j^s$ ,  $s > 2$ , then  $f \in C^2(M)$  a.s.

- Functions on  $\mathbf{T}^2$ :  $h(x) = \sum_m c_m e^{i(x,m)}$ ,  $m \in \mathbf{Z}^2$ . Sobolev norm:  $(\|f\|_{H^s})^2 = \sum_m |c_m|^2 (1 + \|m\|^2)^s$ .

**General surface:**  $f(x) = \sum_j c_j \phi_j(x)$ .

$$\|f\|_{H^s}^2 = \sum_j c_j^2 (1 + \lambda_j)^s.$$

**Sobolev embedding theorem:** If  $s > k + 1$ , and

$$\|f\|_{H^s} < \infty, \text{ then } f \in C^k(M).$$

**Weyl's law:**  $\lambda_j \asymp \text{const} \cdot j$ .

- *Random functions:*  $f$  as in (1), then

$$\mathbb{E}(\|f\|_{H^s}^2) = \sum_j c_j^2 (1 + \lambda_j)^s.$$

- **Proposition 1:** If  $c_j < C/\lambda_j^s$ ,  $s > 1$ , then  $f \in C^0(M)$  a.s.; if  $c_j < C/\lambda_j^s$ ,  $s > 2$ , then  $f \in C^2(M)$  a.s.

- The *covariance function*

$$r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y), \text{ for } x, y \in M.$$

- For  $x \in M$ ,  $f(x)$  is mean zero Gaussian of variance

$$r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2.$$

- **Area change:** Let  $A_0 = \text{area}(M, g_0)$ . If  $g_1 := g_1(a) = e^{2af} g_0$ , then  $dA_1 = e^{2af} dA_0$ . One can show that  $\lim_{a \rightarrow 0} \mathbb{E}[\text{area}(M, g_1(a))] = A_0$ .

- The *covariance function*

$$r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y), \text{ for } x, y \in M.$$

- For  $x \in M$ ,  $f(x)$  is mean zero Gaussian of variance

$$r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2.$$

- **Area change:** Let  $A_0 = \text{area}(M, g_0)$ . If  $g_1 := g_1(a) = e^{2af} g_0$ , then  $dA_1 = e^{2af} dA_0$ . One can show that  $\lim_{a \rightarrow 0} \mathbb{E}[\text{area}(M, g_1(a))] = A_0$ .

- The *covariance function*

$$r_f(x, y) := \mathbb{E}[f(x)f(y)] = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)\phi_j(y), \text{ for } x, y \in M.$$

- For  $x \in M$ ,  $f(x)$  is mean zero Gaussian of variance

$$r_f(x, x) = \sum_{j=1}^{\infty} c_j^2 \phi_j(x)^2.$$

- **Area change:** Let  $A_0 = \text{area}(M, g_0)$ . If  $g_1 := g_1(a) = e^{2af} g_0$ , then  $dA_1 = e^{2af} dA_0$ . One can show that  $\lim_{a \rightarrow 0} \mathbb{E}[\text{area}(M, g_1(a))] = A_0$ .

- Let  $g_1 = e^{2af}g_0$ . Then

$$K_1 = e^{-2af}[K_0 - a\Delta_0 f] \quad (2)$$

$M \neq \mathbf{T}^2$ . Estimate the probability of

$$\{\text{Sgn}(K_1) = \text{Sgn}(K_0)\}$$

- Observation:** If  $K_0 \neq 0$ , then  $\text{Sgn}(K_1) = \text{Sgn}(K_0)\text{Sgn}(1 - a\Delta_0 f/(K_0))$ .
- Let  $P(a) := \text{Prob}\{\exists x : \text{Sgn}K_1(x) \neq \text{Sgn}K_0\}$ , or  $P(a) = \text{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/K_0(x) < 0\}$ . Then

$$P(a) = \text{Prob}\{\sup_{x \in M} (\Delta_0 f)(x)/K_0(x) > 1/a\},$$

Consider the random field  $v = (\Delta_0 f)/K_0$ . Then

$$r_v(x, y) = \frac{\sum_j (c_j \lambda_j)^2 \phi_j(x) \phi_j(y)}{K_0(x) K_0(y)}.$$

- Let  $g_1 = e^{2af}g_0$ . Then

$$K_1 = e^{-2af}[K_0 - a\Delta_0 f] \quad (2)$$

$M \neq \mathbf{T}^2$ . Estimate the probability of

$$\{\text{Sgn}(K_1) = \text{Sgn}(K_0)\}$$

- **Observation:** If  $K_0 \neq 0$ , then  $\text{Sgn}(K_1) = \text{Sgn}(K_0)\text{Sgn}(1 - a\Delta_0 f/(K_0))$ .
- Let  $P(a) := \text{Prob}\{\exists x : \text{Sgn}K_1(x) \neq \text{Sgn}K_0\}$ , or  $P(a) = \text{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/K_0(x) < 0\}$ . Then

$$P(a) = \text{Prob}\{\sup_{x \in M} (\Delta_0 f)(x)/K_0(x) > 1/a\},$$

Consider the random field  $v = (\Delta_0 f)/K_0$ . Then

$$r_v(x, y) = \frac{\sum_j (c_j \lambda_j)^2 \phi_j(x) \phi_j(y)}{K_0(x) K_0(y)}.$$

- Let  $g_1 = e^{2af}g_0$ . Then

$$K_1 = e^{-2af}[K_0 - a\Delta_0 f] \quad (2)$$

$M \neq \mathbf{T}^2$ . Estimate the probability of

$$\{\text{Sgn}(K_1) = \text{Sgn}(K_0)\}$$

- Observation:** If  $K_0 \neq 0$ , then  $\text{Sgn}(K_1) = \text{Sgn}(K_0)\text{Sgn}(1 - a\Delta_0 f/(K_0))$ .
- Let  $P(a) := \text{Prob}\{\exists x : \text{Sgn}K_1(x) \neq \text{Sgn}K_0\}$ , or  $P(a) = \text{Prob}\{\exists x \in M : 1 - a(\Delta_0 f)(x)/K_0(x) < 0\}$ . Then

$$P(a) = \text{Prob}\{\sup_{x \in M} (\Delta_0 f)(x)/K_0(x) > 1/a\},$$

Consider the random field  $v = (\Delta_0 f)/K_0$ . Then

$$r_v(x, y) = \frac{\sum_j (c_j \lambda_j)^2 \phi_j(x) \phi_j(y)}{K_0(x) K_0(y)}.$$

- We shall estimate  $P(a)$  in the limit  $a \rightarrow 0$ . Geometrically, this implies that a.s.  $g_1(a) \rightarrow g_0$ , so  $P(a) \rightarrow 0$ . We want to estimate the *rate*.
- First use **Proposition 2** (Borell, TIS, 1975-76): Let  $v$  be a centered Gaussian process, a.s. bounded on  $M$ , and  $\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2]$ . Let  $\|v\| := \sup_{x \in M} v(x)$ ; then  $E\{\|v\|\} < \infty$ , and  $\exists \alpha$  so that for  $\tau > E\{\|v\|\}$  we have

$$\text{Prob}\{\|v\| > \tau\} \leq e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- Assume that  $K_0 \in C^0$ ,  $s > 2$ , then  $v \in C^0(M)$  a.s. and Proposition 2 applies. In our situation,  $\tau = (1/a) \rightarrow \infty$  as  $a \rightarrow 0$ , so  $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$ .

- We shall estimate  $P(a)$  in the limit  $a \rightarrow 0$ . Geometrically, this implies that a.s.  $g_1(a) \rightarrow g_0$ , so  $P(a) \rightarrow 0$ . We want to estimate the *rate*.
- First use **Proposition 2** (Borell, TIS, 1975-76): Let  $v$  be a centered Gaussian process, a.s. bounded on  $M$ , and  $\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2]$ . Let  $\|v\| := \sup_{x \in M} v(x)$ ; then  $E\{\|v\|\} < \infty$ , and  $\exists \alpha$  so that for  $\tau > E\{\|v\|\}$  we have

$$\text{Prob}\{\|v\| > \tau\} \leq e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- Assume that  $K_0 \in C^0$ ,  $s > 2$ , then  $v \in C^0(M)$  a.s. and Proposition 2 applies. In our situation,  $\tau = (1/a) \rightarrow \infty$  as  $a \rightarrow 0$ , so  $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$ .

- We shall estimate  $P(a)$  in the limit  $a \rightarrow 0$ . Geometrically, this implies that a.s.  $g_1(a) \rightarrow g_0$ , so  $P(a) \rightarrow 0$ . We want to estimate the *rate*.
- First use **Proposition 2** (Borell, TIS, 1975-76): Let  $v$  be a centered Gaussian process, a.s. bounded on  $M$ , and  $\sigma_v^2 := \sup_{x \in M} \mathbb{E}[v(x)^2]$ . Let  $\|v\| := \sup_{x \in M} v(x)$ ; then  $E\{\|v\|\} < \infty$ , and  $\exists \alpha$  so that for  $\tau > E\{\|v\|\}$  we have

$$\text{Prob}\{\|v\| > \tau\} \leq e^{\alpha\tau - \tau^2/(2\sigma_v^2)}.$$

- Assume that  $K_0 \in C^0$ ,  $s > 2$ , then  $v \in C^0(M)$  a.s. and Proposition 2 applies. In our situation,  $\tau = (1/a) \rightarrow \infty$  as  $a \rightarrow 0$ , so  $P(a) \leq \exp[C_2/a - 1/(2a^2\sigma_v^2)]$ .

- To estimate  $P(a)$  from below choose  $x_0 \in M$  where the variance  $r_V(x, x)$  attains its supremum  $\sigma_V^2$ . Clearly,  $\text{Prob}(\|v\| > 1/a) \geq \text{Prob}(v(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_{1/(a\sigma_V)}^\infty e^{-t^2/2} dt$ . Combine the estimates:
- **Theorem 3:** Assume that  $R_0 \in C^0$ ,  $c_j = O(\lambda_j^{-s})$ ,  $s > 2$ . Then  $\exists C_1 > 0, C_2 > 0$  such that

$$(C_1 a) e^{-1/(2a^2\sigma_V^2)} \leq P(a) \leq e^{C_2/a-1/(2a^2\sigma_V^2)},$$

as  $a \rightarrow 0$ . In particular  $\lim_{a \rightarrow 0} a^2 \ln P(a) = \frac{-1}{2\sigma_V^2}$ .

- To estimate  $P(a)$  from below choose  $x_0 \in M$  where the variance  $r_v(x, x)$  attains its supremum  $\sigma_v^2$ . Clearly,  $\text{Prob}(\|v\| > 1/a) \geq \text{Prob}(v(x_0) > 1/a) = \frac{1}{\sqrt{2\pi}} \int_{1/(a\sigma_v)}^\infty e^{-t^2/2} dt$ . Combine the estimates:
- **Theorem 3:** Assume that  $R_0 \in C^0$ ,  $c_j = O(\lambda_j^{-s})$ ,  $s > 2$ . Then  $\exists C_1 > 0, C_2 > 0$  such that

$$(C_1 a) e^{-1/(2a^2\sigma_v^2)} \leq P(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as  $a \rightarrow 0$ . In particular  $\lim_{a \rightarrow 0} a^2 \ln P(a) = \frac{-1}{2\sigma_v^2}$ .

- **Random real-analytic metrics.** Choose the coefficients  $c_j = e^{-\lambda_j T/2}/\lambda_j$ . Then

$$r_V(x, x, T) = e^*(x, x, T)/(K_0(x))^2.$$

where  $e^*(x, x, T)$  is the heat kernel, *without the constant term*.

- **Small  $T$  asymptotics** of  $e^*(x, x, T)$  imply that as  $T \rightarrow 0^+$ ,

$$\sigma_V^2 \sim \frac{1}{4\pi T \inf_{x \in M} (K_0(x))^2}.$$

- **Random real-analytic metrics.** Choose the coefficients  $c_j = e^{-\lambda_j T/2}/\lambda_j$ . Then

$$r_V(x, x, T) = e^*(x, x, T)/(K_0(x))^2.$$

where  $e^*(x, x, T)$  is the heat kernel, *without the constant term*.

- **Small  $T$  asymptotics** of  $e^*(x, x, T)$  imply that as  $T \rightarrow 0^+$ ,

$$\sigma_V^2 \sim \frac{1}{4\pi T \inf_{x \in M} (K_0(x))^2}.$$

- **Theorem 4.**  $M \neq \mathbf{T}^2$ . Let  $g_0$  and  $g_1$  have equal areas,  $R_0$  and  $R_1$  have constant sign,  $K_0 \equiv \text{const}$  and  $K_1 \not\equiv \text{const}$ . Then  $\exists a_0 > 0, T_0 > 0$  (that depend on  $g_0, g_1$ ) such that for any  $0 < a < a_0$  and for any  $0 < t < T_0$ , we have  $P(a, T, g_1) > P(a, T, g_0)$ .
- **Proof:** By Gauss-Bonnet,  $\int_M K_0 dA_0 = \int_M K_1 dA_1$ . Since  $A(M, g_0) = A(M, g_1)$ ; and since  $K_0 \equiv \text{const}$  and  $K_1 \not\equiv \text{const}$ , it follows that  $b_0 := \min_{x \in M} (K_0(x))^2 > \min_{x \in M} (K_1(x))^2 := b_1$ . Accordingly, as  $T \rightarrow 0^+$ , we have

$$\frac{\sigma_V^2(g_1, T)}{\sigma_V^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

The result follows easily from Theorem 3.

- **Theorem 4.**  $M \neq \mathbf{T}^2$ . Let  $g_0$  and  $g_1$  have equal areas,  $R_0$  and  $R_1$  have constant sign,  $K_0 \equiv \text{const}$  and  $K_1 \not\equiv \text{const}$ . Then  $\exists a_0 > 0, T_0 > 0$  (that depend on  $g_0, g_1$ ) such that for any  $0 < a < a_0$  and for any  $0 < t < T_0$ , we have  $P(a, T, g_1) > P(a, T, g_0)$ .
- **Proof:** By Gauss-Bonnet,  $\int_M K_0 dA_0 = \int_M K_1 dA_1$ . Since  $A(M, g_0) = A(M, g_1)$ ; and since  $K_0 \equiv \text{const}$  and  $K_1 \not\equiv \text{const}$ , it follows that  $b_0 := \min_{x \in M} (K_0(x))^2 > \min_{x \in M} (K_1(x))^2 := b_1$ . Accordingly, as  $T \rightarrow 0^+$ , we have

$$\frac{\sigma_V^2(g_1, T)}{\sigma_V^2(g_0, T)} \asymp \frac{b_0}{b_1} > 1.$$

The result follows easily from Theorem 3.

- **Large  $T$  asymptotics:**

$\lambda_1$  - the smallest nonzero eigenvalue of  $-\Delta_0$ . Let  $m = m(\lambda_1)$  be the multiplicity of  $\lambda_1$ , and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{K_0(x)^2}. \quad (3)$$

- One can show that

$$\lim_{T \rightarrow \infty} \frac{\sigma_V^2(T)}{Fe^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let  $g_0$  and  $g_1$  be two metrics (of equal area) on a compact surface  $M$ , such that  $K_0$  and  $K_1$  have constant sign, and such that  $\lambda_1(g_0) > \lambda_1(g_1)$ . Then there exist  $a_0 > 0$  and  $0 < T_0 < \infty$  (that depend on  $g_0, g_1$ ), such that for all  $a < a_0$  and  $T > T_0$  we have  $P(a, T; g_0) < P(a, T; g_1)$ .

- **Large  $T$  asymptotics:**

$\lambda_1$  - the smallest nonzero eigenvalue of  $-\Delta_0$ . Let  $m = m(\lambda_1)$  be the multiplicity of  $\lambda_1$ , and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{K_0(x)^2}. \quad (3)$$

- One can show that

$$\lim_{T \rightarrow \infty} \frac{\sigma_V^2(T)}{F e^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let  $g_0$  and  $g_1$  be two metrics (of equal area) on a compact surface  $M$ , such that  $K_0$  and  $K_1$  have constant sign, and such that  $\lambda_1(g_0) > \lambda_1(g_1)$ . Then there exist  $a_0 > 0$  and  $0 < T_0 < \infty$  (that depend on  $g_0, g_1$ ), such that for all  $a < a_0$  and  $T > T_0$  we have  $P(a, T; g_0) < P(a, T; g_1)$ .

- **Large  $T$  asymptotics:**

$\lambda_1$  - the smallest nonzero eigenvalue of  $-\Delta_0$ . Let  $m = m(\lambda_1)$  be the multiplicity of  $\lambda_1$ , and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{K_0(x)^2}. \quad (3)$$

- One can show that

$$\lim_{T \rightarrow \infty} \frac{\sigma_V^2(T)}{F e^{-\lambda_1 T}} = 1.$$

- **Theorem 5.** Let  $g_0$  and  $g_1$  be two metrics (of equal area) on a compact surface  $M$ , such that  $K_0$  and  $K_1$  have constant sign, and such that  $\lambda_1(g_0) > \lambda_1(g_1)$ . Then there exist  $a_0 > 0$  and  $0 < T_0 < \infty$  (that depend on  $g_0, g_1$ ), such that for all  $a < a_0$  and  $T > T_0$  we have  $P(a, T; g_0) < P(a, T; g_1)$ .

Gauss curvature

Question

Random metrics

$R_1$  changes sign

Using Borell-TIS

Real-analytic metrics

Using A-T

$L^\infty$  bounds

Conclusion

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0: ( $S^2$ , round) extremal for both small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for all  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
- Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
- Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate  $T$ ?

Gauss curvature

Question

Random metrics

$R_1$  changes sign

Using Borell-TIS

Real-analytic metrics

Using A-T

$L^\infty$  bounds

Conclusion

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0: ( $S^2$ , round) extremal for both small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for all  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
- Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
- Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate  $T$ ?

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0:  $(S^2, \text{round})$  extremal for *both* small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for *all*  $T$ .
  - Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
  - Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
  - Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
  - **Question:** Which metrics are extremal for intermediate  $T$ ?

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0: ( $S^2$ , round) extremal for *both* small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for *all*  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
  - Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
  - Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
  - **Question:** Which metrics are extremal for intermediate  $T$ ?

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0: ( $S^2$ , round) extremal for *both* small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for *all*  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
- Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
- Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate  $T$ ?

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0: ( $S^2$ , round) extremal for *both* small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for *all*  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
- Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
- Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate  $T$ ?

- To summarize: Small  $T \Rightarrow$  metrics with  $K_0 \equiv \text{const}$  extremal.
- Large  $T \Rightarrow$  metrics with the largest  $\lambda_1$  extremal.
- Genus 0:  $(S^2, \text{round})$  extremal for *both* small  $T$  and large  $T$  (Hersch). **Conjecture:** extremal for *all*  $T$ .
- Genus  $\gamma \geq 2$ : Small  $T \Rightarrow$  hyperbolic metrics extremal.
- Large  $T$ : By a 1985 theorem of R. Bryant, hyperbolic metrics *never* maximize  $\lambda_1$  in their conformal class.
- Genus 2: Metrics maximizing  $\lambda_1$  for surfaces of genus 2 of fixed area are branched coverings of the round  $S^2$  (J, Levitin, Nigam, Nadirashvili, Polterovich).
- **Question:** Which metrics are extremal for intermediate  $T$ ?

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- We next indicate how to obtain a better estimate for  $P(a)$  for  $M = S^2$ .  $\exists!$  conformal class  $[g_0]$  on  $S^2$ ;  $g_0$  is the round metric,  $K_0 \equiv 1$ .
- The isometry group acts transitively on  $(S^2, g_0)$ , so the random fields  $f(x), v(x)$  are *isotropic* and in particular have *constant variance*. That allows us to apply results of Adler and Taylor and obtain more precise *asymptotic* estimates for  $P(a)$ .

- We next indicate how to obtain a better estimate for  $P(a)$  for  $M = S^2$ .  $\exists!$  conformal class  $[g_0]$  on  $S^2$ ;  $g_0$  is the round metric,  $K_0 \equiv 1$ .
- The isometry group acts transitively on  $(S^2, g_0)$ , so the random fields  $f(x), v(x)$  are *isotropic* and in particular have *constant variance*. That allows us to apply results of Adler and Taylor and obtain more precise *asymptotic* estimates for  $P(a)$ .

- Since  $\Delta_0$  on  $(S^2, g_0)$  is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$  - space of spherical harmonics of degree  $m$ , dimension  $N_m = 2m + 1$ ; the corresponding eigenvalue is  $E_m = m(m + 1)$ . Let  $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$  be an orthonormal basis of  $\mathcal{E}_m$ .
- Let  $f(x) = -\sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$ , where  $a_{m,k}$  are standard Gaussian i.i.d. and  $c_m > 0$  are (suitably decaying) constants satisfying  $\sum_{m=1}^{\infty} c_m = 1$ .
- It follows that  $v = \sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$  has unit variance, and covariance  $r_v(x, y) = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x, y)))$ , where  $P_m$  is the Legendre polynomial.

Gauss curvature

Question

Random metrics

$R_1$  changes sign

Using Borell-TIS

Real-analytic metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Since  $\Delta_0$  on  $(S^2, g_0)$  is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$  - space of spherical harmonics of degree  $m$ , dimension  $N_m = 2m + 1$ ; the corresponding eigenvalue is  $E_m = m(m + 1)$ . Let  $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$  be an orthonormal basis of  $\mathcal{E}_m$ .
- Let  $f(x) = -\sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$ , where  $a_{m,k}$  are standard Gaussian i.i.d. and  $c_m > 0$  are (suitably decaying) constants satisfying  $\sum_{m=1}^{\infty} c_m = 1$ .
- It follows that  $v = \sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$  has unit variance, and covariance  $r_v(x, y) = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x, y)))$ , where  $P_m$  is the Legendre polynomial.

- Since  $\Delta_0$  on  $(S^2, g_0)$  is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$  - space of spherical harmonics of degree  $m$ , dimension  $N_m = 2m + 1$ ; the corresponding eigenvalue is  $E_m = m(m + 1)$ . Let  $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$  be an orthonormal basis of  $\mathcal{E}_m$ .
- Let  $f(x) = -\sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$ , where  $a_{m,k}$  are standard Gaussian i.i.d. and  $c_m > 0$  are (suitably decaying) constants satisfying  $\sum_{m=1}^{\infty} c_m = 1$ .
- It follows that  $v = \sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$  has unit variance, and covariance  $r_v(x, y) = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x, y)))$ , where  $P_m$  is the Legendre polynomial.

- Since  $\Delta_0$  on  $(S^2, g_0)$  is highly degenerate, we normalize our random Fourier series differently.
- $\mathcal{E}_m$  - space of spherical harmonics of degree  $m$ , dimension  $N_m = 2m + 1$ ; the corresponding eigenvalue is  $E_m = m(m + 1)$ . Let  $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$  be an orthonormal basis of  $\mathcal{E}_m$ .
- Let  $f(x) = -\sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$ , where  $a_{m,k}$  are standard Gaussian i.i.d. and  $c_m > 0$  are (suitably decaying) constants satisfying  $\sum_{m=1}^{\infty} c_m = 1$ .
- It follows that  $v = \sqrt{|S^2|} \sum_{m \geq 1, k} \frac{\sqrt{c_m}}{\sqrt{N_m}} a_{m,k} \eta_{m,k}(x)$  has unit variance, and covariance  $r_v(x, y) = \sum_{m=1}^{\infty} c_m P_m(\cos(d(x, y)))$ , where  $P_m$  is the Legendre polynomial.

- In the new normalization, if  $c_m = O(M^{-s})$ ,  $s > 7$ , then  $(\Delta_0 f)(x) \in C^2(S^2)$  a.s.

- Applying results of A-T, we can prove

- **Theorem 6:** Notation as above, let  $c_m = O(m^{-s})$ ,  $s > 7$ . Let  $C = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ . Then there exists  $\alpha > 1$ , s.t. in the limit  $a \rightarrow 0$ ,  $P(a)$  satisfies

$$P(a) = \frac{C}{a} \exp\left(-\frac{1}{2a^2}\right) + \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}\right) + o\left(\exp\left(-\frac{\alpha}{2a^2}\right)\right)$$

- Note that we now have an *asymptotic* expression for  $P(a)$ .

- In the new normalization, if  $c_m = O(M^{-s})$ ,  $s > 7$ , then  $(\Delta_0 f)(x) \in C^2(S^2)$  a.s.

- Applying results of A-T, we can prove

- **Theorem 6:** Notation as above, let  $c_m = O(m^{-s})$ ,  $s > 7$ . Let  $C = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ . Then there exists  $\alpha > 1$ , s.t. in the limit  $a \rightarrow 0$ ,  $P(a)$  satisfies

$$P(a) = \frac{C}{a} \exp\left(-\frac{1}{2a^2}\right) + \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}\right) + o\left(\exp\left(-\frac{\alpha}{2a^2}\right)\right)$$

- Note that we now have an *asymptotic* expression for  $P(a)$ .

- In the new normalization, if  $c_m = O(M^{-s})$ ,  $s > 7$ , then  $(\Delta_0 f)(x) \in C^2(S^2)$  a.s.
- Applying results of A-T, we can prove
- **Theorem 6:** Notation as above, let  $c_m = O(m^{-s})$ ,  $s > 7$ . Let  $C = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ . Then there exists  $\alpha > 1$ , s.t. in the limit  $a \rightarrow 0$ ,  $P(a)$  satisfies

$$P(a) = \frac{C}{a} \exp\left(-\frac{1}{2a^2}\right) + \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}\right) + o\left(\exp\left(-\frac{\alpha}{2a^2}\right)\right)$$

- Note that we now have an *asymptotic* expression for  $P(a)$ .

- In the new normalization, if  $c_m = O(M^{-s})$ ,  $s > 7$ , then  $(\Delta_0 f)(x) \in C^2(S^2)$  a.s.
- Applying results of A-T, we can prove
- **Theorem 6:** Notation as above, let  $c_m = O(m^{-s})$ ,  $s > 7$ . Let  $C = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ . Then there exists  $\alpha > 1$ , s.t. in the limit  $a \rightarrow 0$ ,  $P(a)$  satisfies

$$P(a) = \frac{C}{a} \exp\left(-\frac{1}{2a^2}\right) + \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}\right) + o\left(\exp\left(-\frac{\alpha}{2a^2}\right)\right)$$

- Note that we now have an *asymptotic* expression for  $P(a)$ .

- We next estimate the probability of the event  $\{\|K_1 - K_0\|_\infty < u\}$ ,  $u > 0$ ; we shall do that for  $g_1 = e^{af}g_0$ , in the limit  $a \rightarrow 0$ . The result below hold for any compact orientable surface, including  $\mathbf{T}^2$ .
- To state the result, we define a new random field  $w$  on  $M$ :

$$w = \Delta_0 f + 2K_0 f.$$

We denote its covariance function by  $r_w(x, y)$ , and we define  $\sigma_w^2 = \sup_{x \in M} r_w(x, x)$ .

- We next estimate the probability of the event  $\{\|K_1 - K_0\|_\infty < u\}$ ,  $u > 0$ ; we shall do that for  $g_1 = e^{af}g_0$ , in the limit  $a \rightarrow 0$ . The result below hold for any compact orientable surface, including  $\mathbf{T}^2$ .
- To state the result, we define a new random field  $w$  on  $M$ :

$$w = \Delta_0 f + 2K_0 f.$$

We denote its covariance function by  $r_w(x, y)$ , and we define  $\sigma_w^2 = \sup_{x \in M} r_w(x, x)$ .

- We can now state

**Theorem 7:** Assume that the random metric is chosen so that the random fields  $f, w$  are a.s.  $C^0$ . Let  $a \rightarrow 0$  and  $u \rightarrow 0$  so that  $(u/a) \rightarrow \infty$ . Then

$$\log \text{Prob}(\|K_1 - K_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

- The proof uses Borell-TIS inequality. The condition  $(u/a) \rightarrow \infty$  ensures that the application of Borell-TIS gives an asymptotic result for  $\log \text{Prob}(\|K_1 - K_0\|_\infty > u)$ .
- The condition  $u \rightarrow 0$  is needed to estimate (from above) the probability of certain *exceptional* events.

- We can now state

**Theorem 7:** Assume that the random metric is chosen so that the random fields  $f, w$  are a.s.  $C^0$ . Let  $a \rightarrow 0$  and  $u \rightarrow 0$  so that  $(u/a) \rightarrow \infty$ . Then

$$\log \text{Prob}(\|K_1 - K_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

- The proof uses Borell-TIS inequality. The condition  $(u/a) \rightarrow \infty$  ensures that the application of Borell-TIS gives an asymptotic result for  $\log \text{Prob}(\|K_1 - K_0\|_\infty > u)$ .
- The condition  $u \rightarrow 0$  is needed to estimate (from above) the probability of certain *exceptional* events.

- We can now state

**Theorem 7:** Assume that the random metric is chosen so that the random fields  $f, w$  are a.s.  $C^0$ . Let  $a \rightarrow 0$  and  $u \rightarrow 0$  so that  $(u/a) \rightarrow \infty$ . Then

$$\log \text{Prob}(\|K_1 - K_0\|_\infty > u) \sim -\frac{u^2}{2a^2\sigma_w^2}.$$

- The proof uses Borell-TIS inequality. The condition  $(u/a) \rightarrow \infty$  ensures that the application of Borell-TIS gives an asymptotic result for  $\log \text{Prob}(\|K_1 - K_0\|_\infty > u)$ .
- The condition  $u \rightarrow 0$  is needed to estimate (from above) the probability of certain *exceptional* events.

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
  - Consider “rough” metrics that arise in 2D quantum gravity.
  - Study the case when  $a \rightarrow 0$ .
  - Study Ricci and sectional curvatures in high dimensions.
  - Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
  - Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
  - Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
  - $\Delta$ : small eigenvalues, heat kernel asymptotics.
  - Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss curvature

Question

Random metrics

$R_1$  changes sign

Using Borell-TIS

Real-analytic metrics

Using A-T

$L^\infty$  bounds

Conclusion

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
  - Study the case when  $a \rightarrow 0$ .
  - Study Ricci and sectional curvatures in high dimensions.
  - Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
  - Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
  - Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
  - $\Delta$ : small eigenvalues, heat kernel asymptotics.
  - Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
  - Study Ricci and sectional curvatures in high dimensions.
  - Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
  - Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
  - Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
  - $\Delta$ : small eigenvalues, heat kernel asymptotics.
  - Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- $\Delta$ : small eigenvalues, heat kernel asymptotics.
- Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
  - Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
  - Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
  - $\Delta$ : small eigenvalues, heat kernel asymptotics.
  - Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
  - Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
  - $\Delta$ : small eigenvalues, heat kernel asymptotics.
  - Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- $\Delta$ : small eigenvalues, heat kernel asymptotics.
- Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- $\Delta$ : small eigenvalues, heat kernel asymptotics.
- Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- $\Delta$ : small eigenvalues, heat kernel asymptotics.
- Prove *quantitative* estimates (spectral gaps, level spacing).

Gauss  
curvature

Question

Random  
metrics

$R_1$  changes  
sign

Using  
Borell-TIS

Real-analytic  
metrics

Using A-T

$L^\infty$  bounds

Conclusion

- Improve estimates in [CJW] for the *scalar curvature* in higher dimensions.
- Consider “rough” metrics that arise in 2D quantum gravity.
- Study the case when  $a \rightarrow 0$ .
- Study Ricci and sectional curvatures in high dimensions.
- Consider the space of all metrics, not just those in a conformal class (interesting in dimension  $n \geq 3$ ).
- Study differential geometry of random metrics, e.g. distance between two points, diameter etc.
- Study geodesic and frame flows and their ergodicity; existence of conjugate points; entropy etc.
- $\Delta$ : small eigenvalues, heat kernel asymptotics.
- Prove *quantitative* estimates (spectral gaps, level spacing).