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AN ESCAPE-TIME CRITERION FOR QUEUEING NETWORKS: ASYMPTOTIC RISK-SENSITIVE CONTROL VIA DIFFERENTIAL GAMES

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We consider the problem of risk-sensitive control of a stochastic network. In controlling such a network, an escape-time criterion can be useful if one wishes to regulate the occurrence of large buffers and buffer overflow. In this paper a risk-sensitive escape time criterion is formulated, which in comparison to the ordinary escape-time criteria penalizes exits that occur on short time intervals more heavily. The properties of the risk-sensitive problem are studied in the large buffer limit and related to the value of a deterministic differential game with constrained dynamics. We prove that the game has value and that the value is the (viscosity) solution of a PDE. For a simple network, the value is computed, demonstrating the applicability of the approach.

1. Introduction. In this paper we consider a problem of risk-sensitive control (or rare event control) for queueing networks. The network includes servers that can offer service to two or more classes of customers, and a choice must be made regarding which classes to offer service at each time. We study a stochastic control problem in which this choice is regarded as the control and where the cost is a risk-sensitive version of the time to escape a bounded set. Hence, fixing $c > 0$, and denoting by σ the time when the queue-lengths process first exits a given domain, we consider $E_x e^{-c\sigma}$ as a criterion to be minimized. Such a criterion penalizes short exit times more heavily than ordinary escape time criteria (such as $E_x\sigma$, a criterion to be maximized). There are at least two motivations for the use of such criteria when designing policies for the control of a network. The first is that in many communication networks, system performance is measured in terms of rare event probabilities (e.g., probabilities of data loss or excessive delay). The second motivation follows from the connection between risk-sensitive controls and robust controls. Indeed, as discussed in Dupuis et al. (2000), the optimization of a single fixed stochastic network with respect to a risk-sensitive cost criteria automatically produces controls with specific and predictable robust properties. In particular, these controls give good performance for a family of perturbed network models (where the perturbation is around the design model and the size of the perturbation is measured by relative entropy), and with respect to a corresponding ordinary (i.e., not risk-sensitive) cost.

In many problems, one considers the limit of the risk-sensitive problem as a scaling parameter of the system converges, in the hope that the limit model is more tractable. We follow the same approach here and show that the normalized costs in the risk-sensitive problems converge to the value function of a differential game with constraints. As is well known, the convergence analysis is closely related to the large deviation properties of the sequence of controlled processes. An interesting feature in the setting of stochastic networks is that the asymptotic analysis of a sequence of *controlled* networks is in many ways simpler than the analogous asymptotic analysis of a sequence of *uncontrolled* networks. For example, if one were to fix a particular state feedback service policy at each station, then the

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calculation of the large deviation asymptotics is very difficult. In contrast, it turns out that calculation of the large deviation asymptotics of the optimally controlled network is quite feasible. This is largely due to the fact that a *fixed* service policy invariably includes some *state discontinuities*. For example, a priority policy switches drastically when the highest priority queue empties. When the policy is left as a parameter that is to be optimized, these sharp discontinuities are not dealt with directly because the control and the large deviation behavior are identified simultaneously. The situation is analogous to one found in the control of unconstrained processes such as diffusions. If a fixed, nonsmooth feedback control is considered, then large deviation asymptotics are generally intractable; but when the combined large deviation and optimal control problem is considered, much is possible (Dupuis and McEneaney 1997).

For simplicity, we restrict attention in this paper to a class of Markovian networks and consider just one simple cost structure. Much more general statistical models can be treated with similar arguments, as can a more general cost. A more fundamental restriction is on the routing in the network. We assume a re-entrant line structure, so that the input streams follow a fixed route through the network—we do not allow either randomized or controlled routing. Relaxing the last conditions leads to a problem that is significantly more difficult to analyze and would require a considerable extension of the results we prove.

The deterministic game that is associated with the limit stochastic control problem involves two players. One player allocates service in a way analogous to the control in the stochastic control problem, and the other player perturbs the service and arrival rates. The cost is expressed in terms of the large deviation rate function for the underlying arrival and service processes, cumulated up to the time the dynamics exit the domain. Heuristically, the first player identifies those classes it is most worthwhile to allocate service to, so as to delay the escape as much as possible and thereby maximize the cost. The player who selects the perturbed rates attempts to minimize the cost by driving it out of the domain, while paying a cost for perturbing the rates.

Our main result states that as the scaling parameter of the system converges, the value for the stochastic control problem converges to the value of the game. By way of proving the result, we also show that the Hamilton-Jacobi-Bellman equation associated with the game has a unique (Lipschitz continuous) viscosity solution.

Several works have considered problems of optimal exit probabilities in the context of controlled diffusion processes, in the asymptotically small white-noise intensity regime. Fleming and Souganidis (1986) use viscosity solutions techniques to study a controlled diffusion where the control enters in the drift coefficient. Dupuis and Kushner (1989) extend their results to the case where the diffusion coefficient is possibly degenerate. Their technique relates the stochastic control problem to the game in a more direct way, using time discretization, without involving PDE analysis. The stochastic control problem studied in the current paper has the property that the jump rates in certain directions (those that correspond to services, not to arrivals) can be controlled to assume arbitrarily low values, including zero. It appears to be a more subtle problem than the ones in the above-cited papers, in that it is analogous to a controlled diffusion problem where the control enters also in the *diffusion* coefficient, and where no uniform nondegeneracy condition is assumed. This kind of degeneracy makes it difficult to apply the time discretization idea of Dupuis and Kushner (1989). The main idea of Dupuis and Kushner, in which one directly relates the control problem to the game, is still fruitful in the current setting. Following this approach, we relate the limit inferior (respectively, superior) of the asymptotic value for the control problem to the upper (respectively, lower) value of the game. However, showing that the game has value, and thereby obtaining the full convergence result for the control problem, requires a PDE analysis.

The PDE analysis uses viscosity solutions methods. There are three types of boundary conditions associated with the PDE: Neumann, Dirichlet, and “state-space constraint.”

The first two types of boundary conditions correspond in the game to the nonnegativity constraint on queue lengths and to stopping upon exit from the domain, respectively. The third type of boundary condition arises when there are portions of the boundary where exit can be blocked unilaterally by one of the players, and it is optimal for it to do so. It is well known since Soner (1986) that such a scenario leads to the last boundary condition mentioned above. Combining techniques of Atar and Dupuis (forthcoming) and Soner (1986), we prove uniqueness of viscosity solutions for the PDE and show that the game's upper and lower values are viscosity solutions, thus establishing existence of value. The trivial but crucial fact used in the uniqueness proof is that the Isaacs condition holds (Equation (38)).

As an example, we analyze a case where the domain is a hyperrectangle and where the network consists of one server and many queues, each customer requiring service only once. We find an explicit solution to the corresponding PDE, assuming the parameter c is large enough. This is only an initial result in this direction, but it shows that explicit solutions can be found. The solution turns out to be of particularly simple form (see Equation (49)). The optimal service discipline stemming from the solution corresponds to giving priority to class i whenever the state of the system is within a subset G_i of the domain. The partitioning of the domain into subsets has a simple structure too (see Figure 2 in §5 for an example in two dimensions). See Atar et al. (forthcoming) for explicit solutions in the case of tandem queues, as well as identities relating the perturbed rates with the unperturbed ones in a more general network.

There is relatively little work on risk-sensitive and robust control of networks. Ball et al. (1999b) have considered a robust formulation for network problems arising in vehicular traffic (Ball et al. 1999a) and have explicitly identified the value function in certain instances. Although their model is similar to ours in that the network dynamics are modeled via a Skorokhod Problem, many other features, most notably the cost structure, are qualitatively different. In addition, the model they consider is not naturally related to a risk-sensitive control problem for a jump Markov model of a network.

The organization of the paper is as follows. Section 2 introduces the network and the stochastic control problem, describes a key tool in our analysis (namely the Skorokhod Problem (SP)), introduces the differential game, and states the main result. Section 3 establishes the relation between the control problem asymptotics and the game's upper and lower values. Characterization of the upper and lower values of the game as viscosity solutions of a PDE, as well as uniqueness for this PDE, are established in §4. Section 5 presents an example, and the paper concludes with §6, which gives the proofs of several lemmas. Throughout the paper, numbering such as Lemma $a.b$ refers to the bth item of Lemma a.

2. Problem setting and the main result.

The queueing network control problem. We consider a system with J customer classes, and without loss assume that each class is identified with a queue at one of K servers. Each server provides service to at least one class. Thus, if $C(k)$ denotes the set of classes that are served by k, then the control determines who receives service effort at server k from among $i \in C(k)$. In particular, the sets $C(k)$, $k = 1, \ldots, K$ are disjoint, with $\bigcup_k C(k) = \{1, \ldots, J\}$. The state of the network is the vector of queue lengths, denoted by X. After a customer of class i is served, it turns into a customer of class $r(i)$, where $i = 0$ is used to denote the "outside." We let e_j denote the unit vector in direction j and set $e_0 = 0$ so that following service to class j the state changes by $e_{r(j)} - e_j$. The control will be described by the vector $u = (u_1, \ldots, u_J)$, where $u_i = 1$ if class i customers are given service and $u_i = 0$ otherwise. Because service can be given at any moment to only one class at each station, the control vector must satisfy $\sum_{i \in C(k)} u_i \leq 1$ for each k. We next consider the scaled process X^n under the scaling which accelerates time by a factor of n and shrinks space by the same factor. We are interested in a risk-sensitive cost functional that is associated with exit from a bounded set. Let G be a bounded subset of $\mathbb{R}_+^J$ that contains the origin (additional assumptions on G are given in Condition 1). Define

$$\sigma^n \doteq \inf\{t : X^n(t) \notin G\}.$$

Then the control problem is to minimize the cost $E_x e^{-nc\sigma^n}$, where E_x denotes expectation starting from x, and $c > 0$ is a constant. With this cost structure "risk-sensitivity" means that atypically short exit times are weighted heavily by the cost. A "good" control will avoid such an event with high probability. The significance from the point of view of stabilization of the system is clear. (See also Dupuis and McEneaney 1997 for the robust interpretation.)

A precise description of the stochastic control problem is as follows. Let $G^n \doteq n^{-1}\mathbb{Z}_+^J \cap G$. Define

$$U \doteq \left\{(u_i), i = 1, \ldots, J : \sum_{i\in C(k)} u_i \le 1, k = 1, \ldots, K, u_i \ge 0, i = 1, \ldots, J\right\}.$$

For $u \in U$ and $f : \mathbb{Z}^J \to \mathbb{R}$ let

$$\widetilde{\mathscr{L}}^u f(x) \doteq \sum_{j=1}^J \lambda_j[f(x+e_j) - f(x)] + \sum_{j=1}^J u_j\mu_j 1_{\{x+\tilde{v}_j\in\mathbb{Z}_+^J\}}[f(x+\tilde{v}_j) - f(x)], \tag{1}$$

where $\tilde{v}_j = e_{r(j)} - e_j$. It is assumed that for each i, $\lambda_i \ge 0$, while $\mu_i > 0$. For each $n \in \mathbb{N}$ consider the scaling defined by

$$\widetilde{\mathscr{L}}^{n,u} f(x) \doteq n\widetilde{\mathscr{L}}^u g(nx), \tag{2}$$

where f: $n^{-1}\mathbb{Z}^J \to \mathbb{R}$ and $g(\cdot) = f(n^{-1}\cdot)$. A *controlled Markov process* starting from x will consist of a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P_x^{n,u})$, a state process X^n taking values in G^n that is continuous from the right and with limits from the left, a control process u taking values in U, such that X^n is adapted to $\mathscr{F}_t$, u is measurable and adapted to $\mathscr{F}_t$, $P_x^{u,n}(X^n(0) = x) = 1$, and for every bounded function f: $n^{-1}\mathbb{Z}^J \to \mathbb{R}$

$$f(X^n(t)) - \int_0^t \widetilde{\mathscr{L}}^{n,\,u(s)} f(X^n(s))\, ds$$

is an $\mathscr{F}_t$-martingale. $E_x^{n,u}$ denotes expectation with respect to $P_x^{n,u}$. For a parameter $c > 0$, the value function for the stochastic control problem is defined by

$$V^n(x) \doteq -\inf n^{-1} \log E_x^{u,\,n} e^{-nc\sigma_n}, \qquad x \in G^n. \tag{3}$$

In this definition the infimum is over all controlled Markov processes.

A measurable function $u(x, t)$, u: $G^n \times [0, \infty) \to U$ is said to be a *feedback control*. We will make use of two well-known facts: To each feedback control there corresponds a controlled Markov process with $u(t) = u(X^n(t), t)$; and in the definition of the value function, the infimum can be restricted to feedback controls.

In the formulation just given we allow the maximizing player to choose a control from the convex set U. This is a relaxed formulation, which allows the server to simultaneously split the effort between two or more customer classes. An alternative control space that is more natural in implementation consists of only the vertices of U, in which case the server can only serve one class at a time. In a general game setting, the distinction between such "relaxed" and "pure" control spaces can be significant. However, in the present setting it will turn out that the value is the same for both cases. This is essentially due to the fact that the game arises from a risk-sensitive control problem, which imposes additional structure on the game and will be further commented on below.

Dynamics via the Skorokhod problem. Our main goal will be to study the asymptotics of V^n and, in particular, to show that they are governed by the value of a deterministic differential game. In order to define the dynamics of this game we first need a formulation of the *Skorokhod problem* (SP). We give here the simplest formulation which covers our needs. The reader is referred to Dupuis and Ramanan (1999) for a more general framework. Let

$$D_+([0,\infty):\mathbb{R}^J) \doteq \{\psi \in D([0,\infty):\mathbb{R}^J) : \psi(0) \in \mathbb{R}^J_+\},$$

where $D([0,\infty):\mathbb{R}^J)$ is the space of left-continuous functions with right-hand limits, endowed with the uniform on compacts topology. When restricting to continuous functions, we replace "D" with "C". Let a set of vectors $\{\gamma_i,\ i=1,\dots,J\}$ be given and set $I(x) \doteq \{i : x_i = 0\}$. For each point x on $\partial\mathbb{R}^J_+$—the boundary of $\mathbb{R}^J_+$—let

$$d(x) \doteq \left\{\sum_{i\in I(x)} a_i\gamma_i : a_i \geq 0, \left\|\sum_{i\in I(x)} a_i\gamma_i\right\| = 1\right\}.$$

The Skorokhod map (SM) assigns to every path $\psi \in D_+([0,\infty):\mathbb{R}^J)$ a path ϕ that starts at $\phi(0)=\psi(0)$, but is constrained to $\mathbb{R}^J_+$ as follows. If ϕ is in the interior of $\mathbb{R}^J_+$, then the evolution of ϕ mimics that of ψ, in that the increments of the two functions are the same until ϕ hits $\partial\mathbb{R}^J_+$. When ϕ is on the boundary, a constraining "force" is applied to keep ϕ in the domain, and this force can only be applied in one of the directions $d(\phi(t))$, and only for t such that $\phi(t)$ is on the boundary. The precise definition is as follows. For $\eta \in D([0,\infty):\mathbb{R}^J)$ and $t\in[0,\infty)$ we let $|\eta|(t)$ denote the total variation of η on $[0,t]$ with respect to the Euclidean norm on $\mathbb{R}^J$.

DEFINITION 1. Let $\psi \in D_+([0,\infty):\mathbb{R}^J)$ be given. Then (ϕ,η) solves the SP for ψ (with respect to $\mathbb{R}^J_+$ and $\gamma_i, i=1,\dots,J$) if $\phi(0)=\psi(0)$, and if for all $t\in[0,\infty)$

1. $\phi(t)=\psi(t)+\eta(t)$,
2. $\phi(t)\in\mathbb{R}^J_+$,
3. $|\eta|(t)<\infty$,
4. $|\eta|(t)=\int_{[0,t]} 1_{\{\phi(s)\in\partial\mathbb{R}^J_+\}} d|\eta|(s)$,
5. There exists a Borel measurable function γ: $[0,\infty)\to\mathbb{R}^J_+$ such that $d|\eta|$-almost everywhere $\gamma(t)\in d(\phi(t))$, and such that

$$\eta(t)=\int_{[0,t]}\gamma(s)d|\eta|(s).$$

Under a certain condition on $\{\gamma_i\}$ (known in the literature as the *completely-$\mathcal{S}$* condition, Reiman and Williams 1988), it is known that solutions to the SP exist in all of $D_+([0,\infty):\mathbb{R}^J_+)$. Under further conditions (namely, *existence of the set B*—see Harrison and Reiman 1981, Dupuis and Ishii 1991a, Dupuis and Ramanan 1999, and also Lemma 9 below), it is known that the Skorokhod map is Lipschitz continuous, and consequently the solution is unique. Denoting the map $\psi\mapsto\phi$ in Definition 1 by Γ, the Lipschitz property states that there is a constant K_1 such that

$$\sup_{t\in[0,\infty)}\|\Gamma(\psi_1)(t)-\Gamma(\psi_2)(t)\| \leq K_1 \sup_{t\in[0,\infty)}\|\psi_1(t)-\psi_2(t)\|, \tag{4}$$

$$\psi_1,\psi_2\in D_+([0,\infty):\mathbb{R}^J).$$

The SP that will be considered here is the one for which $\gamma_i=e_i-e_{r(i)}=-\tilde{v}_i$. For this problem, the following is well known.

THEOREM 1 (HARRISON AND REIMAN 1981, DUPUIS AND RAMANAN 1999). *The SP associated with the domain $\mathbb{R}^J_+$ and the constraint vectors γ_i, $i=1,\dots,J$ possesses a unique solution, and the Skorokhod map is Lipschitz continuous on the space $D_+([0,\infty):\mathbb{R}^J_+)$.*

Moreover, the Skorokhod map takes $C_+([0,\infty):\mathbb{R}^J)$ *into* $C_+([0,\infty):\mathbb{R}^J)$, *and therefore* $\Gamma(\phi)$ *is continuous if* ϕ *is.*

We next define a constrained ordinary differential equation. As is proved (in greater generality) in Dupuis and Ishii (1991a), one can define a projection π: $\mathbb{R}^J \to \mathbb{R}^J_+$ that is consistent with the constraint directions $\{\gamma_i, i = 1, \dots, J\}$, in that $\pi(x) = x$ if $x \in \mathbb{R}^J_+$, and if $x \notin \mathbb{R}^J_+$, then $\pi(x) - x = \alpha r$, where $\alpha \geq 0$ and $r \in d(\pi(x))$. With this projection given, we can now define for each point $x \in \partial\mathbb{R}^J_+$ and each $v \in \mathbb{R}^J$ the *projected velocity*

$$\pi(x, v) \doteq \lim_{\Delta \downarrow 0} \frac{\pi(x + \Delta v) - \pi(x)}{\Delta}.$$

For details on why this limit is always well defined and further properties of the projected velocity, we refer to Budhiraja and Dupuis (1999, §3, and Lemma 3.8). Let v: $[0,\infty) \to \mathbb{R}^J$ have the property that each component of v is integrable over each interval $[0, T]$, $T < \infty$. Then the ODE of interest takes the form

$$\dot{\phi}(t) = \pi(\phi(t), v(t)), \qquad \phi(0) = \phi_0 \in \mathbb{R}^J_+. \tag{5}$$

An absolutely continuous function ϕ: $[0,\infty) \to \mathbb{R}^J_+$ is a solution to (5) if the equation is satisfied a.e. in t. By using the regularity properties (4) of the associated Skorokhod map and because of the particularly simple nature of the right-hand side, one can show that ϕ solves (5) if and only if ϕ is the image of $\psi(t) \doteq \int_0^t v(s)\,ds + x$ under the Skorokhod map, and thus all the standard qualitative properties (existence and uniqueness of solutions, stability with respect to perturbations, etc.) hold (Dupuis and Ishii 1991a, Dupuis and Nagurney 1993).

As mentioned above, the SP formulation will be our means of defining the dynamics of a deterministic game. Before discussing this game, let us show how the same formulation is also useful for the stochastic control problem defined earlier in this section. First, because $\tilde{v}_j = e_{r(j)} - e_j = -\gamma_j$, it is easy to verify that for the particular SP considered here $\pi(x, v) = v 1_{x+v \in \mathbb{Z}^J_+}$ for all $x \in \mathbb{Z}^J_+$ and $v \in \{\tilde{v}_j : j = 1, \dots, J\}$. Therefore the generator $\widetilde{\mathscr{L}}^u$ of (1) can also be written as

$$\widetilde{\mathscr{L}}^u f(x) = \sum_{j=1}^J \lambda_j [f(x + e_j) - f(x)] + \sum_{j=1}^J u_j \mu_j [f(x + \pi(x, \tilde{v}_j)) - f(x)].$$

A measurable function $u(t)$, u: $[0,\infty) \to U$ is said to be an *open loop control*. Note that this control has no state feedback. When u is an open loop control, it is possible to write the corresponding controlled process X as $\Gamma(Y)$. The process Y, which will be called *the unconstrained controlled process*, is a controlled Markov process with a simpler structure. To be precise, let

$$\widetilde{\mathscr{L}}^u_0 f(x) = \sum_{j=1}^J \lambda_j [f(x + e_j) - f(x)] + \sum_{j=1}^J u_j \mu_j [f(x + \tilde{v}_j) - f(x)],$$

and let $\widetilde{\mathscr{L}}^{n,u}_0$ be defined analogously to (2). A controlled Markov process X^n on G^n (respectively, Y^n on $n^{-1}\mathbb{Z}^J$) is defined as before, but now using the generator $(\mathscr{L}^{n,u} f)(t) = (\widetilde{\mathscr{L}}^{n,u(t)} f)(x)$ (respectively, $(\mathscr{L}^{n,u}_0 f)(t) = (\widetilde{\mathscr{L}}^{n,u(t)}_0 f)(x)$). The simplification that the SP introduces is that if u is an open loop control, and if Y^n is a controlled Markov process corresponding to $\mathscr{L}^{n,u}_0$ on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P^{n,u}_x)$, then $X^n = \Gamma(Y^n)$ is a controlled Markov process corresponding to $\mathscr{L}^{n,u}$ on the same filtered probability space. The role played by the SP in relating constrained and unconstrained processes is exhibited here in a simple fashion, for introductory purposes. We will, in fact, use it in a slightly more complicated setting later on in Lemma 7 and Lemma 8.

A differential game. In this paper, we prove that the value function $V^n(x)$ for a stochastic control problem associated with our queueing network model is approximately equal (for large n) to the value function of a related differential game. In addition, the dynamics of this game are defined in terms of an associated SP. Before introducing the game formally, we explain why this is to be expected. In a problem with no control, the exponential decay rate of quantities such as $Ee^{-cn\sigma_n}$ is given in terms of the sample path large deviation rate function associated with the process, which in turn can be expressed in terms of the rate function for the Poisson primitives that drive the model. This is supported by the well-known Laplace's principle (Dupuis and Ellis 1997). Heuristically, one thinks of the rate function as a cost paid for changing the measure so as to make the rare event of exiting on short time interval a probable event. Laplace's principle asserts that the decay rate can be expressed as the solution to a deterministic optimization problem involving the cost $-c\sigma$ combined with the cost of changing the measure: cf. Shwartz and Weiss (1995, Equations (5.20)–(5.23)). When the stochastic model involves optimal control, there is one more variable to optimize over in the limit, and this results in a game. The game's deterministic dynamics are the natural law of large numbers limit under the changed measure. Boundary constraints and constraining meachanisms which are present in the prelimit model are represented in the limit model by the SP. The cost for the game involves the large deviation rate function for the Poisson primitives, the time until the dynamics exit the domain, and the parameter c.

We thus consider a zero-sum game involving two players. One (which we call the maximizing player) selects the service allocation and attempts to maximize. The other (called the minimizing player) chooses the perturbed arrival and service rates and attempts to minimize. Throughout, the perturbed rates will be denoted by an overbar, as in $\bar{\lambda}_i, \bar{\mu}_i$.

Recall that for $u \in U$, u_i stands for the fraction of service effort given to class i. The control space for the maximizing player is

$$\overline{U} \doteq \{u\colon [0,\infty) \to U;\ u \text{ is measurable}\}.$$

Let $l\colon \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ be defined as

$$l(x) \doteq \begin{cases} x\log x - x + 1 & x \ge 0, \\ +\infty & x < 0, \end{cases}$$

where $0\log 0 \doteq 0$. Denoting $M = [0,\infty)^{2J}$, the control space for the minimizing player will be

$$\overline{M} = \big\{m = (\bar{\lambda}_1, \dots, \bar{\lambda}_J, \bar{\mu}_1, \dots, \bar{\mu}_J)\colon [0,\infty) \to M;\ m \text{ is measurable, } l \circ m \text{ is locally integrable}\big\}. \tag{6}$$

For $u \in U$ and $m \in M$ define

$$v(u,m) \doteq \sum_{j=1}^{J} \bar{\lambda}_j v_j + \sum_{i=1}^{J} u_i \bar{\mu}_i \tilde{v}_i,$$

where $v_j = e_j$, and as before $\tilde{v}_i = e_{r(i)} - e_i$. Then the dynamics are given by

$$\begin{cases} \dot{\phi}(t) = \pi(\phi(t), v(u(t), m(t))), \\ \phi(0) = x. \end{cases}$$

To define the cost for the game, let $\rho\colon U \times M \to \mathbb{R}_+ \cup \{+\infty\}$ be

$$\rho(u,m) \doteq \sum_{i=1}^{J} \lambda_i l\left(\frac{\bar{\lambda}_i}{\lambda_i}\right) + \sum_{i=1}^{J} u_i \mu_i l\left(\frac{\bar{\mu}_i}{\mu_i}\right).$$

By convention, if $\lambda_i = 0$ and $\bar{\lambda}_i > 0$ for some i, we let $\rho = \infty$ (recall that by assumption, $\mu_i > 0$). Let the exit time be defined by

$$\sigma \doteq \inf\{t : \phi(t) \notin G\}.$$

With $c > 0$ as in (3), the cost is given by

$$C(x, u, m) = \int_0^\sigma [c + \rho(u(t), m(t))]\, dt.$$

As in Elliott and Kalton (1972), we need the notion of strategies. We endow both $\overline{U}$ and $\overline{M}$ with the metric $\tilde{\rho}(\omega_1, \omega_2) = \sum_n 2^{-n}(\int_0^n |\omega_1(t) - \omega_2(t)|\, dt \wedge 1)$, and with the corresponding Borel σ-fields. A mapping α: $\overline{M} \to \overline{U}$ is called a *strategy for the maximizing player* if it is measurable and if for every $m, \tilde{m} \in \overline{M}$ and $t > 0$ such that

$$m(s) = \tilde{m}(s) \qquad \text{for a.e. } s \in [0, t],$$

one has

$$\alpha[m](s) = \alpha[\tilde{m}](s) \qquad \text{for a.e. } s \in [0, t].$$

In an analogous way, one defines a mapping β: $\overline{U} \to \overline{M}$ to be a *strategy for the minimizing player*. The set of all strategies for the maximizing (respectively, minimizing) player will be denoted by A (respectively, B). The lower value for the game is defined as

$$V^-(x) = \inf_{\beta \in B} \sup_{u \in \overline{U}} C(x, u, \beta[u]),$$

and the upper value as

$$V^+(x) = \sup_{\alpha \in A} \inf_{m \in \overline{M}} C(x, \alpha[m], m).$$

To avoid confusion, we remark that despite the terms "upper" and "lower" value, it is not in general obvious that $V^- \le V^+$.

Main result. We make the following assumption on the domain G. Let

$$\mathcal{J}_+ \doteq \{i \in \{1, \dots, J\} : \lambda_i > 0\}.$$

CONDITION 1. We assume that the domain G satisfies one of the following.

1. G is a rectangle given by

$$G = \{(x_1, \dots, x_J) : 0 \le x_i < z_i, i \in \mathcal{J}_+;\ 0 \le x_j \le z_j, j \notin \mathcal{J}_+\},$$

for some $z_i > 0$, $i = 1, \dots, J$.

2. G is simply connected and bounded, and given by

$$G = \bigcap_{i \in \mathcal{J}_+} G_i,$$

where for $i \in \mathcal{J}_+$, we are given positive Lipschitz functions ϕ_i: $\mathbb{R}^{J-1} \to \mathbb{R}$, and

$$G_i = \{(x_1, \dots, x_J) \in \mathbb{R}_+^J : 0 \le x_i < \phi_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_J)\}.$$

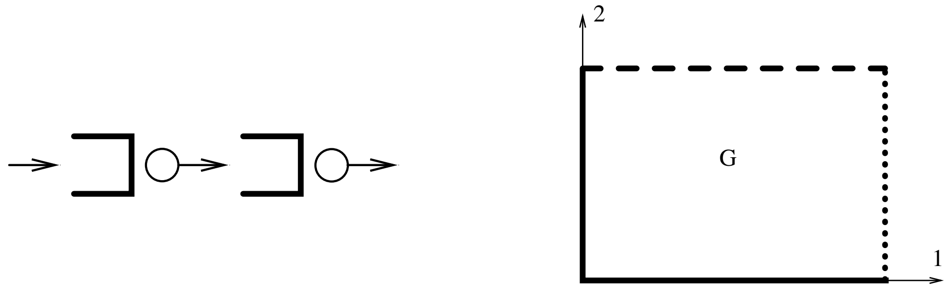

FIGURE 1. A simple queueing network, a rectangular domain and three types of boundary. Full line: $\partial_+ G$, dashed line: $\partial_c G$, and dotted line: $\partial_o G$.

This condition covers many typical constraints one would consider on buffer size, including separate constraints on individual queues (Condition 1.1) and one constraint on the sum of the queues (Condition 1.2).

The shape of the domain is simpler in Condition 1.1 in that it is restricted to a hyper-rectangle. On the other hand, it is also possible under this condition for the maximizing player to unilaterally prevent an exit through a certain portion of $\partial G \backslash \partial \mathbb{R}_+^J$. Although it is in principle possible that the dynamics could exit through this portion of the boundary, it will always be optimal for the maximizing player to not allow it. Consider the simple network illustrated in Figure 1. The maximizing player can prevent exit through the dashed portion of the boundary simply by stopping service at the first queue. As a consequence, there are in general three different types of boundary: the constraining boundary due to nonnegativity constraints on queue length, the part of the boundary where exit can be blocked, and the remainder. These three types of boundary behavior will result, in the PDE analysis, in three types of boundary conditions. We now define the three portions of the boundary. Under Condition 1.1, let

$$\partial_c G = \{(x_1, \dots, x_J) \in G : x_j = z_j, \text{ some } j \notin \mathcal{J}_+\}.$$

For notational convenience, we let $\partial_c G = \varnothing$ under Condition 1.2. In both cases, we then set

$$\partial_o G = \partial G \backslash G, \qquad \partial_+ G = (G \cap \partial \mathbb{R}_+^J) \backslash \partial_c G.$$

Note that in both cases, $\partial_c G$, $\partial_o G$, and $\partial_+ G$ partition ∂G. Also, $\partial_c G \subset G$ while $\partial_o G \subset G^c$. As usual, we will denote $G^o = G \backslash \partial G$ and $\overline{G} = G \cup \partial G$. $\partial_c G$ is the part of the boundary where the maximizing player can prevent the dynamics from exiting, and $\partial_o G$ is the part where it cannot. Finally, it will be convenient to denote

$$\partial_{co} G = \partial_o G \cup \partial_c G.$$

Our main result is the following.

THEOREM 2. *Let Condition* 1 *hold. Then* $V^+ = V^- \doteq V$ *on* G. *Moreover, if* $x_n \in G^n$, $n \in \mathbb{N}$ *are such that* $x_n \to x \in G$, *then* $\lim_{n\to\infty} V^n(x_n) = V(x)$.

REMARK. A stronger form of the convergence statement in fact holds. Namely,

$$\limsup_{\epsilon \downarrow 0} \limsup_{n\to\infty} \sup\{|V^n(x) - V(y)| : x \in G^n, y \in G, |x-y| \le \epsilon\} = 0.$$

This is an immediate consequence of Theorem 2 and the fact that for each n, V^n is Lipschitz on G^n, with a constant that does not depend on n (Lemma 2). □

The proof is established in two major steps. Step 1 will be an immediate consequence of the main results of Section 3, and Step 2 will follow from Section 4.

PROOF.

Step 1. We define a version of the game, technically easier to work with, in which all perturbed rates $(\bar{\lambda}_i, \bar{\mu}_i)$ are bounded by $b < \infty$. The corresponding upper and lower values, defined analogously, are denoted by $V^{b,+}$ and $V^{b,-}$. Then we show that for all b large enough (cf. Theorem 3),

$$V^{b,+}(x) \le \liminf_{n\to\infty} V^n(x_n) \le \limsup_{n\to\infty} V^n(x_n) \le V^{b,-}(x).$$

Step 2. We show that for b large, $V^{b,+} = V^{b,-}$ on G. To this end, we formulate a PDE for which we show that uniqueness of (Lipschitz) viscosity solutions holds (Theorem 5), and also show that both $V^{b,+}$ and $V^{b,-}$ are viscosity solutions (Theorem 6). Since $V^n(x)$ does not depend on b, neither do $V^{b,\pm}(x)$. Theorem 2 follows. $\square$

3. The control problem and the game. We begin by stating some basic properties of the stochastic control problem and of the deterministic game. The proofs of these properties are deferred to §6.

Consider the following generators, defined for any $u \in U$ and $m \in M$, for constrained and unconstrained controlled Markov processes:

$$\mathscr{L}^{n,u,m} f(x) = \sum_{j=1}^{J} n\bar{\lambda}_j \left[f\left(x + \frac{1}{n} v_j\right) - f(x)\right] + \sum_{i=1}^{J} n\bar{\mu}_i u_i \left[f\left(x + \frac{1}{n}\pi(x, \tilde{v}_i)\right) - f(x)\right],$$

$$\mathscr{L}_0^{n,u,m} f(x) = \sum_{j=1}^{J} n\bar{\lambda}_j \left[f\left(x + \frac{1}{n} v_j\right) - f(x)\right] + \sum_{i=1}^{J} n\bar{\mu}_i u_i \left[f\left(x + \frac{1}{n}\tilde{v}_i\right) - f(x)\right].$$

The definition of the corresponding controlled processes will be made precise in Lemmas 7 and 8.

Owing to the logarithmic transform in (3), one expects V^n to satisfy an Isaacs equation (Fleming and Souganidis 1986). In fact, V^n satisfies

$$\begin{cases} 0 = \sup_{u\in U} \inf_{m\in M} [\mathscr{L}^{n,u,m} V^n(x) + c + \rho(u,m)], & x \in G^n \\ V^n(x) = 0, & x \notin G^n. \end{cases} \tag{7}$$

We comment that this is also the dynamic programming equation (DPE) for an associated stochastic game that is related to the deterministic game via a law of large numbers scaling and limit and will not be further considered in this paper.

LEMMA 1. *The value function V^n of* (3) *uniquely solves the DPE* (7).

The following lemma gives a key estimate on the value function.

LEMMA 2. *Under Condition* 1, *$V^n(x)$ satisfies the Lipschitz property on $(n^{-1}\mathbb{Z}_+^J) \cap \overline{G}$ with a constant that does not depend on $n \in \mathbb{N}$. Consequently,* $\sup_{n,x\in G^n} V^n(x) < \infty$.

We comment that the above result is, in general, not valid for V^n on $n^{-1}\mathbb{Z}_+^J$ because V^n changes abruptly near the portion $\partial_c G$ of the boundary.

For each fixed $u \in U$, the mapping $m \to \rho(u,m)$, when restricted to $\bar{\mu}_i$ such that $u_i > 0$, is strictly convex with compact level sets. We conclude that the infimum over m in the DPE is achieved, and denote such a point by $m^n(x,u)$. Although Part 1 of the following lemma is not used elsewhere, it indicates why the Isaacs condition should hold in (7).

LEMMA 3. *Let Condition* 1 *hold. Then*:

1. *$m^n(x,u)$ can be chosen independently of u, and*
2. *there is $b_0 < \infty$ such that for all x, n and u, $m^n(x,u) \le b_0$.*

We introduce two parametric variations of the game defined in §2. The first will be associated with domain perturbation (parameterized by the symbol a) and the second with a bound on the perturbed rates (parameterized by the symbol b).

For some fixed $a_0 > 0$, consider perturbations G_a, $a \in (-a_0, a_0)$ of the domain G defined as follows. If G satisfies Condition 1.1, then G_a is defined as G, but with z_i replaced by $z_i + a$, $i = 1, \dots, J$. If G is as in Condition 1.2, then G_a is defined as G, but where ϕ_i is replaced by $\phi_i + a$, $i \in \mathcal{J}_+$.

For any $b \in (0, \infty)$, let $M^b = [0, b]^{2J}$. Analogously to the definition (6) of $\overline{M}$, let

$$\overline{M}^b = \{m = (\bar{\lambda}_1, \dots, \bar{\lambda}_J, \bar{\mu}_1, \dots, \bar{\mu}_J) : [0, \infty) \to M^b;\ m \text{ is measurable}\}. \tag{8}$$

Strategies and values for the game are then defined analogously to the way strategies and values are defined for the original game, using $\overline{M}^b$ in place of $\overline{M}$. It will be convenient to set $M^\infty \doteq M$ and $\overline{M}^\infty \doteq \overline{M}$, and to refer to the original game of §2 as the case $b = \infty$.

The cost, sets of strategies, and lower and upper values of the games resulting by the introduction of the parameters a and b will be denoted as $C_a(x, u, m)$, A^b, B^b, $V_a^{b,-}$, and $V_a^{b,+}$. When $a = 0$ (respectively, $b = \infty$), the dependence on a (b) will be eliminated from the notation, as in V_a^-, $V^{b,-}$.

Let b_0 be as in Lemma 3. Denote

$$b^* \doteq \max\{b_0, \lambda_i, \mu_i, i = 1, \dots, J\} + 1. \tag{9}$$

LEMMA 4. *Assume Condition* 1. *Then*

1. $\operatorname{dist}(\partial_{co} G_a, \partial_{co} G) \doteq \inf\{|x - y| : x \in \partial_{co} G_a, y \in \partial_{co} G\} > 0$ *if* $0 < |a| < a_0$;
2. *the values* $V^{b,\pm}$ *are bounded on* G, *uniformly for* $b \in [b^*, \infty]$, *and there is a constant* c_0 *such that for any* $x \in G$, $|a| < \epsilon$ (*where* ϵ *depends on* x), *and* $b \in [b^*, \infty]$, *one has* $|V_a^{b,-}(x) - V^{b,-}(x)| \le c_0|a|$ *and* $|V_a^{b,+}(x) - V^{b,+}(x)| \le c_0|a|$.

The following lemma shows that any nearly optimal strategy for the minimizing player will satisfy a uniform upper bound on the integrated running cost. Moreover, there is a finite time T_0 such that for each such minimizing strategy, any open loop control used by the maximizing player leads to exit by T_0. Similarly, given any strategy for the maximizing player, the minimizing player can restrict to open loop controls that force exit by T_0.

LEMMA 5. *Fix* $b \in [b^*, \infty]$. *Given* $\beta \in B^b$, *write* $(\bar{\lambda}_i(\cdot), \bar{\mu}_i(\cdot)) = \beta[u](\cdot)$. *For* $z, T > 0$ *let* $B^{z,T}$ *denote the set of* $\beta \in B^b$ *which satisfy*

$$\int_0^T \sum_i [\lambda_i l(\bar{\lambda}_i(t)/\lambda_i) + u_i(t)\mu_i l(\bar{\mu}_i(t)/\mu_i)]\, dt \le z,$$

for all $u \in \overline{U}$. *For* $\alpha \in A^b$, *let* $\overline{M}(\alpha, T)$ *denote the set of* $m \in \overline{M}$ *for which* $\sigma(x, \alpha[m], m) \le T$. *Then there are constants* $z_0, T_0 > 0$ *such that*

$$V^-(x) = \inf_{\beta \in B^{z_0, T_0}} \sup_{u \in \overline{U}} \int_0^{\sigma \wedge T_0} [c + \rho(u(t), \beta[u](t))]\, dt,$$

and

$$V^+(x) = \sup_{\alpha \in A} \inf_{m \in \overline{M}(\alpha, T_0)} \int_0^{\sigma \wedge T_0} [c + \rho(\alpha[m](t), m(t))]\, dt.$$

In the rest of the section the strategies β will be assumed (without loss) to be in B^{z_0,T_0}, where z_0, T_0 are as in Lemma 5, and are fixed throughout. Also, $m \in \overline{M}$ will be assumed to be in $\overline{M}(\alpha, T_0)$ whenever it is clear which α is considered. With an abuse of notation, we denote B^{z_0,T_0} by B.

LEMMA 6. *Under Condition* 1, $V^{b,-}$ *and* $V^{b,+}$ *are Lipschitz on* G, *uniformly for* $b \in [b^*, \infty]$.

We are now ready to prove the following result.

THEOREM 3. *Let Condition* 1 *hold, and let* $b \in [b^*, \infty)$. *Then for any* $x \in G$ *and any* $\{x_n\}$ *converging to* x (*with* $x_n \in G^n$),

$$V^{b,+}(x) \le \liminf_{n\to\infty} V^n(x_n) \le \limsup_{n\to\infty} V^n(x_n) \le V^{b,-}(x).$$

PROOF OF THEOREM 3. The result is established by considering a sequence of stochastic processes, defined using the constrained ODEs, but for which the controls u and m are governed by, on one hand, a nearly optimal strategy for the game, and on the other hand, a nearly optimal control for the stochastic control problem. The technique uses standard martingale estimates and is based on the construction (deferred to §6) of an auxiliary controlled Markov process that is controlled by the selected strategy and stochastic control.

Upper bound. Fix $b \in [b^*, \infty)$. The dependence on b will be suppressed in the notation for V^-, V_a^-, etc. We first show that

$$\limsup_{n\to\infty} V^n(x_n) \le V^-(x). \tag{10}$$

According to Lemma 4.2, it is enough to show that for all $a > 0$,

$$\limsup_{n\to\infty} V^n(x_n) \le V_a^-(x).$$

Let $\beta \in B^b$ and $a > 0$ be given, and set $C_a(x, \beta) = \sup_{u\in\overline{U}} C_a(x, u, \beta[u])$. It is enough to show that

$$\limsup_{n\to\infty} V^n(x_n) \le C_a(x, \beta), \qquad a > 0. \tag{11}$$

We therefore fix β throughout and turn to prove (11). We can assume without loss that

$$C_a(x, \beta) < \infty. \tag{12}$$

Note that in the DPE (7) the supremum is with respect to u in a compact set U and that the function being maximized is continuous in u (for each y). Let $u^n(y)$ denote a point where it is achieved. Then, for any $m \in M$ and $y \in G^n$,

$$0 \le \mathscr{L}^{n,u^n(y),m} V^n(y) + c + \rho(u^n(y), m). \tag{13}$$

LEMMA 7. *Let* n *be fixed, and let* $b \in [b^*, \infty)$, β *and* x_n *be as above. Then there is a filtered probability space* $(\overline{\Omega}, \overline{F}, (\overline{F}_t), \overline{P})$, *and* $\overline{F}_t$-*adapted RCLL processes* $\overline{X}^n$, $\overline{Y}^n$, *and* m^n *such that with* $\overline{P}$-*probability one* $m^n(t) = \beta[\bar{u}^n](t)$ *a.e.* t, $\bar{u}^n(t) = u^n(\overline{X}^n(t))$, $\overline{X}^n = \Gamma(\overline{Y}^n)$, $\overline{X}^n(0) = \overline{Y}^n(0) = x_n$, *and for any* f

$$f(\overline{X}^n(t)) - \int_0^t \mathscr{L}^{n,\bar{u}^n,m^n(s)} f(\overline{X}^n(s))\, ds,$$

and

$$f(\overline{Y}^n(t)) - \int_0^t \mathscr{L}_0^{n,\bar{u}^n,m^n(s)} f(\overline{Y}^n(s))\, ds$$

are (F_t)-*martingales. Moreover, with* T_0 *as in Lemma* 5, *let* N_n *denote the total number of jumps of* $\overline{Y}^n$ *on* $[0, T_0]$. *Then*

$$EN_n \le 2JT_0bn. \tag{14}$$

PROOF. See §6.

Returning to the proof of (11), let $\bar{u}^n(t) \doteq u^n(\overline{X}^n(t))$ and let $\bar{\sigma}^n$ be the first exit time of $\overline{X}^n$ from G. Combining (13) and Lemma 7, for any bounded stopping time $S \le \bar{\sigma}^n$,

$$V^n(x_n) \le \overline{E}_{x_n}\left[V^n(\overline{X}^n(S)) + \int_0^S [c + \rho(\bar{u}^n(s), \beta[\bar{u}^n](s))]\,ds\right]. \tag{15}$$

Denoting $\beta[\bar{u}^n](t) = \{(\bar{\lambda}_i^n(t), \bar{\mu}_i^n(t))\}$, define ϕ^n as $\phi^n = \Gamma(\psi^n)$, where

$$\psi^n(t) = x + \int_0^t v(\bar{u}^n, \beta[\bar{u}^n])\,ds,$$

and let

$$\hat{\sigma}_a^n \doteq \inf\{t : \phi^n(t) \notin G_a\}.$$

Then the definition of $C_a(x, \beta)$ implies

$$\int_0^{\hat{\sigma}_a^n} [c + \rho(\bar{u}^n(s), \beta[\bar{u}^n](s))]\,ds \le C_a(x, \beta).$$

Apply (15) with $S = \hat{\sigma}_a^n \wedge \bar{\sigma}^n \wedge T$. If T is sufficiently large, then (12) and the fact that $c > 0$ imply $\hat{\sigma}_a^n \le T$. Thus, using $\overline{E}_{x_n}(V^n(\overline{X}^n(\bar{\sigma}^n))) = 0$,

$$V^n(x_n) \le \overline{E}_{x_n}\left[V^n(\overline{X}^n(\hat{\sigma}_a^n))1_{\{\hat{\sigma}_a^n < \bar{\sigma}^n\}} + \int_0^{\hat{\sigma}_a^n} [c + \rho(\bar{u}^n(s), \beta[\bar{u}^n](s))]\,ds\right].$$

Again using the uniform boundedness of $V^n(x)$ (Lemma 2), there is a constant $b_2 < \infty$ such that for all n

$$V^n(x_n) \le b_2 \overline{P}_{x_n}(\hat{\sigma}_a^n \le \bar{\sigma}^n) + C_a(x, \beta). \tag{16}$$

In what follows, we show that $\overline{P}_{x_n}(\hat{\sigma}_a^n \le \bar{\sigma}^n)$ tends to zero. To this end, note that $\mathscr{L}_0^{n,u,m}\,\mathrm{id}(y) = \sum_i \bar{\lambda}_i v_i + \sum_i u_i \bar{\mu}_i \tilde{v}_i$, where id is the identity map. Therefore, again using Lemma 7,

$$\overline{Y}^n(t) - x_n = \int_0^t \left[\sum_{i=1}^J \bar{\lambda}_i^n(s) v_i + \sum_{i=1}^J \bar{u}_i^n(s) \bar{\mu}_i^n(s) \tilde{v}_i\right] ds + \eta^n(t),$$

where η^n is a zero mean martingale. To prove that

$$\sup_{t \in [0,T]} |\eta^n(t)| \to 0 \quad \text{in distribution,} \tag{17}$$

it is enough, by Doob's maximal inequality, to show that

$$\overline{E}|\eta^n(T)|^2 \to 0.$$

Let $[x](t) = \sum_{s \in [0,t]} |\Delta x_s|^2$. By the Burkholder-Davies-Gundy inequality (see Dellacherie and Meyer 1980, VII.92),

$$\overline{E}|\eta^n(T)|^2 \le c_1 \overline{E}[\eta^n](T),$$

where c_1 is a constant. Because each jump is bounded by $c_2 n^{-1}$ (c_2 a constant) and the total number of jumps $N_n(T)$ satisfies (14),

$$\overline{E}|\eta^n(T)|^2 \le c_3 n^{-2} \overline{E} N_n(T) \le c_4 n^{-1},$$

which proves (17). This implies that $\sup_{[0,T]}|\overline{Y}^n(t)-\psi^n(t)|\to 0$ in distribution, and therefore the continuity of Γ implies $\sup_{[0,T]}|\overline{X}^n(t)-\phi^n(t)|\to 0$ in distribution. By Lemma 4.1,

$$\begin{aligned}\overline{P}_{x_n}(\hat{\sigma}_a^n\le\bar{\sigma}^n) &\le \overline{P}_{x_n}(\overline{X}^n(\hat{\sigma}_a^n)\in G, \phi^n(\hat{\sigma}_a^n)\in\partial_{co}G_a)\\ &\le \overline{P}_{x_n}\Bigl(\sup_{t\in[0,T]}|\overline{X}^n(t)-\phi^n(t)|\ge b_1\Bigr),\end{aligned}$$

where $b_1>0$ depends only on a. Hence, by (17), $\overline{P}_{x_n}[\hat{\sigma}_a^n\le\bar{\sigma}^n]\to 0$ as $n\to\infty$. Therefore, (16) implies

$$\limsup_{n\to\infty}V^n(x_n)\le C_a(x,\beta).$$

This gives (11) and completes the proof of (10).

Lower bound
Next we prove

$$\liminf_{n\to\infty}V^n(x_n)\ge V^+(x). \tag{18}$$

By Lemma 4.2, it is enough to show that for all $a<0$

$$\liminf_{n\to\infty}V^n(x_n)\ge V_a^+(x).$$

Let $\alpha\in A$ be given, and set $C_a(x,\alpha)=\inf_{m\in\overline{M}^b}C_a(x,\alpha[m],m)$. Then it suffices to show

$$\liminf_{n\to\infty}V^n(x_n)\ge C_a(x,\alpha),\qquad a<0. \tag{19}$$

Fixing α, we now prove (19).

Interchanging the order of infimum and supremum in Equation (7) (see Rockafellar 1970, Corollary 37.3.2), and noting that the infimum over m is of a continuous function with compact level sets, we denote by $m^n(y)$ a point where the infimum is achieved. By Lemma 3, the components $\bar{\lambda}_i^n(y)$ and $\bar{\mu}_i^n(y)$ of $m^n(y)$ are all bounded by b_0. For $u\in U$ and $y\in G^n$,

$$0\ge\mathscr{L}^{n,u,m^n(y)}V^n(y)+c+\rho(u,m^n(y)). \tag{20}$$

LEMMA 8. *Let n be fixed, and let α and x_n be as above. Then there is a filtered probability space $(\overline{\Omega},\overline{F},(\overline{F}_t),\overline{P})$, and $\overline{F}_t$-adapted RCLL processes $\overline{X}^n$, $\overline{Y}^n$, and u^n such that with $\overline{P}$-probability one $u^n(t)=\alpha[\bar{m}^n](t)$ a.e. t, $\bar{m}^n(t)=m^n(\overline{X}^n(t))$, $\overline{X}^n=\Gamma(\overline{Y}^n)$, $\overline{X}^n(0)=\overline{Y}^n(0)=x_n$, and for any f,*

$$f(\overline{X}^n(t))-\int_0^t\mathscr{L}^{n,u^n(s),m^n}f(\overline{X}^n(s))\,ds,$$

and

$$f(\overline{Y}^n(t))-\int_0^t\mathscr{L}_0^{n,u^n(s),m^n}f(\overline{Y}^n(s))\,ds$$

are $(\overline{F}_t)$-martingales.

PROOF. See §6.

Let $\bar{m}^n(t)=m^n(\overline{X}^n(t))$ and let $\bar{\sigma}^n$ be the first exit time of $\overline{X}^n$ from G. By (20) and Lemma 8, for any bounded stopping time $S\le\bar{\sigma}^n$,

$$V^n(x_n)\ge\overline{E}_{x_n}\biggl[V^n(\overline{X}^n(S))+\int_0^S[c+\rho(\alpha[\bar{m}^n](s),\bar{m}^n(s))]\,ds\biggr]. \tag{21}$$

Denoting $\bar{m}^n(t) = ((\bar{\lambda}_i^n(t), \bar{\mu}_i^n(t))$, define ϕ^n as $\phi^n = \Gamma(\psi^n)$, where

$$\psi^n = x + \int_0^{\cdot} v(\bar{u}^n, \bar{m}^n)\, ds,$$

and let

$$\hat{\sigma}_a^n \doteq \inf\{t : \phi^n(t) \notin G_a\}.$$

Then the definition of $C_a(x, \alpha)$ implies

$$\int_0^{\hat{\sigma}_a^n} [c + \rho(\alpha[\bar{m}^n](s), \bar{m}^n(s))]\, ds \geq C_a(x, \alpha).$$

Apply (21) with $S = \hat{\sigma}_a^n \wedge \bar{\sigma}^n \wedge T$, with large enough T, using the fact that $V^n \geq 0$ to get

$$\begin{aligned} V^n(x_n) &\geq \overline{E}_{x_n}\left[\int_0^{\hat{\sigma}_a^n \wedge \bar{\sigma}^n \wedge T} [c + \rho(\alpha[\bar{m}^n](s), \bar{m}^n(s))]\, ds\right] \\ &\geq \overline{E}_{x_n}\left[1_{\hat{\sigma}_a^n \leq \bar{\sigma}^n} \int_0^{\hat{\sigma}_a^n \wedge T} [c + \rho(\alpha[\bar{m}^n](s), \bar{m}^n(s))]\, ds\right] \\ &\geq \overline{P}_{x_n}(\hat{\sigma}_a^n \leq \bar{\sigma}^n) C_a(x, \alpha). \end{aligned}$$

The proof that $\overline{P}_{x_n}(\hat{\sigma}_a^n \leq \bar{\sigma}^n)$ tends to one is analogous to the proof of the that $\overline{P}_{x_n}(\hat{\sigma}_a^n \leq \bar{\sigma}^n) \to 0$ in the upper bound. It is therefore omitted. Hence

$$\liminf_{n \to \infty} V^n(x_n) \geq C_a(x, \alpha).$$

This gives (19), and the proof of (18) is established. □

In fact, the value of the game is independent of b for large b, so that the game has a value with the unbounded action space M. As the result depends on Theorems 3, 5, and 6, we postpone the proof to §6.

THEOREM 4. *For all* $b \in [b^*, \infty]$, $V^{b,+} = V^+ = V^{b,-} = V^-$.

PROOF. See §6.

4. The PDE. In this section we show that the upper and lower values of the game are the unique Lipschitz viscosity solutions of the PDE (23). Throughout, the parameter $b \in [b^*, \infty)$ is fixed. Let

$$H(q) = \inf_m \sup_u [\langle q, v(u, m)\rangle + \rho(u, m) + c]. \tag{22}$$

It will be useful to note that the infimum is over the compact set M^b, and the map $(q, u, m) \mapsto [\langle q, v(u, m)\rangle + \rho(u, m) + c]$ is continuous. The PDE of interest is

$$\begin{cases} H(DV(x)) = 0, & x \in G^o, \\ \langle DV(x), \gamma_i\rangle = 0, & i \in I(x),\ x \in \partial_+ G, \\ V(x) = 0, & x \in \partial_o G. \end{cases} \tag{23}$$

Here, γ_i are the directions of constraint that were introduced in §2.

DEFINITION 2. Let a Lipschitz continuous function u: $X \to \mathbb{R}$ be given (where $X \subset G$). We say that u is a subsolution (respectively, supersolution) to (23) on X if the following conditions hold. Let θ: $X \to \mathbb{R}$ be continuously differentiable on $\overline{X}$. Let $y \in X$ be a local maximum (minimum) of the map $x \mapsto u(x) - \theta(x)$. Then

$$H(D\theta(y)) \vee \max_{i\in I(y)} \langle D\theta(y), \gamma_i \rangle \geq 0, \tag{24}$$

$$\Big(H(D\theta(y)) \wedge \min_{i\in I(y)} \langle D\theta(y), \gamma_i \rangle \leq 0\Big), \tag{25}$$

and

$$V(x) \leq 0, \qquad x \in \overline{X} \cap \partial_o G, \tag{26}$$

$$(V(x) \geq 0, \qquad x \in \overline{X} \cap \partial_o G). \tag{27}$$

We say that V is a viscosity solution to (23) if it is both a subsolution on G and a supersolution on $G \backslash \partial_c G$.

REMARK. In case that $\partial_c G \neq \emptyset$, a viscosity solution is often called a constrained viscosity solution (cf. Soner 1986, Capuzzo-Dolcetta and Lions 1990). The requirement that V is a subsolution up to the boundary $\partial_c G$—the part of the boundary where exit can be unilaterally blocked—serves as a boundary condition on this part of the boundary. Note that in the current paper, the term "constrained" refers to the part $\partial_+ G$ of the boundary, where it is the mechanism associated with the Skorokhod problem that constrains the dynamics to G. □

First, we address uniqueness of solutions to (23).

THEOREM 5. *Let u be a subsolution and v a supersolution to* (23). *Then $u \leq v$ on G.*

The proof combines ideas from two sources, namely Atar and Dupuis (forthcoming) (which is based on Dupuis and Ishii 1991b, and discusses how to deal with the constrained dynamics on $\partial_+ G$), and Soner (1986) (to accommodate the fact that under Condition 1.1 part of the boundary ($\partial_c G$) can be thought of as imposing a state-space constraint on the maximizing player).

The following lemma will be used in proving Theorem 5. In the interest of consistency with previous publications, we use B in Lemma 9 below to denote a certain subset of $\mathbb{R}^J$ (although everywhere except in this section, B denotes a set of strategies). Part 1 states that the "Set B" condition holds, namely, a condition under which it was proved in Dupuis and Ishii (1991a) that the SM enjoys the regularity property (4). The proof that this condition holds in the current setting can be found in Dupuis and Ramanan (1999). The existence of a smooth version of the set B is proved in Atar and Dupuis (forthcoming) (before Lemma 2.1). For Parts 2 and 3, see Lemmas 2.1 and 2.2 of Atar and Dupuis (note that the condition that γ_i are independent holds).

LEMMA 9. 1. *There exists a compact, convex, and symmetric set $B \subset \mathbb{R}^J$ with $0 \in B^o$, such that if $z \in \partial B$ and if n is an outward normal to B at z, then for all $i \in \{1, \ldots, J\}$*

$$\langle z, e_i \rangle \geq -1 \text{ implies } \langle \gamma_i, n \rangle \geq 0 \quad \textit{and} \quad \langle z, e_i \rangle \leq 1 \text{ implies } \langle \gamma_i, n \rangle \leq 0.$$

In addition, the unit outward normal $n(x)$ to B at $x \in \partial B$ is unique and continuous (*as a function on ∂B*).

2. *Let $\bar{n}$ be the extension of n to $\mathbb{R}^J$ satisfying $\bar{n}(x) = n(y)$ whenever $ax = y \in \partial B$, some $a \in (0, \infty)$* (*and define $\bar{n}(0)$ arbitrarily*). *Let Ξ: $\mathbb{R}^J \to \mathbb{R}_+$ be defined via*

$$\Xi(x) = a \;\Leftrightarrow\; x \in \partial(aB)$$

for all $a \in [0, \infty)$, and let $\xi(x) = (\Xi(x))^2$. Then there exist constants $m, M \in (0, \infty)$ and a function ϱ: $\mathbb{R}^J \to [m, M]$ such that the $C^1(\mathbb{R}^J)$ function ξ satisfies $m\|x\|^2 \leq \xi(x) \leq M\|x\|^2$, and $D\xi(x) = \varrho(x)\Xi(x)\bar{n}(x)$.

3. *There exists a constant* $m_1 \in (0, \infty)$ *and a continuously differentiable function* $\mu\colon \mathbb{R}_+^J \to [0, m_1]$ *such that* $\|D\mu\| \le m_1$ *on* $\mathbb{R}_+^J$, *and*

$$\langle D\mu(x), \gamma_i \rangle < 0, \qquad x \in \mathbb{R}_+^J, \qquad i \in I(x).$$

In what follows, we keep the notation of Lemma 9 for $B, \bar{n}, \Xi, \xi, \varrho$, and μ.

PROOF OF THEOREM 5. For $a > 0$, let

$$U(x) = u(x) - a\mu(x),$$
$$V(x) = v(x) + a\mu(x).$$

Let $\delta > 0$. Then it suffices to show that for all small $a > 0$, $\delta > 0$, one has $U \le (1+\delta)V$ on G. Arguing by contradiction, we assume that this is not true. Then there are a and δ arbitrarily small, such that

$$\rho = \sup_{x \in G}[U(x) - (1+\delta)V(x)] > 0.$$

Below we let c_i, $i = 1, 2, \dots$, denote positive constants. Consider Condition 1.1 first. Let

$$\Phi(x, y) = U(x) - (1+\delta)V(y) - \frac{1}{\epsilon}\xi(x - y - \epsilon^{1/2}y). \tag{28}$$

Let $(\bar{x}, \bar{y}) \in \overline{G}^2$ achieve the maximum of Φ in $\overline{G} \times \overline{G}$. By continuity of U and V, there exists $\bar{z} \in \overline{G}$ so that $\rho = U(\bar{z}) - (1+\delta)V(\bar{z})$. Note that

$$(1+\epsilon^{1/2})^{-1}\bar{z} \in \overline{G}. \tag{29}$$

Hence, by the Lipschitz continuity of V,

$$\begin{aligned} \Phi(\bar{x}, \bar{y}) &\ge \Phi\left(\bar{z}, \frac{\bar{z}}{1+\epsilon^{1/2}}\right) \\ &= U(\bar{z}) - (1+\delta)V\left(\frac{\bar{z}}{1+\epsilon^{1/2}}\right) \\ &\ge \rho - c_1\epsilon^{1/2}. \end{aligned} \tag{30}$$

By Lipschitz continuity of U and the lower bound on ξ given in Lemma 9,

$$\begin{aligned} \Phi(\bar{x}, \bar{y}) &= U(\bar{x}) - (1+\delta)V(\bar{y}) - \frac{1}{\epsilon}\xi(\bar{x} - \bar{y} - \epsilon^{1/2}\bar{y}) \\ &\le U(\bar{y}) + c_2|\bar{x} - \bar{y}| - (1+\delta)V(\bar{y}) - \frac{m}{\epsilon}|\bar{x} - \bar{y} - \epsilon^{1/2}\bar{y}|^2 \\ &\le \rho + c_2|\bar{x} - \bar{y}| - \frac{c_3}{\epsilon}|\bar{x} - \bar{y} - \epsilon^{1/2}\bar{y}|^2. \end{aligned} \tag{31}$$

By (30) and (31),

$$\begin{aligned} c_2|\bar{x} - \bar{y}| + c_1\epsilon^{1/2} &\ge \frac{c_3}{\epsilon}|\bar{x} - \bar{y} - \epsilon^{1/2}\bar{y}|^2 \\ &\ge \frac{c_4}{\epsilon}|\bar{x} - \bar{y}|^2 - c_4|\bar{y}|^2. \end{aligned} \tag{32}$$

Because $\bar{x}$ and $\bar{y}$ are bounded, (32) implies $|\bar{x} - \bar{y}|^2 \le c_5\epsilon$, so

$$|\bar{x} - \bar{y}| \le c_6\epsilon^{1/2}. \tag{33}$$

Using this in (32) we have

$$|\bar{x}-\bar{y}-\epsilon^{1/2}\bar{y}| \le c_7 \epsilon^{3/4}. \tag{34}$$

By (33), $\bar{x}-\bar{y} \to 0$ as $\epsilon \downarrow 0$. Also, we claim that for all $\epsilon > 0$ small, $\bar{x}$ and $\bar{y}$ are bounded away from $\partial_o G$. To see this, assume the contrary. Then, along a subsequence, both $\bar{x}$ and $\bar{y}$ must converge to the same point on $\partial_o G$. Using the continuity of u and v, (26)–(27), and the nonnegativity of ξ, $\limsup \Phi(\bar{x}, \bar{y}) \le \limsup[u(\bar{x}) - (1+\delta)v(\bar{y})] \le 0$, where the limit superior is taken along this subsequence. However, by (30), for all small ϵ, $\Phi(\bar{x}, \bar{y}) \ge \rho/2 > 0$, which gives a contradiction.

Let

$$\theta(x) = \frac{1}{\epsilon}\xi(x-\bar{y}-\epsilon^{1/2}\bar{y}) + a\mu(x),$$

and note that the map $x \mapsto u(x) - \theta(x)$ has a maximum at $\bar{x} \in G$. Because u is a subsolution, (24) must be satisfied at $\bar{x}$. Denoting

$$q^\epsilon = \varrho(\bar{x}-\bar{y}-\epsilon^{1/2}\bar{y})\,\Xi(\bar{x}-\bar{y}-\epsilon^{1/2}\bar{y})\,\bar{n}(\bar{x}-\bar{y}-\epsilon^{1/2}\bar{y}), \tag{35}$$

we have from Lemma 9.2 that

$$D\theta(\bar{x}) = \frac{1}{\epsilon}q^\epsilon + aD\mu(\bar{x}).$$

Suppose i is such that $\bar{x}_i = 0$. Then $\langle \bar{x}-\bar{y}-\epsilon^{1/2}\bar{y}, e_i\rangle \le 0$, and so by Lemma 9.1,

$$\langle \gamma_i, \bar{n}(\bar{x}-\bar{y}-\epsilon^{1/2}\bar{y})\rangle \le 0.$$

Because by Lemma 9.3, $\langle \gamma_i, D\mu(\bar{x})\rangle < 0$, it follows that $\langle \gamma_i, D\theta(\bar{x})\rangle < 0$. It follows from (24) that

$$H(D\theta(\bar{x})) \ge 0,$$

namely,

$$H\left(\frac{1}{\epsilon}q^\epsilon + aD\mu(\bar{x})\right) \ge 0. \tag{36}$$

On the other hand, let

$$\alpha(y) = -a\mu(y) - \frac{1}{\epsilon(1+\delta)}\xi(\bar{x}-y-\epsilon^{1/2}y).$$

Note that

$$D\alpha(\bar{y}) = -aD\mu(\bar{y}) + \frac{1+\epsilon^{1/2}}{\epsilon(1+\delta)}q^\epsilon,$$

and that the map $y \mapsto v(y) - \alpha(y)$ has a minimum at $\bar{y}$. Because $\bar{x} \in \overline{G}$, (34) implies that $\bar{y} \in G\backslash\partial_c G$ for all small ϵ. Because v is a supersolution, (25) is satisfied at y. An argument as above shows that

$$H(D\alpha(\bar{y})) \le 0,$$

and therefore

$$H\left(\frac{1}{1+\delta}\left(\frac{1+\epsilon^{1/2}}{\epsilon}q^\epsilon - a(1+\delta)D\mu(\bar{y})\right)\right) \le 0.$$

It follows from the definition of H, using $\rho(u,m)\ge 0$, that

$$H\left(\frac{1}{1+\delta}p\right)\ge\frac{1}{1+\delta}H(p)+\frac{\delta}{1+\delta}c,$$

and therefore

$$H\left(\frac{1}{\epsilon}q^{\epsilon}+\frac{1}{\epsilon^{1/2}}q^{\epsilon}-a(1+\delta)D\mu(\bar{y})\right)+\delta c\le 0. \tag{37}$$

Now $D\mu$ is bounded, and by (34), boundedness of n and ϱ, and the Lipschitz continuity of Ξ, it follows that $\epsilon^{-1/2}q^{\epsilon}$ converges to zero as $\epsilon\to 0$. Note that by (22) and the following comment, H is uniformly continuous on $\mathbb{R}^J$. Therefore, (36) and (37) give a contradiction when $a>0$ and $\epsilon>0$ are small and $\delta>0$ fixed.

Under Condition 1.2, the above argument is not valid because (29) may not hold. However, in this case the minimizing player can force exit from any point on $\partial G\backslash\partial_+G$, and the additional complications due to the "state-space constraint" used under Part 1 are no longer needed. In other words, instead of (28) we can consider

$$\Phi(x,y)=U(x)-(1+\delta)V(y)-\frac{1}{\epsilon}\xi(x-y),$$

and a review of the above proof shows that (36) and (37) still hold if all terms with $\epsilon^{1/2}$ in (35) and (37) are removed. A contradiction is then obtained analogously. $\square$

We next consider the upper and lower values of the game and remind the reader that in this section the rates m are assumed to be bounded.

THEOREM 6. *V^- and V^+ are solutions to* (23).

Recall that from Lemma 6, $V^{\pm}$ are Lipschitz.

PROOF OF THEOREM 6. We use the specific form of $v(u,m)$ and $\rho(u,m)$. These can be written as

$$v(u,m)=b_0(m)+\sum_{i=1}^{J}u_ib_i(m),$$

and

$$\rho(u,m)=c_0(m)+\sum_{i=1}^{J}u_ic_i(m).$$

We have $\sum_{i\in C(k)}u_i\le 1$. The b_i are linear, and the c_i are convex. Hence, as a direct consequence of Rockafellar (1970, Corollary 37.3.2), the *Isaacs condition* holds, namely

$$H(q)=\inf_m\sup_u[\langle q,v(u,m)\rangle+c+\rho(u,m)]=\sup_u\inf_m[\langle q,v(u,m)\rangle+c+\rho(u,m)]. \tag{38}$$

Another fact we will use is that for any $y\in G\backslash\partial_cG$, there is $\delta_0=\delta_0(y)>0$, which serves as a lower bound on the exit time. Namely, if ϕ solves $\dot{\phi}=\pi(\phi,v(u,m))$, $\phi(0)=y$, then

$$\sigma\doteq\inf\{t\ge 0:\phi(t)\notin G\}\ge\delta_0,\qquad u\in\overline{U},\qquad m\in\overline{M}. \tag{39}$$

The bound is an immediate consequence of the u and m being uniformly bounded.

By definition, $V^{\pm}(x)=0$ for $x\in\partial_oG$. Thus, we need only to establish (24)–(25). The proof consists of four parts.

Proof that V^- is a supersolution on $G\backslash\partial_c G$. Standard dynamic programming arguments show that for $\delta > 0$,

$$V^-(x) = \inf_\beta \sup_u \left[\int_0^{\sigma\wedge\delta} (c + \rho(u, \beta[u]))\, dt + V^-(\phi(\sigma \wedge \delta)) \right], \tag{40}$$

where ϕ is the solution to $\dot\phi = \pi(\phi, v(u, \beta[u]))$, with $\phi(0) = x$. Let θ be smooth, and let $y \in G\backslash\partial_c G$ be a local minimum of $V^- - \theta$. We can assume without loss that $V^-(y) = \theta(y)$. We need to show

$$H(D\theta(y)) \wedge \min_{i\in I(y)} \langle D\theta(y), \gamma_i\rangle \le 0. \tag{41}$$

We shall assume the contrary and reach a contradiction. Thus, there exists $a > 0$ such that $H(D\theta(y)) \ge a$, and

$$\langle D\theta(y), \gamma_i\rangle \ge a, \qquad i \in I(y). \tag{42}$$

From the definition of H and (38),

$$\sup_u \inf_m [\langle D\theta(y), v(u,m)\rangle + c + \rho(u,m)] \ge a,$$

and therefore there exists a u_0 such that for all m,

$$\langle D\theta(y), v(u_0, m)\rangle + c + \rho(u_0, m) \ge a/2.$$

For any strategy β, if $\bar u(t) \equiv u_0$,

$$\big\langle D\theta(y), v(\bar u(t), \beta[\bar u](t))\big\rangle + c + \rho(\bar u(t), \beta[\bar u](t)) \ge a/2 \tag{43}$$

for all t. Let ϕ denote the dynamics corresponding to $\bar u$ and a generic β, starting from y. Note that the mapping $z \mapsto I(z)$ is upper semicontinuous in the sense that for any z there is a neighborhood of z on which $I(\cdot) \subset I(z)$. Using the boundedness of m, this implies that for any $\beta \in B$, one has $I(\phi(r)) \subset I(y)$ for $r \in [0, \delta]$, if $\delta > 0$ is chosen small enough. We now use that ϕ is a solution to the SP. Choosing such a $\delta > 0$, for any $r \in [0, \delta]$ there exist $a_i \ge 0$ (that may depend on r) such that

$$\dot\phi(r) = v(\bar u(r), \beta[\bar u](r)) + \sum_{i\in I(y)} a_i\gamma_i.$$

Using the continuity of $D\theta$ and taking $\delta > 0$ smaller if necessary, (42) and (43) imply, for $t \in [0, \delta]$,

$$\begin{aligned} \frac{d}{dt}\theta(\phi(t)) &= \langle D\theta(\phi(t)), \dot\phi(t)\rangle \\ &\ge -c - \rho(\bar u(t), \beta[\bar u](t)) + a/4. \end{aligned}$$

Taking δ even smaller if necessary (so that it is at most δ_0), we have from (39) that

$$\theta(\phi(\delta)) - \theta(y) \ge -\int_0^\delta (c + \rho(\bar u(t), \beta[\bar u](t)))\, dt + a\delta/4.$$

From (40), one can find a β such that

$$V^-(y) \ge \sup_u \left[\int_0^\delta (c + \rho(u, \beta[u]))\, dt + V^-(\phi(\delta)) - a\delta/8 \right].$$

Letting $u = \bar{u}$, the last two displays give (using $\theta(y) = V^-(y)$)

$$\theta(\phi(\delta)) \geq V^-(\phi(\delta)) + a\delta/8,$$

so that $V^-(\phi(\delta)) - \theta(\phi(\delta)) < 0$ for all $\delta > 0$ small, contradicting the assumption that y is a local minimum of $V^- - \theta$. This proves that V^- is a supersolution on $G \backslash \partial_c G$.

Proof that V^- is a subsolution on G. Let θ be smooth and $y \in G$ a local maximum of $V^- - \theta$. In case that $y \in \partial_c G$, let $\overline{U}_{y,\beta,\delta}$ be the set of controls $u \in \overline{U}$ for which the trajectory ϕ determined by u and $\beta[u]$ and starting from y does not exit G on $[0, \delta]$. Given $y \in \partial_c G$, it is clear that $\overline{U}_{y,\beta,\delta}$ is not empty for all δ small and all β, by considering the control $u = 0$. Moreover, for all δ small enough, (40) is valid where the supremum extends only over $u \in \overline{U}_{y,\beta,\delta}$. Indeed, given $u \notin \overline{U}_{y,\beta,\delta}$, consider u' that agrees with u on $[0, \sigma]$ and $u' = 0$ on $(\sigma, \delta]$. Then the expression in brackets in (40) is identical under u and under u', but $u' \in \overline{U}_{y,\beta,\delta}$.

Assume without loss that $V^-(y) = \theta(y)$. We would like to show that

$$H(D\theta(y)) \vee \max_{i \in I(y)} \langle D\theta(y), \gamma_i \rangle \geq 0. \tag{44}$$

Assuming the contrary, there exists $a > 0$ such that $H(D\theta(y)) \leq -a$, and

$$\langle D\theta(y), \gamma_i \rangle \leq -a, \qquad i \in I(y). \tag{45}$$

Using the definition of H and (38), for all u there exists m_u such that

$$\langle D\theta(y), v(u, m_u) \rangle + c + \rho(u, m_u) \leq -a/2. \tag{46}$$

Note that it is possible to choose m_u so that it depends continuously on u. Define $\bar{\beta}$ as $\bar{\beta}[u](t) = m_{u(t)}$ for all t. Since $\bar{\beta}[u]$ is measurable if u is, $\bar{\beta}$ maps $\overline{U}$ into $\overline{M}$. Let ϕ be the trajectory corresponding to $\bar{\beta}$ and a generic $u \in \overline{U}$, (or a generic $u \in \overline{U}_{y,\beta,\delta}$ if $y \in \partial_c G$) starting from y. Arguing as before by upper semicontinuity of $I(\cdot)$, if δ is small enough, then

$$\dot{\phi}(r) = v(u(r), \bar{\beta}[u](r)) + \sum_{i \in I(y)} a_i \gamma_i, \qquad r \in [0, \delta],$$

where $a_i \geq 0$ may depend on r. By possibly taking δ smaller, and smaller than δ_0, we have, using the continuity of $D\theta$ and (45), (46) that

$$\begin{aligned} \frac{d}{dt}\theta(\phi(t)) &= \langle D\theta(\phi(t)), \dot{\phi}(t) \rangle \\ &\leq -c - \rho(u(t), \bar{\beta}[u](t)) - a/4, \end{aligned}$$

and

$$\theta(\phi(\delta)) - \theta(y) \leq -\int_0^\delta (c + \rho(u(t), \bar{\beta}[u](t)))dt - a\delta/4.$$

Now, (40) implies that for any β there is u such that

$$V^-(y) \leq \int_0^\delta (c + \rho(u, \beta[u]))dt + V^-(\phi(\delta)) + a\delta/8.$$

Specializing to $\beta = \bar{\beta}$, the last two displays show that $V^-(\phi(\delta)) - \theta(\phi(\delta)) > 0$ for all $\delta > 0$ small. This contradicts the assumption that y is a local maximum of $V^- - \theta$, and as a result, V^- is a subsolution.

Proof that V^+ is a supersolution on $G\backslash\partial_c G$. The proof is analogous to the proof that V^- is a subsolution. Most details are therefore skipped. The dynamic programming principle states that for $\delta > 0$,

$$V^+(x) = \sup_{\alpha} \inf_{m} \left[\int_0^{\sigma\wedge\delta} (c + \rho(\alpha[m], m))\, dt + V^+(\phi(\sigma \wedge \delta)) \right], \tag{47}$$

where ϕ is the dynamics corresponding to α and m, starting from x. Taking a smooth θ, and letting $y \in G\backslash\partial_c G$ be a local minimum of $V^+ - \theta$, showing

$$H(D\theta(y)) \wedge \min_{i\in I(y)} \langle D\theta(y), \gamma_i \rangle \le 0$$

can be obtained by an argument analogous to that used to prove (44), using (47) in place of (40).

Proof that V^+ is a subsolution on G. We need to show that

$$H(D\theta(y)) \vee \max_{i\in I(y)} \langle D\theta(y), \gamma_i \rangle \ge 0, \tag{48}$$

where θ is smooth, and $y \in G$ is a local maximum of $V^+ - \theta$. In the special case where $y \in \partial_c G$, we can assume without loss that the supremum in (47) extends only over $\alpha \in A_{y,\delta}$, the set of strategies under which, for any $m \in \overline{M}^b$, the dynamics associated with α and m, and starting from y, does not leave G before δ. The proof of (48) is analogous to the proof of (41) and is skipped.

This completes the proof that V^- and V^+ are solutions to (23). □

5. A competing queues example. Consider a queueing network with only one server, providing service to J classes. Each customer requires service once. In this example all arrival rates are positive: $\lambda_i > 0$ for all i, hence $\mathcal{J}_+ = \{1, \dots, J\}$. This network, "the k competing queues," has been studied extensively, in discrete and continuous time (see Baras et al. 1985, Walrand 1984, and references therein). When the criterion (to be minimized) is either the average cost or the discounted cost, and the one-step cost is a positive linear combination $\sum_i c_i x_i$ of the queue sizes x_i, the optimal policy is the μ-c rule, which is a priority discipline, giving absolute priority to the nonempty queue for which $\mu_i c_i$ is maximal. Under the cost studied here, the optimal policy is quite different.

PROPOSITION 1. *Consider the case where G is a hyperrectangle, given as $G = \{x : 0 \le x < z_i\}$, where $z_i > 0$ are constants. Assume that $\lambda_i > 0$ for all $i = 1, \dots, J$. If c is large enough, then the viscosity solution to the PDE* (23) *is given as*

$$V(x) = \min_i \alpha_i (z_i - x_i), \tag{49}$$

where $\alpha_i > 0$ are constants depending on c.

We remark that the constants α_i are uniquely defined by (51) below. In the totally symmetric case, where $\mu_i = \mu$, $\lambda_i = \lambda$, and $z_i = z$ for all i, the solution takes the form $V(x) = \alpha \min_i (z - x_i)$. In this case, the optimal service discipline can be interpreted as "serve the longest queue." An asymmetric two-dimensional example is given in Figure 2, where the domain G is divided into two subdomains G_1 and G_2 in accordance with the Structure (49), and the optimal service discipline corresponds to giving priority to class i when the state is within G_i, $i = 1, 2$. Thus, the optimal control under our escape-time criterion is very different from the optimal controls for the average or discounted cost criteria.

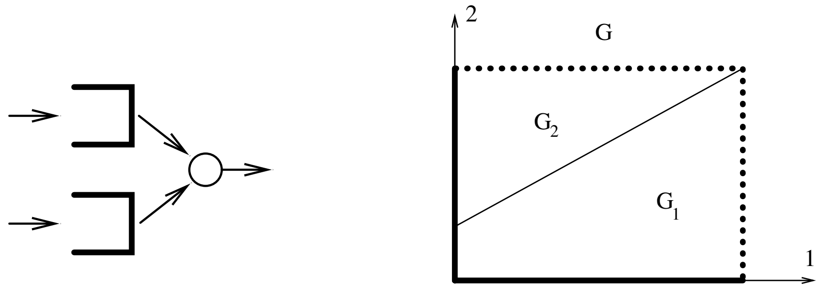

FIGURE 2. Priority to class i when the state is in G_i, $i = 1, 2$.

PROOF. The constraint directions are given by $\gamma_i = e_i$. The Hamiltonian is given by

$$H(p) = \sup_u \inf_m H(p, u, m),$$

where

$$H(p, u, m) = c + \sum_i \left[p_i(\bar{\lambda}_i - u_i\bar{\mu}_i) + \lambda_i l\left(\frac{\bar{\lambda}_i}{\lambda_i}\right) + u_i\mu_i l\left(\frac{\bar{\mu}_i}{\mu_i}\right)\right].$$

Using strict convexity and smoothness of the map $m \mapsto H(p, u, m)$, the minimum over m is attained at $\bar{\lambda}_i = \lambda_i e^{-p_i}$, $\bar{\mu}_i = \mu_i e^{p_i}$. Thus,

$$H(p, u) \doteq \inf_m H(p, u, m) = c + \sum_i [\lambda_i(1 - e^{-p_i}) + u_i\mu_i(1 - e^{p_i})].$$

For the proposed solution, $DV(x) \in -\mathbb{R}^J_+$ wherever the gradient is defined. For $p \in -\mathbb{R}^J_+$, maximizing $H(p, u)$ over u clearly gives

$$H(p) = \sup_u H(p, u) = c + \sum_i \lambda_i(1 - e^{-p_i}) + \max_i \mu_i(1 - e^{p_i}). \tag{50}$$

We use the well-known fact that the definition of viscosity solutions can be equivalently stated in terms of sub- and superdifferentials (see Dupuis and Nagurney 1993, Lemma II.1.7). Note that (26) and (27) hold because $V = 0$ on $\partial_o G$. Hence, it suffices to verify that (24) (respectively, (25)) holds where $D\theta(y)$ is replaced by any superdifferential (subdifferential) of V at y.

We first show that the equation $H(DV(x)) = 0$ holds wherever DV is defined. The proposed Form (49) satisfies $DV(x) = -\alpha_i e_i$ wherever the gradient is defined, with $i = i_x$ depending on x. By the special form of the gradient, the equation $H(DV(x)) = 0$ takes the form

$$H(DV(x)) = c + \lambda_i(1 - e^{\alpha_i}) + \mu_i(1 - e^{-\alpha_i}) = 0, \tag{51}$$

where $i = i_x$. Denote $c_i = c/(\lambda_i + \mu_i)$. Then, equivalently, $1 + c_i - F_i(\alpha_i) = 0$, where

$$F_i(\alpha_i) = \frac{\lambda_i e^{\alpha_i} + \mu_i e^{-\alpha_i}}{\lambda_i + \mu_i}.$$

The function F_i is strictly convex, $F_i(0) = 1$, and $F_i(\alpha_i) \to \infty$ as $\alpha_i \to \infty$. Because $c_i > 0$, it follows that there are unique positive constants α_i where $F_i(\alpha_i) = 1 + c_i$, $i = 1, \ldots, J$. These are the constants in (49). In particular, (51) holds for $i = i_x$, and $H(DV(x)) = 0$.

Next consider any interior point x at which the gradient is not defined. Clearly, there are no subdifferentials at that point, and any superdifferential is given as a convex combination of $-\alpha_i e_i$, $i = 1, \ldots, J$. Let $B(e_i, \epsilon)$ be the open ball of radius ϵ about e_i. Denote

$\widetilde{S}=\{\nu\in\mathbb{R}^J:\nu_i\geq 0,\sum\nu_i=1\}$, $S=\{\nu\in\mathbb{R}^J:\nu_i\geq 0,\sum\nu_i\leq 1\}$, $S_\epsilon=S\cap\bigcup_i B(e_i,\epsilon)$, and $S_\epsilon^c=S-S_\epsilon$. Let $q=-\sum_i\nu_i\alpha_i e_i$. It suffices to show that $H(q)\geq 0$ for $\nu\in\widetilde{S}$, but since we later need a stronger statement than that, we show that in fact $H(q)\geq 0$ holds for $\nu\in S$. By (50),

$$H(q)=c+\sum_i\lambda_i(1-e^{\nu_i\alpha_i})+\max_i\mu_i(1-e^{-\nu_i\alpha_i}).$$

Define

$$H^1(q)=c+\sum_i\lambda_i(1-e^{\nu_i\alpha_i})+\mu_1(1-e^{-\nu_1\alpha_1}),$$

and

$$\overline{H}(q)=c+\sum_i[\lambda_i(1-e^{\nu_i\alpha_i})+\mu_i(1-e^{-\nu_i\alpha_i})].$$

By (51), $c+\lambda_i(1-e^{\alpha_i})+\mu_i\geq 0$ and $c+\lambda_i(1-e^{\alpha_i})\leq 0$, and it follows that there are constants A_1,A_2,A_3, and A_4 such that for all c and $i=1,\dots,J$,

$$A_1+\log(c+A_2)\leq\alpha_i\leq A_3+\log(c+A_4). \tag{52}$$

We first consider small perturbations ν of e_1. To show that $H(q)\geq 0$, it suffices to show that $H^1(q)\geq 0$. Note that (51) implies $H^1(q)|_{\nu=e_1}=0$. Also,

$$\nabla_\nu H^1(q)|_{\nu=e_1}=(-\lambda_1\alpha_1 e^{\alpha_1}+\mu_1\alpha_1 e^{-\alpha_1})e_1-\sum_{i\neq 1}\lambda_i\alpha_i e_i.$$

Hence, for $\gamma=e_i-e_1$ (where $i\neq 1$), using $c+\lambda_1(1-e^{\alpha_1})\leq 0$, (51) and (52),

$$\begin{aligned}\nabla_\nu H^1(q)|_{\nu=e_1}\cdot\gamma&=\lambda_1\alpha_1 e^{\alpha_1}-\mu_1\alpha_1 e^{-\alpha_1}-\alpha_i\lambda_i\\&=\alpha_1(2\lambda_1 e^{\alpha_1}-\mu_1-c-\lambda_1)-\alpha_i\lambda_i\\&\geq\alpha_1(c-\mu_1)-\alpha_i\lambda_i\\&\geq[A_1+\log(c+A_2)](c-\mu_1)-[A_3+\log(c+A_4)]\lambda_i\\&\geq 1,\end{aligned}$$

for all c large. Analogous calculations give $\nabla_\nu H^1(q)|_{\nu=e_1}\cdot\gamma\geq 1$ for $\gamma=-e_1$ as well. As a result, the directional derivatives $(\partial/\partial\tilde{\gamma})H^1(q)|_{\nu=e_1}$ in the direction $\tilde{\gamma}$, where $\tilde{\gamma}$ are of the form $\tilde{\gamma}=(y-e_1)/\|y-e_1\|$, $y\in S$, are bounded below by $1/2$. Hence, $H^1(q)\geq 0$ for $\nu\in S$ within a neighborhood of e_1 and c large. Consequently, a similar statement holds for $H(q)$. Because the same argument holds for neighborhoods of e_i, $i=2,\dots,J$, we conclude that there is $\epsilon>0$ and c_0 such that $H(q)\geq 0$ for $\nu\in S_\epsilon$ and $c\geq c_0$.

Next consider $\nu\in S_\epsilon^c$. We first provide a lower bound on $(\partial/\partial c)\overline{H}(q)$. Differentiating (51) with respect to c, $\dot{\alpha}_i\doteq\partial\alpha_i/\partial c=(\lambda_i e^{\alpha_i}-\mu_i e^{-\alpha_i})^{-1}$. Using (52), for all c large, $0\leq\dot{\alpha}_i\leq(\lambda_i e^{\alpha_i}-1)^{-1}$. Using this, the fact that ν is bounded away from $\bigcup_i\{e_i\}$, and by taking c large, one has

$$\begin{aligned}\frac{\partial}{\partial c}\overline{H}(q)&\geq 1-\sum_i\nu_i\lambda_i\dot{\alpha}_i e^{\nu_i\alpha_i}\\&\geq 1-\sum_i\nu_i[e^{(1-\nu_i)\alpha_i}-1]^{-1}\\&\geq\frac{1}{2}.\end{aligned}$$

Note that the above bound holds for all $c \geq c_1$ and all $\nu \in S_\epsilon^c$, where c_1 is a constant. It follows that there is c_2 such that for all $c \geq c_2$ and all $\nu \in S_\epsilon^c$, one has $\overline{H}(q) \geq \sum_i \mu_i$. Since $H \geq \overline{H} - \sum_i \mu_i$, $H(q) \geq 0$. We conclude that $H(q) \geq 0$ for all $\nu \in S$. In particular, $H(q) \geq 0$ where q is any superdifferential of V at any interior point.

Finally, consider a point $x \in G \cap \partial\mathbb{R}_+^J$. Any superdifferential of V at x is given as $q = \sum_{i\in I(x)} \eta_i e_i - \sum_{j=1}^J \nu_j \alpha_j e_j$, where $\eta_i \geq 0$. If $\max_{i\in I(x)} \langle q, \gamma_i\rangle \geq 0$, then (24) holds. Otherwise, $\langle q, \gamma_i\rangle < 0$ for all $i \in I(x)$. Consequently, any q of the form above is given as $-\sum_{j=1}^J \nu_j' \alpha_j e_j$, with $\nu' \in S$. As we have shown, in this case, $H(q) \geq 0$. Therefore, (24) holds.

Similarly, any subdifferential of V at $x \in G \cap \partial\mathbb{R}_+^J$ is of the form $-\sum_{i\in I(x)} \eta_i e_i - \sum_{j=1}^J \nu_j \alpha_j e_j$. In particular, $\langle q, e_i\rangle \leq 0$ for all i, and (25) holds. $\square$

6. Proofs of lemmas.

PROOF OF LEMMA 1. Let

$$W^n(x) \doteq \inf E_x^{u,n} e^{-nc\sigma_n}.$$

Because $c > 0$, W^n is well defined. Standard iterative methods can be used to construct a solution to the DPE

$$0 = \inf_{u\in U}\big[\widetilde{\mathscr{L}}^{n,u}\overline{W}^n(x) - nc\overline{W}^n(x)\big], \qquad x \in G^n, \tag{53}$$

and the boundary condition $\overline{W}^n(x) = 1$ if $x \notin G^n$. We claim that this solution coincides with the risk-sensitive cost. To see this, consider a controlled Markov process (X^n, u) that starts at x. Then

$$Y(t) \doteq \overline{W}^n(X^n(t)) - \overline{W}^n(x) - \int_0^t \widetilde{\mathscr{L}}^{n,u(s)}\overline{W}^n(X^n(s))\,ds$$

is a martingale. Equation (53) implies $\mathscr{L}^{n,u(s)}\overline{W}^n(X^n(s)) \geq nc\overline{W}^n(X^n(s))$, so

$$\overline{W}^n(X^n(t)) - \overline{W}^n(x) - \int_0^t nc\overline{W}^n(X^n(s))\,ds = \int_0^t Z(s)ds + Y(t)$$

for some nonnegative process Z. Using Gronwall's lemma, we obtain that for each $t < \infty$

$$E_x^{n,u}\overline{W}^n(X^n(t\wedge\sigma^n))e^{-nc(t\wedge\sigma^n)} \geq \overline{W}^n(x),$$

and by the Lebesgue Dominated Convergence Theorem,

$$E_x^{n,u}e^{-nc\sigma^n} \geq \overline{W}^n(x).$$

If we define u in terms of the feedback control that minimizes in (53), then all the inequalities above become equalities, thus showing that $\overline{W}^n = W^n$.

The definition of W^n implies $W^n(x) = \exp[-nV^n(x)]$. If we insert this into the DPE of $\overline{W}^n$ and multiply by $\exp[nV^n(x)]$, then the equation

$$\begin{aligned}0 = \inf_{u\in U}\Bigg[&\sum_{j=1}^J n\lambda_j\left(\exp\left[-nV^n\left(x+\frac{1}{n}v_j\right)+nV^n(x)\right]-1\right)\\ &+\sum_{i=1}^J n\mu_i u_i\left(\exp\left[-nV^n\left(x+\frac{1}{n}\pi(x,\tilde{v}_i)\right)+nV^n(x)\right]-1\right)-nc\Bigg]\end{aligned}$$

results. Recall the definition $l(x) = x\log x - x + 1$ for $x > 0$. We now divide throughout by n and use the convex duality relation

$$[e^y - 1] = \sup_{x>0}[xy - l(x)]$$

to represent the terms in the previous display. For example, in the sum on j we take $x = \bar{\lambda}_j/\lambda_j$ and $y = -[nV^n(x + \frac{1}{n}v_j) - nV^n(x)]$. Representing each term in this way and multiplying by -1 produces the first line in (7). The boundary condition that is the second line in (7) follows directly from the relation between W^n and V^n. $\square$

PROOF OF LEMMA 2. We reduce the Lipschitz property on $(n^{-1}\mathbb{Z}_+^J) \cap \overline{G}$ to a Lipschitz property near the boundary. To this end we use the following coupling. For $z \in G^n$, let $u^n(z)$ be a minimizer in (53). Given a point x on the lattice, let X^x denote the process corresponding to the generator $\mathscr{L}^{n,u^n}$ and starting at x (see the discussion following (2)). To simplify the notation, we will not explicitly denote the dependence of quantities such as X^x on n. Let $u(t) = u^n(X^x(t))$, and let F_t be the filtration generated by X^x.

Fix a point $y \neq x$ and let X^y denote the queueing process on this probability space that starts at y and uses the control u. In other words, X^y is the image, under the Skorokhod map, of $y + X^x(\cdot) - x$. The evolution of the processes X^x and X^y are identical, save that jumps which would cause X^y to leave $\mathbb{Z}_+^J$ are deleted. Automatically, u is suboptimal for the control problem starting from y. Define

$$V^n(y; u) = -n^{-1} \log E_x^{u,n} e^{-nc\sigma^y}, \tag{54}$$

where σ^y is the exit time of X^y from G. Note that due to the coupling, we may take expectations with respect to $E_x^{u,n}$ rather then with respect to $E_y^{u,n}$. Because (X^y, u) is a (possibly suboptimal) controlled Markov process, we have

$$V^n(x) - V^n(y) \le V^n(x) - V^n(y; u). \tag{55}$$

Define $\sigma = \min\{\sigma^x, \sigma^y\}$. By Theorem 1 on the Lipschitz continuity of the Skorokhod map we have

$$\mathrm{dist}(X^x(\sigma), \partial G) \le K_1|x - y|, \qquad \mathrm{dist}(X^y(\sigma), \partial G) \le K_1|x - y|, \tag{56}$$

because at least one of the processes has left G by σ. In the last display, K_1 is the constant appearing in (4). We claim that

$$V^n(x) - V^n(y) \le \sup\{V^n(z) : z \in S\}, \tag{57}$$

where $S \doteq \{z \in n^{-1}\mathbb{Z}_+^J \cap \overline{G} : \mathrm{dist}(z, \partial_{co}G) \le K_1|x - y|\}$. To establish this, note that

$$\begin{aligned}
V^n(x) - V^n(y, u) &= -\frac{1}{n}\big[\log E_x^{u,n} e^{-nc\sigma^x} - \log E_x^{u,n} e^{-nc\sigma^y}\big] \\
&\le -\frac{1}{n}\big[\log E_x^{u,n}\big[e^{-nc\sigma} E_x^{u,n}\big(e^{-nc(\sigma^x - \sigma)}\big| X^x(\sigma)\big)\big] - \log E_x^{u,n} e^{-nc\sigma}\big] \\
&\le \sup_{z \in S} -\frac{1}{n}\big[\log E_x^{u,n}\big[e^{-nc\sigma} E_x^{u,n}\big(e^{-nc(\sigma^x - \sigma)}\big| X^x(\sigma) = z\big)\big] - \log E_x^{u,n} e^{-nc\sigma}\big] \\
&= \sup_{z \in S} -\frac{1}{n}\big[\log\big[E_x^{u,n}\big(e^{-nc(\sigma^x - \sigma)}\big| X^x(\sigma) = z\big) E_x^{u,n} e^{-nc\sigma}\big] - \log E_x^{u,n} e^{-nc\sigma}\big] \\
&= \sup_{z \in S} -\frac{1}{n}\big[\log E_x^{u,n}\big(e^{-nc(\sigma^x - \sigma)}\big| X^x(\sigma) = z\big)\big].
\end{aligned}$$

However, by the strong Markov property,

$$-\frac{1}{n}\Big[\log E_x^{u,n}\Big(e^{-nc(\sigma^x - \sigma)} \,\Big|\, X^x(\sigma) = z\Big)\Big] \le \sup_u -\frac{1}{n} \log\Big(E_z^{u,n} e^{-nc\sigma^z}\Big) = V^n(z),$$

and together with (55) we have (57).

To prove the lemma, one needs to show that $|V_n(x) - V_n(y)| \le c_0|x-y|$ for all n and all $x, y \in (n^{-1}\mathbb{Z}_+^J) \cap \overline{G}$, where c_0 does not depend on x, y and n. It suffices to prove this inequality for x, y such that $|x-y| = n^{-1}$. Because the roles of x and y are symmetric, and in view of (57), it suffices to show that for $\{x \in G : \mathrm{dist}(x, \partial_{co}G) \le K_1 n^{-1}\}$,

$$V_n(x) = -n^{-1} \log \inf_u E_x^{u,n} e^{-nc\sigma^x} \le c_1 n^{-1}, \tag{58}$$

where $c_1 > 0$ is a constant.

Let us first treat the case where G is not a rectangle. In that case, Condition 1 implies that for any x with $\mathrm{dist}(x, \partial_{co}G) \le K_1 n^{-1}$,

$$\text{there is} \quad i \in \mathcal{J}_+ \quad \text{such that} \quad x + c' n^{-1} e_i \notin G, \tag{59}$$

where c' is a constant. Let such i be fixed. To show (58), it is enough to show that for any x such that $\mathrm{dist}(x, \partial_{co}G) \le K_1 n^{-1}$, and any n and u,

$$E_x^{u,n} e^{-nc\sigma^x} \ge c_2 > 0. \tag{60}$$

Recall that $\lambda_i > 0$. Let S_t denote the event that all service processes and all arrival processes, except for the one corresponding to i, do not increase on $[0, t]$. Recall that the expected time until a Poisson process of rate λ hits level K is K/λ. Then, for any $\alpha \in (0, 1)$

$$\begin{aligned} E_x^{u,n} e^{-nc\sigma^x} &\ge \alpha P_x^{u,n}(e^{-nc\sigma^x} > \alpha) \\ &= \alpha P_x^{u,n}\left(\sigma^x < -\frac{\log\alpha}{nc}\right). \end{aligned}$$

Choosing $t_0 = -(\log\alpha)/nc = 2\tilde{c}/n\lambda_i$ and using $P_x^{u,n}(\sigma^x < 2E\sigma^x) \ge 1/2$,

$$\begin{aligned} E_x^{u,n} e^{-nc\sigma^x} &\ge \alpha P_x^{u,n}(\sigma^x < t_0 | S_{t_0}) P_x^{u,n}(S_{t_0}) \\ &\ge e^{-2c\tilde{c}/\lambda_i} \frac{1}{2} c_3, \end{aligned}$$

where $c_3 > 0$ is the probability that a Poisson process with rate nc_4 has not jumped by time $t_0 = 2\tilde{c}/\lambda_i n$. This proves (60), which implies (58), and hence the statement of the lemma holds.

In the case where G is a rectangle, the bound (57) does not suffice because $V^n(x)$ is discontinuous near $\partial_c G$. We therefore prove that a similar bound applies, where the supremum is over $S = \{z \in (n^{-1}\mathbb{Z}_+^J) \cap \overline{G} : \mathrm{dist}(z, \partial_o G) \le K_1|x-y|\}$. To apply the previous argument, we need to show that if $X^x(t)$ is close to $\partial_c G$, then neither X^x nor X^y will exit (locally) through that boundary. This is clear for X^x: The only way for the process to leave G is due to a service to one of the queues, say, queue j, leading to an increase in queue i. However, allowing this service is certainly not optimal: it is better to avoid this control, as our objective is to increase σ^x. To prevent X^y from exiting, we need to modify the coupling argument as follows. The control u^y used by X^y avoids a jump that leads X^y to exit through $\partial_c G$ (that is, queue j above will not be served if $X_i^y(t) = z_i - 1$). Note that this is the only possible type of jump that leads the process out of G. Moreover, the ℓ_1 distance between X^x and X^y may only decrease due to this change in control: the control is changed only if $X_i^x(t) < X_i^y(t)$, and following the service X_i^x increases by 1 so that $|X_i^x(t) - X_i^y(t)|$ decreases by 1, while X_j^x decreases by 1.

Condition 1 still implies (59), but only for x such that

$$z_i - x_i \le n^{-1}, \quad \text{for some } i \in \mathcal{J}_+.$$

For such x, the argument in the last paragraph holds. However, for x near $\partial_c G$ there is nothing to prove, since the process never exits through such a boundary. $\square$

PROOF OF LEMMA 3. The first part is an immediate consequence of the fact that $u_i \geq 0$ and both $\mathscr{L}^{n,u,m}V^n(x)$ and $\rho(u,m)$ depend on u as $\sum_i u_i\eta_i$, where η_i is a function of m_i, x, n, but not of u.

For the second part of the lemma, one can explicitly solve for m^n in terms of V^n and get $m^n(x,u) = m^n(x) = ((\bar{\lambda}_i^n(x)), (\bar{\mu}_i^n(x)))$, where

$$\bar{\lambda}_i^n(x) = \lambda_i e^{-n\delta_i V^n(x)}, \qquad \bar{\mu}_i^n = \mu_i e^{-n\tilde{\delta}_i V^n(x)},$$

and

$$\delta_i V^n(x) \doteq V^n(x + n^{-1}v_i) - V^n(x), \qquad \tilde{\delta}_i V^n(x) \doteq V^n(x + n^{-1}\pi(x, \tilde{v}_i)) - V^n(x).$$

The result follows from Lemma 2, because it shows that there is a constant b_2 independent of x, n where

$$n\delta_i V^n(x) \geq -b_2, \qquad n\tilde{\delta}_i V^n(x) \geq -b_2. \quad \square$$

PROOF OF LEMMA 4. We fix $b \in [b^*, \infty]$ and suppress it from the notation throughout the proof. Item 1 of the lemma is trivial under Condition 1.1. Under Condition 1.2, by continuity of the functions ϕ_i, we need only to show is that $\partial_{co}G_a$ and $\partial_{co}G$ do not intersect. Consider first $a > 0$, and let $x \in \partial_{co}G_a$. Then $x_i = a + \phi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_J)$ for some $i \in \mathscr{J}_+$, and therefore x cannot belong to the closure of G. The proof for $a < 0$ is similar.

Let $\beta_0[u](t) = m_0$ for all u, t, where m_0 sets all $\bar{\lambda}_i = \lambda_i$ and $\bar{\mu}_i = 0$. Then $\rho(u(t), \beta_0[u](t))$ is bounded by a constant, and the dynamics, unaffected by u, follow $X(t) = x + \sum_i \lambda_i e_i t$ and leave the bounded set G within a finite time bounded by $\operatorname{diam}(G)/\max_i \lambda_i < \infty$. Therefore,

$$V^-(x) \leq \sup_u C(x, u, m_0) = C(x, \beta_0) \leq c_1 < +\infty.$$

Similarly,

$$V^+(x) \leq \sup_\alpha C(x, \alpha(m_0), m_0) \leq c_1 < +\infty.$$

It is useful to notice that for all $a \in (0, a_0)$ and $y \in \partial_o G$ there is $i = i_y \in \mathscr{J}_+$ such that $y + 2ae_{i_y} \notin G_a$. Similarly, for all $a \in (-a_0, 0)$ and $y \in \partial_o G_a$ there is $i = i_y \in \mathscr{J}_+$ such that $y + 2ae_{i_y} \notin G$.

First consider $a > 0$ and recall that σ_a (respectively, σ) is the exit time from G_a (respectively, G), so that for any fixed u and $\beta[u]$ we have $\sigma \leq \sigma_a$. Therefore, since c and ρ are positive,

$$\begin{aligned} V_a^-(x) &= \inf_\beta \sup_u \int_0^{\sigma_a} (c + \rho(u(s), \beta[u](s))\, ds \\ &\geq \inf_\beta \sup_u \int_0^{\sigma} (c + \rho(u(s), \beta[u](s))\, ds \\ &= V^-(x). \end{aligned}$$

Thus, to prove the Lipschitz property a one-sided bound suffices. Recall that $\phi(\sigma)$ is the exit point from G, and for each β define the extension β_a by

$$\beta_a[u] = \begin{cases} \beta[u](t) & t \in [0, \sigma), \\ \hat{m} & t \in [\sigma, \infty), \end{cases}$$

where $\hat{m}$ sets all $\bar{\mu}_j = 0$ and $\bar{\lambda}_j = 1_{j = i_{\phi(\sigma)}}$. Then for any β,

$$\begin{aligned} V_a^-(x) &= \inf_\beta \sup_u C_a(x, \beta[u], u) \\ &\leq \inf_\beta \sup_u C_a(x, \beta_a[u], u) \end{aligned}$$

$$= \inf_\beta \sup_u \left[C(x, \beta[u], u) + \int_\sigma^{\sigma_a} (c + \rho(u(s), \hat{m})\, ds \right]$$
$$\leq V^-(x) + c_1 a,$$

where the last line follows because $\rho(u(s), \hat{m})$ is bounded and because by the previous paragraph, $\sigma_a - \sigma \leq 2a$. Note that c_1 does not depend on $b \in [b^*, \infty]$. For $a < 0$ the same argument shows that $V^-(x) \leq V_a^-(x) + c_3|a|$, by interchanging the roles of G and G_a.

For $V_a^+(x)$, note that an argument as above gives $V_a^+(x) \geq V^+(x)$. For each m, define m_a by

$$m_a = \begin{cases} m(t) & t \in [0, \sigma), \\ \hat{m} & t \in [\sigma, \infty), \end{cases}$$

where $\hat{m}$ is as above. Let α_ϵ be an ϵ-optimal strategy. Then, because for any fixed u and m we have $\sigma \leq \sigma_a$,

$$\begin{aligned} V_a^+(x) &= \sup_\alpha \inf_m C_a(x, m, \alpha[m]) \\ &\leq \inf_m C_a(x, m, \alpha_\epsilon[m]) + \epsilon \\ &\leq \inf_m C_a(x, m_a, \alpha_\epsilon[m_a]) + \epsilon, \end{aligned}$$

because we are taking the infimum over a smaller class of controls. By the definition of C_a,

$$\begin{aligned} V_a^+(x) &\leq \inf_m C(x, m, \alpha_\epsilon[m]) + \int_\sigma^{\sigma_a} (c + \rho(\alpha_\epsilon[m_a](s), \hat{m}(s)))\, ds + \epsilon \\ &\leq \sup_\alpha \inf_m C(x, m, \alpha[m]) + c_2 a + \epsilon \end{aligned}$$

by the previous argument, where c_2 does not depend on x, ϵ, and b. Because ϵ is arbitrarily small, the proof for $a > 0$ and $V_a^+(x)$ is established. $\square$

PROOF OF LEMMA 5. We suppress b from the notation, throughout the proof. It is obvious that one can restrict the infimum over $\beta \in B$ to the class of strategies β for which $C(x, \beta) \leq V^-(x) + 1$. Within this class, for every β and u,

$$\sigma c \leq \int_0^\sigma [c + \rho(u(t), \beta[u](t))]\, dt \leq V^-(x) + 1,$$

and therefore one always has that $\sigma \leq T_0 \doteq (V(x) + 1)/c$. Lemma 5 asserts an upper bound on the cost until a fixed time T_0, and so we must define the strategy for times $t \in [\sigma, T_0]$. Let $\hat{m}$ be an arbitrary fixed element of M. Then the extended β is just

$$\hat{\beta}[u](t) = \begin{cases} \beta[u](t) & t < \sigma, \\ \hat{m} & t \geq \sigma. \end{cases}$$

With this definition, one has that $C(x, u, \beta[u]) = C(x, u, \hat{\beta}[u])$. One can therefore further restrict to strategies β satisfying $\hat{\beta} = \beta$. For such β, it follows that

$$\int_0^{T_0} \rho(u(t), \beta[u](t))\, dt \leq c_1 T_0,$$

where c_1 does not depend on u, β. The result regarding V^- follows.

Regarding V^+, let m_0 be a control which sets all μ_i and λ_i to zero, except that $\lambda_{i_0} = 1$ for some $i_0 \in \mathcal{J}_+$. Then for any $\alpha \in A$ and m for which

$$C(x, \alpha[m], m) \le C(x, \alpha[m_0], m_0), \tag{61}$$

one has $c\sigma(x, \alpha[m], m) \le C(x, \alpha[m], m) \le C(x, \alpha[m_0], m_0) \le c_1 < \infty$. Note that c_1 can be chosen independent of α because the dynamics and running cost under m_0 are independent of α. Clearly, for each α it suffices to consider, in optimizing over m, only those m that satisfy (61). It follows that it suffices to consider only those m for which $\sigma(x, \alpha[m], m) \le c_1/c$. This completes the proof of the lemma. $\square$

PROOF OF LEMMA 6. Fix $b \in [b^*, \infty]$ which we omit from the notation. Recall from Lemma 4 that $V^{\pm}$ are bounded on G. We first show that V^- is Lipschitz. Assume first that Condition 1.2 holds. Recall that for $x \in G$,

$$V^-(x) = \inf_{\beta} \sup_{u} C(x, \beta[u], u).$$

Let β_ϵ^x be an ϵ-optimal strategy starting from x, i.e.,

$$\sup_{u} C(x, \beta_\epsilon^x[u], u) \le V^-(x) + \epsilon.$$

For any $z \in G$, let $\sigma_z = \inf\{t : \phi_z \notin G\}$, where ϕ_z is the solution to $\dot{\phi} = \pi(\phi, v(u, \beta_\epsilon^x[u]))$, with $\phi(0) = z$. Note that $C(x, u, \beta_\epsilon^x[u]) = \int_0^{\sigma_x} [c + \rho(u(t), \beta_\epsilon^x[u](t))]\, dt$ (with possibly $\sigma_x = \infty$). Now let $y \in G$. Note that on $[0, \sigma_x \wedge \sigma_y]$, one has by the Lipschitz property of the Skorokhod map that $|\phi_x(t) - \phi_y(t)| \le c_1|x - y|$, where c_1 is some constant. Recall that we are considering the case of Condition 1.2. Therefore, at $\sigma_x \wedge \sigma_y$, both ϕ_x and ϕ_y are within a distance of $c_1|x - y|$ of the boundary $\partial_o G$. Because of the assumptions on the domain G, there exists a constant c_2 such that at time $\sigma_x \wedge \sigma_y$, both $\phi_x + c_2 e_{i^*} \notin G$ and $\phi_y + c_2 e_{i^*} \notin G$, where $i^* \in \mathcal{J}_+$ and, moreover, c_2 is independent of $x, y, b \ge b^*$ and i^*.

Define $\beta_\epsilon^{x,y}$ as $\beta_\epsilon^{x,y}[u] = \beta_\epsilon^x[u]$ on $[0, \sigma_x)$, and, if $\sigma_y \ge \sigma_x$, set $\beta_\epsilon^{x,y}[u] = m_0$ on $[\sigma_x, \sigma_y]$. Here, m_0 sets all $\bar{\mu}_i$ and all $\bar{\lambda}_i$ to zero, except that it sets $\bar{\lambda}_{i^*} = \lambda_{i^*}$, where i^* is as above. Consequently, $\sigma_y < \sigma_x + c'|x - y|$ for some $c' > 0$, and there exists u_ϵ such that

$$\begin{aligned}
V^-(y) &\le \sup_u C(y, u, \beta_\epsilon^{x,y}[u]) \\
&\le \int_0^{\sigma_y} [c + \rho(u_\epsilon(s), \beta_\epsilon^{x,y}[u_\epsilon](s))]\, ds + \epsilon \\
&\le \int_0^{\sigma_x} [c + \rho(u_\epsilon(s), \beta_\epsilon^{x,y}[u_\epsilon](s))]\, ds + 1_{\sigma_x < \sigma_y} \int_{\sigma_x}^{\sigma_y} [c + \rho(u_\epsilon(s), \beta_\epsilon^{x,y}[u_\epsilon](s))]\, ds + \epsilon \\
&\le C(x, u_\epsilon, \beta_\epsilon^x[u_\epsilon]) + c_3|x - y| + \epsilon \\
&\le \sup_u C(x, u, \beta_\epsilon^x[u]) + c_3|x - y| + \epsilon \\
&\le V^-(x) + c_3|x - y| + 2\epsilon.
\end{aligned}$$

Because $x, y \in G$ and $\epsilon > 0$ are arbitrary, and c_3 does not depend on them or on b, V^- is Lipschitz, uniformly for $b \in [b^*, \infty]$.

In case that Condition 1.1 holds, the same argument shows that $V_a^-(y) \le V^-(x) + c_3|x - y| + 2\epsilon$, where $a = c_1|x - y|$. By Lemma 4, this implies that $V^-(y) \le V^-(x) + c_4|x - y| + 2\epsilon$, some constant c_4, and therefore V^- is Lipschitz.

Next, consider the upper value

$$V^+(x) = \sup_{\alpha} \inf_{m} C(y, \alpha[m], m)$$

under Condition 1.2. Let $x, y \in G$. Note that there is an α_ϵ^x such that

$$V^+(x) \le \inf_m C(x, \alpha_\epsilon^x[m], m) + \epsilon,$$

and an $m_\epsilon = m_\epsilon(x, y)$ for which

$$\begin{aligned} V^+(y) &\ge \inf_m C(y, \alpha_\epsilon^x[m], m) \\ &\ge C(y, \alpha_\epsilon^x[m_\epsilon], m_\epsilon) - \epsilon. \end{aligned}$$

Let $\sigma_z = \inf\{t : \phi_z \notin G\}$, where ϕ_z is the solution to $\dot\phi = \pi(\phi, v(\alpha_\epsilon^x[m_\epsilon], m_\epsilon))$, with $\phi(0) = z$. Let i^* be defined in an analogous way to that in the first paragraph of the proof. Now define $\bar m_\epsilon = \bar m_\epsilon(x, y)$ as follows. If $\sigma_x \le \sigma_y$, let $\bar m_\epsilon = m_\epsilon$. If $\sigma_y < \sigma_x$, let $\bar m_\epsilon$ agree with m_ϵ on $[0, \sigma_y)$ and with m_0 on $[\sigma_y, \sigma_x]$. Here, m_0 sets all $\bar\mu_i$ and all $\bar\lambda_i$ to zero, except that it sets $\bar\lambda_{i^*} = \lambda_{i^*}$. Because m_ϵ and $\bar m_\epsilon$ agree on $[0, \sigma_y)$, the restrictions to $[0, \sigma_y]$ of $\alpha_\epsilon^x[m_\epsilon]$ and of $\alpha_\epsilon^x[\bar m_\epsilon]$ agree a.e. on $[0, \sigma_y]$, and therefore,

$$C(y, \alpha_\epsilon^x[m_\epsilon], m_\epsilon) = C(y, \alpha_\epsilon^x[\bar m_\epsilon], \bar m_\epsilon).$$

Arguing again by the Lipschitz property of the Skorokhod map and the definition of m_0, there is a constant c_4 for which $(\sigma_x - \sigma_y)^+ \le c_4|x - y|$. Hence,

$$\begin{aligned} V^+(y) &\ge C(y, \alpha_\epsilon^x[\bar m_\epsilon], \bar m_\epsilon) - \epsilon \\ &\ge \int_0^{\sigma_x} [c + \rho(\alpha_\epsilon^x[\bar m_\epsilon^x](s), \bar m_\epsilon^x(s))]\, ds - 1_{\sigma_y < \sigma_x} \int_{\sigma_y}^{\sigma_x} [c + \rho(\alpha_\epsilon^x[\bar m_\epsilon^x](s), \bar m_\epsilon^x(s))]\, ds - \epsilon \\ &\ge C(x, \alpha_\epsilon^x[\bar m_\epsilon], \bar m_\epsilon) - c_5|x - y| - \epsilon \\ &\ge \inf_m C(x, \alpha_\epsilon^x[m], m) - c_5|x - y| - \epsilon \\ &\ge V^+(x) - c_5|x - y| - 2\epsilon. \end{aligned}$$

Because c_5 does not depend on x, y, ϵ, or b, we have that V^+ is Lipschitz uniformly for $b \in [b^*, \infty]$.

Under Condition 1.1, the same argument shows that $V_a^+(y) \ge V^+(x) - c_5|x - y| - 2\epsilon$, where $a = c_1|x - y|$, and again one argues by Lemma 4. $\square$

PROOF OF LEMMA 7. The processes are constructed recursively, using a sequence of standard exponential clocks. Recall that $\mathscr{L}^{n,u,m}$ is given for every n, $u \in U$, $m \in M$ by

$$\mathscr{L}^{n,u,m} f(x) = \sum_{j=1}^J n\bar\lambda_j [f(x + n^{-1} v_j) - f(x)] + \sum_{i=1}^J n\bar\mu_i u_i [f(x + n^{-1}\pi(x, \tilde v_i)) - f(x)].$$

Given n, x_n, and β, we construct a filtered probability space and three processes, $\overline{X}(t)$, $\bar u(t)$, and $\bar m(t)$ (to simplify notation, we do not write the superscript n in the notation of $\overline{X}^n$, $\bar u^n$, and $\bar m^n$) such that (a) $\overline{X}$, $\bar u$, and $\bar m$ are $(\overline{F}_t)$-adapted; (b) $\bar m(t) = \beta[\bar u](t)$ a.e. $t \ge 0$, a.s.; (c) $\bar u(\cdot) = u^n(X(\cdot))$ a.s. (where u^n is as in the statement before the lemma); and (d) for any f, the process

$$f(\overline{X}(t)) - \int_0^t \mathscr{L}^{n, \bar u(s), \bar m(s)} f(\overline{X}(s))\, ds$$

is an $(\overline{F}_t)$-martingale. For (a)–(d) to hold, it suffices that (a)–(c) hold, and (e) on any finite interval the process $\overline{X}$ jumps finitely many times—we denote the kth jump by τ_k and let $\tau_0 = 0$; (f) the random times (τ_k) are stopping times on $(\overline{F}_t)$, and (g) denoting $X_k = \overline{X}(\tau_k)$, for any k,

$$\overline{E}[f(X_{k+1}) - f(X_k) | \overline{F}_{\tau_k}] = \sum_{i=1}^J \overline{E}[A_i^{k, \bar u, \bar m} + B_i^{k, \bar u, \bar m} | \overline{F}_{\tau_k}],$$

where

$$A_i^{k,\bar{u},\bar{m}} = n\int_{\tau_k}^{\tau_{k+1}} \bar{\lambda}_i(s)\,ds[f(X_k + n^{-1}v_i) - f(X_k)],$$

$$B_i^{k,\bar{u},\bar{m}} = n\int_{\tau_k}^{\tau_{k+1}} \bar{\mu}_i(s)u_i(s)\,ds[f(X_k + n^{-1}\pi(X_k,\tilde{v}_i)) - f(X_k)]\ .$$

The construction is recursive. On a complete probability space $(\overline{\Omega}, \overline{F}, \overline{P})$, we are given $2J$ independent i.i.d. standard Poisson processes, denoted a_i and b_i, $i = 1, \ldots, J$. Let $T_i^a(k)$ (respectively, $T_i^b(k)$) denote the first time a_i (respectively, b_i) equals k. For each $\omega \in \Omega$ we construct recursively a sequence of times (τ_k) and the processes $\overline{X}$, $\bar{u}$ and $\bar{m}$ up to time τ_k. Once these processes are defined, we will define $(\overline{F}_t)$, $\overline{F}_t \subset \overline{F}$, $t \ge 0$, and verify that items (a)–(c) and (e)–(g) are satisfied on $(\overline{\Omega}, \overline{F}, (\overline{F}_t), \overline{P})$.

We set $\overline{X}(0) = x_n$ and $\bar{u}(0) = u^n(x_n)$. Because $\bar{m}$ need only be defined almost everywhere on $[0,\infty)$, we do not define it at zero nor at any τ_k, $k = 1, 2, \ldots$. Now assume that we have constructed τ_i, $i \le k$ as well as the processes $\overline{X}$ and $\bar{u}$ on $[0, \tau_k]$ and $\bar{m}$ a.e. on $[0, \tau_k]$. Let $\hat{u}^k(t) = u^n(\overline{X}(t \wedge \tau_k))$, $t \ge 0$. Also let $\hat{m}^k = \beta[\hat{u}^k]$. With $\hat{u}^k(\cdot) = (\hat{u}_i^k(\cdot))$ and $\hat{m}^k(\cdot) = ((\hat{\lambda}_i^k(\cdot)), (\hat{\mu}_i^k(\cdot)))$, let

$$p_i^k(t) = n\int_0^t \hat{\lambda}_i^k(s)\,ds, \quad q_i^k(t) = n\int_0^t \hat{\mu}_i^k(s)\hat{u}_i^k(s)\,ds, \qquad i = 1, \ldots, J,\ t \ge 0.$$

Denoting $\Delta z(s) = z(s) - z(s-)$, also let

$$\tau_{k+1} = \inf\bigl\{t > \tau_k : \text{ either } \Delta a_i(p_i^k(t)) > 0 \text{ or } \Delta b_i(q_i^k(t)) > 0 \text{ for some } i = 1, \ldots, J\bigr\},$$

where $\inf \varnothing = +\infty$. We first consider the case that $\tau_{k+1} < +\infty$. In this case,

$$\text{there is } i \text{ such that either } \Delta a_i(p_i^k(\tau_{k+1})) > 0 \quad \text{or} \quad \Delta b_i(q_i^k(\tau_{k+1})) > 0. \tag{62}$$

In the former case we let $\hat{v}^k = v_i$; otherwise, we let $\hat{v}^k = \tilde{v}_i$.

The three processes are defined on the next interval as follows. Let $\overline{X}(t) = \overline{X}(\tau_k)$ for $t \in (\tau_k, \tau_{k+1})$, and $\overline{X}(\tau_{k+1}) = \overline{X}(\tau_k) + n^{-1}\pi(\overline{X}(\tau_k), \hat{v}^k)$. Let $\bar{u}(t) = u^n(\overline{X}(t))$ for $t \in (\tau_k, \tau_{k+1}]$. Let $\check{u}(t) = u^n(\overline{X}(t \wedge \tau_{k+1}))$ and define $\bar{m}(t) = \beta[\check{u}](t)$, $t \in [0, \tau_{k+1}]$. Note that since β is a strategy, this definition of $\bar{m}$ on $[0, \tau_{k+1}]$ is consistent with its definition on $[0, \tau_k]$, because the same is true for $\bar{u}$. For the same reason, for a.e. $t \le \tau_{k+1}$, $\bar{m}(t) = \hat{m}^k(t)$. In particular, the equations for p_i^k, q_i^k still hold if we replace hats by bars, namely,

$$p_i^k(t) = n\int_0^t \bar{\lambda}_i(s)\,ds, \qquad q_i^k(t) = n\int_0^t \bar{\mu}_i(s)\bar{u}_i(s)\,ds, \qquad i = 1, \ldots, J,\ \tau_k \le t \le \tau_{k+1}. \tag{63}$$

Note that the above relations are consistent in the sense that for a given k, they hold not only for $t \in [\tau_k, \tau_{k+1}]$, but in fact for $t \in [0, \tau_{k+1}]$. Hence, on the event $\tau_k \to \infty$, one can equivalently consider the processes

$$p_i(t) = n\int_0^t \bar{\lambda}_i(s)\,ds, \qquad q_i(t) = n\int_0^t \bar{\mu}_i(s)\bar{u}_i(s)\,ds, \qquad i = 1, \ldots, J,\ t \ge 0. \tag{64}$$

This completes the definition of the three processes on $[0, \tau_{k+1}]$.

In case that $\tau_{k+1} = +\infty$, the definitions above of $\overline{X}$, $\bar{u}$ and $\bar{m}$ all apply on (τ_k, τ_{k+1}), and there is nothing else to define.

To complete the construction of the three processes on $\overline{\Omega} \times [0, +\infty)$, we must consider the set $\overline{\Omega}_0$ of $\omega \in \overline{\Omega}$, for which $\overline{T} \doteq \sup \tau_k$ is finite. We show that this set is $\overline{P}$-null, owing to the fact that the range $\overline{M}^b$ of β consists of bounded functions. Suppose $\overline{T}$ is finite. The construction above defines $\overline{X}$, $\bar{u}$, and $\bar{m}$ on $[0, \overline{T})$. Let $\bar{u}'(t) = \bar{u}(t)$ for $t < \overline{T}$ and define

$\bar{u}'(t)$ arbitrarily on $[\overline{T}, +\infty)$ but such that $\bar{u}' \in \overline{U}$. Then $\bar{m}' = \beta[\bar{u}']$ agrees with $\bar{m}$ a.e. on $[0, \overline{T}]$. Because each component of $\bar{m}'$ is bounded by b,

$$n^{-1} \max_{i=1}^{J} \big[p_i(\overline{T}) \vee q_i(\overline{T})\big] \le 2J\overline{T}b < +\infty. \tag{65}$$

However, by construction, $\overline{T} < \infty$ implies that either $a_i(p_i(t)) \to \infty$ or $b_i(q_i(t)) \to \infty$ as $t \uparrow \overline{T}$, for some i. Hence, $\overline{T} < \infty$ must be a null set. We let $\overline{X}$, $\bar{u}$, and $\bar{m}$ be defined arbitrarily on Ω_0.

The definition of the process $\overline{Y}$ is similar to that of $\overline{X}$, but where $\pi(x, v)$ is replaced by v throughout. The relation $\overline{X} = \Gamma(\overline{Y})$ is clear from the construction.

Define for each $t \ge 0$, $\overline{F}_t$ to be the σ-field generated by $\{\overline{Y}(s), s \in [0, t]\}$. Note that it is equivalently defined as the σ-field generated by $\{a_i(p_i(t)), b_i(q_i(t)), i = 1, \dots, J\}$, where p_i, q_i are as in (64). By construction, $\bar{u}(t) = u^n(\overline{X}(t))$, $t \ge 0$ and item (c) holds. Item (b), namely that $\bar{m} = \beta[\bar{u}]$, also holds by construction. $\overline{X}$ and $\bar{u}$ are therefore $(\overline{F}_t)$-adapted, and since β is a strategy, so is $\bar{m}$, and item (a) holds. Items (e) and (f) are trivial. Concerning (g), let $i^k \in \{1, \dots, 2J\}$ denote the index i satisfying (62) in case that $\Delta a_i(p_i^k(t)) > 0$ holds, and let it denote $i + J$ in the case $\Delta b_i^k(q_i^k(t)) > 0$. It suffices to show that for every $i \in \{1, \dots, 2J\}$,

$$\overline{P}(i^k = i \,|\, \overline{F}_{\tau_k}) = \begin{cases} \overline{E}\Big[\int_{\tau_k}^{\tau_{k+1}} p_i(s)\, ds \Big| \overline{F}_{\tau_k}\Big] \Big/ Z_k & i \le J, \\ \overline{E}\Big[\int_{\tau_k}^{\tau_{k+1}} q_{i-J}(s)\, ds \Big| \overline{F}_{\tau_k}\Big] \Big/ Z_k & i > J, \end{cases}$$

where Z_k is a normalization factor (not depending on i). For $k = 0$ ($\tau_k = 0$), this is a well-known property of exponential clocks. For $k > 0$, the same argument holds, merely because conditional on $\overline{F}_{\tau_k}$, the processes $\int_{\tau_k}^{\cdot} p_i(s)\, ds$, $\int_{\tau_k}^{\cdot} q_i(s)\, ds$ are independent, and moreover, $a_i(\cdot - \tau_k) - a_i(\tau_k)$, $b_i(\cdot - \tau_k) - b_i(\tau_k)$ are still independent Poisson processes (which is a statement on the lack of memory for exponential random variables).

The proof of the claim regarding the martingale associated with $\mathscr{L}_0$ is similar (only simpler). This completes the proof of the first part of the lemma.

Clearly,

$$\max_i p_i(T_0) \vee q_i(T_0) \le nT_0 b,$$

where T_0 is as in Lemma 5. Thus, if $N_n = \max\{k : \tau_k \le T_0\}$, then

$$N_n \le \sum_i a_i(nT_0 b) + b_i(nT_0 b),$$

and (14) follows. $\square$

PROOF OF LEMMA 8. The proof is completely analogous to that of Lemma 7 and is therefore omitted. $\square$

PROOF OF THEOREM 4. By Theorems 5 and 6, $V^{b,-} = V^{b,+}$ for all $b \in [b^*, \infty)$. As a result, Theorem 3 implies that $V^n \to V^{b,-}$ for all $b \in [b^*, \infty)$, as $n \to \infty$. In particular, $V^{b,-}$ does not depend on $b \in [b^*, \infty)$. It remains to show that for all x, $V^{b,-}(x) \to V^-(x)$ and $V^{b,+}(x) \to V^+(x)$ as $b \to \infty$.

Proof that $V^{b,-} \to V^-$. It is immediate from the definitions that $V^- \le V^{b,-}$.

Let $\beta \in B$, and let $\sigma = \sigma(x, u, \beta)$ be the exit time of ϕ from G where $\dot{\phi} = \pi(\phi, v(u, \beta[u]))$, $\phi(0) = x$. Let $\bar{\beta}$ be defined by

$$\bar{\beta}[u](t) = \begin{cases} \min\{b, \beta[u](t)\} & t \le \sigma, \\ \hat{m} & t > \sigma, \end{cases}$$

where $\hat{m}$ sets all $\bar{\mu}_j = 0$ and $\bar{\lambda}_j = 1_{j=i_{\phi(\sigma)}}$, and the minimum is componentwise. It is clear that $\bar{\beta}$ is a strategy. Let $\bar{u}$ be any extension of u to $[0,\infty)$, and denote by $\bar{\phi}$ and $\bar{\sigma}$ the dynamics and exit time corresponding to $x, \bar{\beta}, \bar{u}$. Recall that by (9) b is greater than all λ_i and μ_i. Thus

$$\begin{aligned} C(x,\bar{u},\bar{\beta}[\bar{u}]) &= \int_0^{\sigma} (c+\rho(u,\bar{\beta}[u])\,ds + 1_{\bar{\sigma}>\sigma}\int_{\sigma}^{\bar{\sigma}} (c+\rho(\bar{u},\hat{m}))\,ds \\ &\le C(x,u,\beta[u]) + c_1(\bar{\sigma}-\sigma)^+. \end{aligned}$$

Moreover, by the Lipschitz property of the Skorokhod map, and denoting $\beta[u] = (\bar{\lambda}_i, \bar{\mu}_i)$,

$$(\bar{\sigma}-\sigma)^+ \le c_2|\phi(\sigma)-\bar{\phi}(\sigma)| \le c_2\int_0^{\sigma}\sum_i [(\bar{\lambda}_i - b)^+ + (u_i\bar{\mu}_i - b)^+]\,ds.$$

Because it is enough to consider β for which (for any u) λ_i and $u_i\mu_i$ are uniformly integrable over $[0,\sigma]$, we have that $(\bar{\sigma}-\sigma)^+ \le \delta(b)$, where $\delta(b)\to 0$ as $b\to\infty$. This shows that $\lim_{b\to\infty} V^{b,-}(x) \le V^-(x)$.

Proof that $V^{b,+}\to V^+$. It is immediate that $V^+ \le V^{b,+}$.

To show that $V^+(x) \ge \lim_{b\to\infty} V^{b,+}(x)$ it is enough to show that for $b \ge b^*$ and a small, $V^+(x) \ge V^{b,+}_{-a}(x)$. For any $m \in \overline{M}$, let m^b denote the pointwise and componentwise truncation of m at level b. For any $\alpha \in A$, let $\alpha^b \in A$ be defined by $\alpha^b[m] = \alpha[m^b]$. We will write $m \in \overline{M}(\alpha, a)$ if m, α, a satisfy $C_a(x,\alpha[m],m) \le V_a^+(x)+1$. In the expression for $V_a^+(x)$,

$$\sup_{\alpha}\inf_{m} C_a(x,\alpha[m],m),$$

it is enough to consider $\alpha \in A$ and $m \in \overline{M}(\alpha,a)$ (including for $a=0$). For such α, m, the functions $\bar{\lambda}_i, u_i\bar{\mu}_i$ are uniformly integrable over $[0,T]$. Let $\alpha\in A$ and $m \in \overline{M}(\alpha^b,0)$. Consider a truncation of $\bar{\lambda}_i$ and $\bar{\mu}_i$ at b. Denote by ϕ (respectively, ϕ^b) the dynamics that correspond to (x,α^b,m), (respectively, (x,α^b,m^b)). Then the effect of the truncation on ϕ is such that for all $a>0$, there is b such that $\sup_{[0,T]}|\phi-\phi^b| \le a$ (by uniform integrability). In particular, $|\phi^b(\sigma)-\phi(\sigma)| \le a$. Hence, using the monotonicity of the running cost for large values of the rates, and that $\alpha^b[m^b] = \alpha^b[m]$,

$$C_{-a}(x,\alpha^b[m^b],m^b) \le C(x,\alpha^b[m],m).$$

We therefore have

$$C_{-a}(x,\alpha[m^b],m^b) \le C(x,\alpha^b[m],m).$$

Because $m \in \overline{M}(\alpha^b,0)$ implies that $m^b \in \overline{M}(\alpha^b,-a)$,

$$\begin{aligned} \inf_{m\in\overline{M}^b} C_{-a}(x,\alpha[m],m) &\le \inf_{m:m^b\in\overline{M}(\alpha^b,-a)} C_{-a}(x,\alpha[m^b],m^b) \\ &\le \inf_{m\in\overline{M}(\alpha^b,0)} C(x,\alpha^b[m],m) \\ &= \inf_{m\in\overline{M}} C(x,\alpha^b[m],m). \end{aligned}$$

Hence,

$$\sup_{\alpha\in A}\inf_{m\in\overline{M}^b} C_{-a}(x,\alpha[m],m) \le \sup_{\alpha\in A}\inf_{m\in\overline{M}} C(x,\alpha[m],m).$$

Taking $a\to 0$ by letting $b\to\infty$, we have from Lemma 4 that $\lim_b V^{b,+}(x) \le V^+(x)$. $\square$

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