Robust and Consistent Sampling

Tsvi G. Dvorkind and Yonina C. Eldar, *Senior Member, IEEE*

***Abstract*—We address a sampling problem in which the goal is to approximate a signal from its nonideal (generalized) samples. The reconstruction is constrained to lie in a subspace, and to be consistent with the measured samples. It is well known how to obtain a consistent approximation, if the sampling and reconstruction spaces satisfy a certain direct sum condition. Here we show when consistency can be achieved without the need for this condition. The proposed solution provides geometrical insight into the structure of the problem and extends previous treatments by finding a consistent and simultaneously robust approximation of the signal from its samples.**

***Index Terms*—Consistent sampling, generalized sampling, robust approximations.**

I. Introduction

DIGITAL signal processing entails representing a signal by a set of discrete numbers and relies on the existence of methods for reconstructing the signal from these samples. The most common setting is the one introduced by the Shannon–Whittaker theorem, in which the input signal is bandlimited and the samples are ideal, i.e., they are point-wise evaluations of the continuous-time function at a set of sampling points. A more recent approach is to consider generalized sampling, in which the samples are modelled as inner products of the input $\mathbf{x}$ with a set of sampling vectors associated with the acquisition device [1]–[4]. Reconstruction is obtained by forming linear combinations of given reconstruction vectors, spanning a reconstruction space $\mathcal{W}$, which is possibly different than the sampling space.

In this framework, the reconstructed signal $\hat{\mathbf{x}}$ is constrained to lie in $\mathcal{W}$. Therefore, if the input $\mathbf{x}$ does not lie in $\mathcal{W}$, then perfect reconstruction cannot be achieved. A natural question, is whether the samples can be processed so that $\hat{\mathbf{x}} \in \mathcal{W}$ is close to $\mathbf{x}$ in some sense.

A common approach is to form an approximation $\hat{\mathbf{x}}$ which is consistent with the measured samples. Namely, $\hat{\mathbf{x}}$ yields the same samples as $\mathbf{x}$, when re-injected into the acquisition device. This idea of consistency was first introduced in [1] for the shift invariant setup, and was then extended to arbitrary Hilbert spaces in [5]. In previous works, consistency was treated under a direct-sum condition on the sampling and reconstruction spaces. In this case, the consistent reconstruction is unique and is given by an oblique projection operation. An extension, which does not require the direct sum assumption but assumes a finite-dimensional Hilbert space setting was developed in [6]. There, the solution was obtained by oblique-projecting the samples onto a prespecified portion of the reconstruction space. However, the exact choice of the subset within the reconstruction space was assigned to the user, and the form of all consistent solutions in such a setting was not derived.

Another strategy to approximate the input, is to consider a reconstruction which is closest to $\mathbf{x}$ in the squared-norm sense. Since the reconstruction is constrained to $\mathcal{W}$, the best squared-norm approximation is the orthogonal projection $\hat{\mathbf{x}} = P_{\mathcal{W}}\mathbf{x}$. Unfortunately, for general sampling and reconstruction spaces, the minimal squared-norm error cannot be obtained over the entire Hilbert space of input signals [7]. Instead, in [7], a minimax regret approximation was considered, where $\hat{\mathbf{x}}$ was optimized to approximate $P_{\mathcal{W}}\mathbf{x}$ over the group of all possible inputs. Although the solution in [7] is robust, in general, it is not consistent with the measured samples.

In this letter, we reexamine the ideas of consistency and robustness. Our emphasis is on the case where a consistent solution exists, but may be nonunique. We begin by developing necessary and sufficient conditions on the input signal to ensure that it has a consistent reconstruction. We then characterize the class of all consistent solutions. Similarly to [6], our derivations apply to finite-dimensional Hilbert spaces, and do not require a direct sum condition between the spaces. However, in contrast to [6], our strategy does not require the user to specify a portion of the reconstruction space in order to retrieve a consistent solution. Instead, we propose a closed form procedure for obtaining a robust approximation of the input signal, which is also consistent with the samples.

After stating the notations, we formulate the consistency and robustness problems in Section II. In Section III we derive a necessary and sufficient condition to have a consistent solution, and characterize the set of all such solutions. In Section IV we develop a robust approximation of a signal in a subspace, which also turns out to be consistent with the samples. An example is presented in Section V, followed by concluding remarks.

II. Problem Formulation

Notations

We denote vectors in a Hilbert space $\mathcal{H}$ by bold lowercase letters. The elements of a sequence $c \in \mathbb{C}^N$ will be written with square brackets, e.g., $c[n]$. Operators are denoted by upper case letters. The operator $P_{\mathcal{W}}$ represents the orthogonal projection onto a closed subspace $\mathcal{W}$, and $\mathcal{W}^{\perp}$ is the orthogonal complement of $\mathcal{W}$. We denote by $P_{\mathcal{W},\mathcal{S}^{\perp}}$ an oblique (i.e., not necessarily orthogonal) projection operator [8] with range space $\mathcal{W}$ and null space $\mathcal{S}^{\perp}$. It is defined as the unique operator satisfying $P_{\mathcal{W},\mathcal{S}^{\perp}}\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathcal{W}$ and $P_{\mathcal{W},\mathcal{S}^{\perp}}\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{S}^{\perp}$.

Manuscript received January 15, 2009; revised April 28, 2009. First published May 26, 2009; current version published June 24, 2009. This work was supported by the Israel Science Foundation under Grant 1081/07 and by the European Commission in the framework of the FP7 Network of Excellence in Wireless COMmunications NEWCOM++ (Contract 216715).

T. G. Dvorkind is with the Rafael Company, Haifa 2250, Israel (e-mail: dvorkind@gmail.com).

Y. C. Eldar is with the Department of Electrical Engineering Technion—Israel Institute of Technology, Haifa 32000, Israel (e-mail: yonina@ee.technion.ac.il).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LSP.2009.2023481

The Moore–Penrose pseudo inverse [9] and the adjoint of a bounded transformation T are written as $T^\dagger$ and T^*, respectively. Inner products and norms are denoted by $\langle \mathbf{a}, \mathbf{b} \rangle$ and $||\mathbf{a}||$. All inner products are linear with respect to the second argument. An easy way to describe linear combinations and inner products is by utilizing set transformations. A set transformation S corresponding to frame [10] vectors $\{\mathbf{s}_n(t)\}_{n=1}^N$ defines a linear combination $Sc = \sum_n c[n]\mathbf{s}_n(t)$ for all $c \in \mathbb{C}^N$. From the definition of the adjoint, if $c = S^*\mathbf{x}$, then $c[n] = \langle \mathbf{s}_n, \mathbf{x} \rangle$. We define $\{\mathbf{s}_n\}_{n=1}^N$ ($\{\mathbf{w}_n\}_{n=1}^M$) to be the sampling (reconstruction) vectors, which form a frame for their span, denoted as the sampling (reconstruction) space $\mathcal{S}$ ($\mathcal{W}$). We define by S (W) the corresponding set transformation.

A direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2$ between two closed subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ of a Hilbert space $\mathcal{H}$ is the sum set $\mathcal{W}_1 + \mathcal{W}_2 = \{\mathbf{w}_1 + \mathbf{w}_2; \mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2\}$ with the property $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$.

A. Generalized Sampling

Suppose we are given the generalized samples [1], [3] of some function $\mathbf{x}$ in a finite-dimensional Hilbert space $\mathcal{H}$, described by a set of inner products

$$c[n] = \langle \mathbf{s}_n, \mathbf{x} \rangle, 1 \leq n \leq N. \tag{1}$$

Equivalently, using operator notation, $c = S^*\mathbf{x}$. For example, an analog to digital converter which is modelled by a filter with impulse response $\mathbf{h}(t)$ followed by ideal sampling at times $t_n = 1, \ldots, N$ fits this description with $\mathbf{s}_n(t) = \mathbf{h}(n - t)$ [1]. To approximate $\mathbf{x}$ from its samples, we construct a function $\hat{\mathbf{x}}$ which lies within some predefined reconstruction space $\mathcal{W} \subseteq \mathcal{H}$. We focus on finding a consistent reconstruction [1], [3] of $\mathbf{x}$, *i.e.,* a function $\hat{\mathbf{x}} \in \mathcal{W}$ such that it gives the same samples when re-injected into the acquisition device:

$$c = S^*\hat{\mathbf{x}}.$$

It is well known [1], [3], [5] that if the sampling and reconstruction spaces satisfy the direct sum condition

$$\mathcal{W} \oplus \mathcal{S}^\perp = \mathcal{H} \tag{2}$$

then a unique consistent solution exists, which is given by an oblique projection operation

$$\hat{\mathbf{x}} = P_{\mathcal{W},\mathcal{S}^\perp}\mathbf{x} = P_{\mathcal{W},\mathcal{S}^\perp}\mathbf{x}_s. \tag{3}$$

Here we denote

$$\mathbf{x}_s = P_{\mathcal{S}}\mathbf{x} = S\left(S^*S\right)^\dagger c \tag{4}$$

the orthogonal projection of $\mathbf{x}$ onto the sampling space, which is uniquely determined by the samples c.

In this letter we treat the case in which (2) does not necessarily hold. For such setups, a consistent solution may not exist, or there may be infinitely many consistent reconstructions. Assuming the latter, we also consider obtaining a robust approximation of the unknown signal.

If $\mathbf{x}$ lies in the reconstruction space $\mathcal{W}$, then, assuming (2), $\hat{\mathbf{x}}$ of (3) is also the perfect reconstruction of $\mathbf{x}$. If $\mathbf{x} \in \mathcal{W}$, but the direct sum condition is not satisfied, then perfect reconstruction of $\mathbf{x}$ cannot be obtained since the input signal is no longer uniquely determined by its samples. For example, if $\mathcal{W} \cap \mathcal{S}^\perp \neq \{0\}$, then for any nonzero $\mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp$, both $\mathbf{x}$ and $\mathbf{x} + \mathbf{v}$ produce the same samples. In this setup, we take a robust approach and among all finite norm functions $\mathbf{x} \in \mathcal{W}$, which can produce the known samples, we find an approximation $\hat{\mathbf{x}}$ which minimizes the worst case error. Thus, we seek $\hat{\mathbf{x}}$ that is the solution to the minimax objective

$$\min_{\hat{\mathbf{x}}} \max_{\mathbf{x} \in \mathcal{C}} ||\hat{\mathbf{x}} - \mathbf{x}||^2 \tag{5}$$

where we define

$$\mathcal{C} = \{\mathbf{x} | c = S^*\mathbf{x}, \mathbf{x} \in \mathcal{W}, ||\mathbf{x}|| \leq B\} \tag{6}$$

to be the set of all finite norm functions, consistent with our data. As we will show, the solution $\hat{\mathbf{x}}$ of (5) is not a function of the norm bound B, and it turns out to be an element of the set $\mathcal{C}$. Consequently, in addition to obtaining a robust approximation of the input, our solution is also consistent.

III. Consistency Revised

As we have seen, if the direct sum condition (2) holds, then there is a unique consistent solution. Unfortunately, in general, (2) might be violated. In the following proposition, we show that the largest group of signals for which consistent approximations in $\mathcal{W}$ exist, is given by $\mathcal{W} + \mathcal{S}^\perp$.

Proposition 1: Let $c = S^*\mathbf{x}$ be the samples of some $\mathbf{x}$ in a finite-dimensional Hilbert space $\mathcal{H}$. A necessary and sufficient condition to have a consistent approximation of $\mathbf{x}$ in $\mathcal{W} \subseteq \mathcal{H}$, *i.e.,* a function $\hat{\mathbf{x}} \in \mathcal{W}$ such that $c = S^*\hat{\mathbf{x}}$, is

$$\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp. \tag{7}$$

Proof: To show sufficiency, take an $\mathbf{x} \in \mathcal{W} + \mathcal{S}^\perp$ which produces the samples $c = S^*\mathbf{x}$. Thus, there is a decomposition $\mathbf{x} = \mathbf{w} + \mathbf{v}$, with $\mathbf{w} \in \mathcal{W}$ and $\mathbf{v} \in \mathcal{S}^\perp$. As a result, $c = S^*\mathbf{w}$, so that $\mathbf{w}$ is a consistent solution within $\mathcal{W}$.

To show necessity, take some $\mathbf{x}$ for which there is a consistent solution, *i.e.,* if $c = S^*\mathbf{x}$ then there exists $\mathbf{w} \in \mathcal{W}$ such that $c = S^*\mathbf{w}$. By the frame assumption on the sampling functions, since $\mathbf{x}$ and $\mathbf{w}$ produce the same samples $P_{\mathcal{S}}\mathbf{x} = P_{\mathcal{S}}\mathbf{w}$, or equivalently, $\mathbf{x} - \mathbf{w} \in \mathcal{S}^\perp$. This leads to $\mathbf{x} \in \mathcal{W} + \mathcal{S}^\perp$, as required. ■

Assuming that (7) holds but $\mathcal{W} \cap \mathcal{S}^\perp \neq \{0\}$, a consistent solution in $\mathcal{W}$ exists, but it is no longer unique. Indeed, if $\mathbf{w} \in \mathcal{W}$ is a consistent solution, then so is $\mathbf{w} + \mathbf{v}$ for any $\mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp$. Furthermore, in such a setup, the oblique projection operator (3) is no longer well defined and it is not clear how to find a consistent approximation of $\mathbf{x}$ from its samples.

This problem was first considered in [6], where it was suggested to extend the form (3), by oblique projecting $\mathbf{x}_s$ along $\mathcal{S}^\perp$ onto a portion $L \subset \mathcal{W}$ of the reconstruction space which satisfies the direct sum condition: $L \oplus \mathcal{S}^\perp = \mathcal{H}$. The choice of L is user dependent, with the ability to incorporate predefined reconstruction vectors of interest into this subspace.

In order to free the user from choosing the subspace $L \subset \mathcal{W}$, we suggest a different strategy, which also leads to geometrical insight into the problem. Instead of specifying a portion of the reconstruction space, and oblique-projecting onto it, we first study the geometrical form of all consistent solutions. As we show, this enables us to choose a consistent approximation $\hat{\mathbf{x}}$ which is the "center" of all possible solutions. In the following theorem, we characterize all consistent solutions in $\mathcal{W}$,

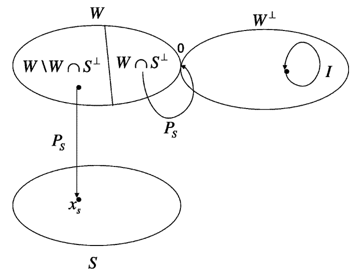

Fig. 1. Illustration of $I - P_{\mathcal{S}^\perp} P_{\mathcal{W}}$. Inverting $I - P_{\mathcal{S}^\perp} P_{\mathcal{W}}$ on $\mathbf{x}_s$ is equivalent to retrieving the origin of $\mathbf{x}_s$ within $\mathcal{W} \backslash \mathcal{W} \cap \mathcal{S}^\perp$.

assuming (as in [6]) that our Hilbert space satisfies only the sum condition

$$\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp. \tag{8}$$

Note that due to Proposition 1, (8) implies that for any $\mathbf{x} \in \mathcal{H}$ a consistent approximation exists.

Theorem 1: Assume that a finite-dimensional Hilbert space $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp$. Then, the set of consistent solutions in $\mathcal{W}$ is

$$F^\dagger \mathbf{x}_s + \mathcal{W} \cap \mathcal{S}^\perp \tag{9}$$

where $\mathbf{x}_s = P_{\mathcal{S}} \mathbf{x}$ and $F = I - P_{\mathcal{S}^\perp} P_{\mathcal{W}}$.

Before proving the theorem, we discuss the role of the operator F. It is not hard to show that

$$F\mathbf{f} = \begin{cases} \mathbf{f}, & \forall\, \mathbf{f} \in \mathcal{W}^\perp \\ P_{\mathcal{S}} \mathbf{f}, & \forall\, \mathbf{f} \in \mathcal{W}. \end{cases}$$

Furthermore, under condition (8), the orthogonal projection of $\mathcal{W}$ to $\mathcal{S}$ is onto (*i.e.,* $F(\mathcal{W}) = \mathcal{S}$). Indeed, for any $\mathbf{s} \in \mathcal{S}$ there is some $\mathbf{s}_w \in \mathcal{W}$ and $\mathbf{s}_v \in \mathcal{S}^\perp$ such that $\mathbf{s} = \mathbf{s}_w + \mathbf{s}_v$. Therefore, $\mathbf{s} = P_{\mathcal{S}} \mathbf{s} = P_{\mathcal{S}} \mathbf{s}_w = F \mathbf{s}_w$. Thus, the operator F maps $\mathcal{W}$ onto the sampling space by an orthogonal projection, and $\mathcal{N}(F) = \mathcal{W} \cap \mathcal{S}^\perp$, as illustrated in Fig. 1. We also comment that due to our finite-dimensional setting, the range of F is closed, so that the pseudo-inverse of F is well defined.

Proof: To see that the set of consistent solutions is given by (9), we first take $\mathbf{f} = F^\dagger \mathbf{x}_s + \mathbf{v}$ for some $\mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp$. We need to show that $c = S^* \mathbf{f}$ (or equivalently, due to the frame assumption, $\mathbf{x}_s = P_{\mathcal{S}} \mathbf{f}$) and also $\mathbf{f} \in \mathcal{W}$. Indeed, (8) implies that there exists $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{x}_s = P_{\mathcal{S}} \mathbf{w} = F\mathbf{w}$ and we may rewrite

$$\mathbf{f} = F^\dagger F \mathbf{w} + \mathbf{v} = P_{\mathcal{N}^\perp(F)} \mathbf{w} + \mathbf{v}. \tag{10}$$

Now, $P_{\mathcal{N}^\perp(F)} \mathbf{w} = \mathbf{w} - P_{\mathcal{N}(F)} \mathbf{w} = \mathbf{w} - P_{\mathcal{W} \cap \mathcal{S}^\perp} \mathbf{w}$. Since $P_{\mathcal{N}^\perp(F)} \mathbf{w}$ is the difference of two vectors in $\mathcal{W}$, $P_{\mathcal{N}^\perp(F)} \mathbf{w} \in \mathcal{W}$. From (10), this implies that $\mathbf{f} \in \mathcal{W}$. In addition,

$$\begin{aligned} P_{\mathcal{S}} \mathbf{f} &= F\mathbf{f} = F(F^\dagger \mathbf{x}_s + \mathbf{v}) = F F^\dagger \mathbf{x}_s \\ &= F F^\dagger F \mathbf{w} = F\mathbf{w} = \mathbf{x}_s \end{aligned}$$

as required.

In the other direction, we need to show that if we take some $\mathbf{w} \in \mathcal{W}$ which is consistent with the samples, then it is of the form $F^\dagger \mathbf{x}_s + \mathbf{v}$ for some $\mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp$. Indeed, decomposing $\mathbf{w}$ along $\mathcal{W} \cap \mathcal{S}^\perp$ and $\left(\mathcal{W} \cap \mathcal{S}^\perp\right)^\perp$ we have $\mathbf{w} = P_{(\mathcal{W} \cap \mathcal{S}^\perp)^\perp} \mathbf{w} + \mathbf{v}$ for some $\mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp$. Thus,

$$\begin{aligned} \mathbf{w} &= P_{\mathcal{N}^\perp(F)} \mathbf{w} + \mathbf{v} = F^\dagger F \mathbf{w} + \mathbf{v} \\ &= F^\dagger P_{\mathcal{S}} \mathbf{w} + \mathbf{v} = F^\dagger \mathbf{x}_s + \mathbf{v} \end{aligned}$$

where we used $\mathbf{x}_s = P_{\mathcal{S}} \mathbf{w}$ in the last equality. ■

IV. Robust Approximation on a Subspace

We now consider the case where $\mathbf{x} \in \mathcal{W}$. If $\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp$ but $\mathcal{W} \cap \mathcal{S}^\perp \neq \{0\}$, then in general $\mathbf{x}$ cannot be perfectly reconstructed from its samples, since there are many functions $\mathbf{x} \in \mathcal{W}$ which can produce the known samples. Instead, we seek a robust approximation $\hat{\mathbf{x}}$ as in (5).

Theorem 2: Suppose that a finite-dimensional Hilbert space $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{W} + \mathcal{S}^\perp$. Then the solution of

$$\min_{\hat{\mathbf{x}}} \max_{\mathbf{x} \in \mathcal{C}} \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \tag{11}$$

with $\mathcal{C} = \{\mathbf{x} | c = S^* \mathbf{x}, \mathbf{x} \in \mathcal{W}, \|\mathbf{x}\| \leq B\}$ is given by

$$\hat{\mathbf{x}} = F^\dagger \mathbf{x}_s = \left(I - P_{\mathcal{S}^\perp} P_{\mathcal{W}}\right)^\dagger \mathbf{x}_s. \tag{12}$$

Proof: By Theorem 1, we may rewrite the consistency set $\mathcal{C}$ as

$$\mathcal{C} = \left\{ F^\dagger \mathbf{x}_s + \mathbf{v} | \mathbf{v} \in \mathcal{W} \cap \mathcal{S}^\perp, \|\mathbf{v}\| \leq \tilde{B} \right\}.$$

The norm bound $\tilde{B}$ can be calculated from the original norm bound B of $\mathbf{x}$. Indeed, $\|\mathbf{x}\| = \left\| F^\dagger \mathbf{x}_s + \mathbf{v} \right\| \leq B$. Since $F^\dagger \mathbf{x}_s = F^\dagger F \mathbf{w} = P_{\mathcal{N}^\perp(F)} \mathbf{w}$ for some consistent $\mathbf{w} \in \mathcal{W}$, and $\mathbf{v} \in \mathcal{N}(F)$, we must have that $\|\mathbf{v}\|^2 \leq \tilde{B}^2 = B^2 - \left\| F^\dagger \mathbf{x}_s \right\|^2$.

The proof can now be directly obtained from geometric considerations, as the maximization within (11) is carried over a shifted balanced set. From the set theoretic literature (e.g., [11], [12]) it is known that the minimax center of a shifted balanced set is given by its shift vector. For completeness, however, we derive the solution explicitly. We first rewrite

$$\max_{\mathbf{x} \in \mathcal{C}} \|\hat{\mathbf{x}} - \mathbf{x}\|^2 = \max_{\mathbf{v} \in \mathcal{G}} \left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s - \mathbf{v} \right\|^2 \tag{13}$$

by defining $\mathcal{G} = \left\{ \mathbf{v} : \mathbf{v} \in \mathcal{S}^\perp \cap \mathcal{W}, \|\mathbf{v}\| \leq \tilde{B} \right\}$. Now, (13) equals

$$\max_{\mathbf{v} \in \mathcal{G}} \max \left\{ \left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s - \mathbf{v} \right\|^2, \left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s + \mathbf{v} \right\|^2 \right\}$$

because if $\mathbf{v} \in \mathcal{G}$, then also $-\mathbf{v} \in \mathcal{G}$. Lower bounding the maximal value of two numbers by their mean, we obtain

$$\begin{aligned} &\max_{\mathbf{x} \in \mathcal{C}} \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \\ &\quad \geq \max_{\mathbf{v} \in \mathcal{G}} \frac{1}{2} \left(\left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s - \mathbf{v} \right\|^2 + \left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s + \mathbf{v} \right\|^2 \right) \\ &\quad = \max_{\mathbf{v} \in \mathcal{G}} \left(\left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s \right\|^2 + \|\mathbf{v}\|^2 \right) = \left\| \hat{\mathbf{x}} - F^\dagger \mathbf{x}_s \right\|^2 + \tilde{B}^2. \end{aligned} \tag{14}$$

As a result, $\min_{\hat{\mathbf{x}}} \max_{\mathbf{x} \in \mathcal{C}} \|\hat{\mathbf{x}} - \mathbf{x}\|^2 \geq \tilde{B}^2$ since with $\hat{\mathbf{x}} = F^\dagger \mathbf{x}_s$ we minimize the lower bound. Finally, the lower bound is tight, since it is achieved by $\hat{\mathbf{x}} = F^\dagger \mathbf{x}_s$. ■

Note that the minimax solution $\hat{\mathbf{x}} = F^{\dagger}\mathbf{x}_s$ of (11) is independent of the norm bound B. Hence, the set $\mathcal{C}$ of possible input signals may have arbitrarily large norm. Furthermore, since $\hat{\mathbf{x}}$ is an element of the consistency set $\mathcal{C}$, the minimax solution is also consistent. Therefore, solving (11) leads to a consistent and robust approximation of the input signal.

We now establish a link between the minimax consistent solution $\hat{\mathbf{x}} = F^{\dagger}\mathbf{x}_s$ and [6]. Specifically, we show that $\hat{\mathbf{x}}$ corresponds to oblique projecting $\mathbf{x}_s$ onto a portion $L \subseteq \mathcal{W}$ of the reconstruction space, where the subspace L is determined by orthogonally projecting the sampling space $\mathcal{S}$ onto $\mathcal{W}$.

Theorem 3: Under the conditions of Theorem 2, the solution $\hat{\mathbf{x}} = F^{\dagger}\mathbf{x}_s$ of (11) can be also expressed as

$$\hat{\mathbf{x}} = P_{L,\mathcal{S}^{\perp}}\mathbf{x}_s \tag{15}$$

where $L = P_{\mathcal{W}}(\mathcal{S})$ is the orthogonal projection of $\mathcal{S}$ onto $\mathcal{W}$.

Proof: We first show that the oblique projection operator $P_{L,\mathcal{S}^{\perp}}$ is well defined, by proving that $L \oplus \mathcal{S}^{\perp} = \mathcal{H}$. Indeed, let $\mathbf{w} \in L \cap \mathcal{S}^{\perp}$. Then, there is some $\mathbf{s} \in \mathcal{S}$ such that $\mathbf{w} = P_{\mathcal{W}}\mathbf{s}$. Since $\mathbf{w} \in \mathcal{S}^{\perp}$ we must have that $0 = \langle \mathbf{w}, \mathbf{s}\rangle = \langle P_{\mathcal{W}}\mathbf{s}, \mathbf{s}\rangle = \|P_{\mathcal{W}}\mathbf{s}\|^2 = \|\mathbf{w}\|^2$ implying that $\mathbf{w} = 0$, i.e., $L \cap \mathcal{S}^{\perp} = \{0\}$. To show that $L + \mathcal{S}^{\perp} = \mathcal{H}$ it is equivalent to prove that the perpendicular subspace is trivial, i.e., $(L + \mathcal{S}^{\perp})^{\perp} = L^{\perp} \cap \mathcal{S} = \{0\}$. Indeed, if $\mathbf{s} \in L^{\perp} \cap \mathcal{S}$, then also $0 = \langle P_{\mathcal{W}}\mathbf{s}, \mathbf{s}\rangle = \|P_{\mathcal{W}}\mathbf{s}\|^2$, such that $\mathbf{s} \in \mathcal{W}^{\perp}$. But the assumption $\mathcal{H} = \mathcal{W} + \mathcal{S}^{\perp}$ is equivalent to $\mathcal{W}^{\perp} \cap \mathcal{S} = \{0\}$, implying $\mathbf{s} = 0$.

Since $\hat{\mathbf{x}} = F^{\dagger}\mathbf{x}_s$ is consistent with the samples, and the direct sum condition $L \oplus \mathcal{S}^{\perp} = \mathcal{H}$ implies that there is only one consistent solution in L, we just need to show that $\hat{\mathbf{x}} \in L$. Take any $\mathbf{f} \in L^{\perp}$. Then, $\left\langle \mathbf{f}, F^{\dagger}\mathbf{x}_s \right\rangle = \left\langle P_{\mathcal{W}}\mathbf{f}, F^{\dagger}\mathbf{x}_s \right\rangle$, since $F^{\dagger}\mathbf{x}_s \in \mathcal{W}$. Noticing that for any $\mathbf{f} \in L^{\perp}$ we must have $P_{\mathcal{W}}\mathbf{f} \in \mathcal{S}^{\perp}$, we obtain that $P_{\mathcal{W}}\mathbf{f} = P_{\mathcal{W}\cap\mathcal{S}^{\perp}}\mathbf{f} = P_{\mathcal{N}(F)}\mathbf{f}$. But $F^{\dagger}\mathbf{x}_s \in \mathcal{N}^{\perp}(F)$, resulting in $\left\langle P_{\mathcal{W}}\mathbf{f}, F^{\dagger}\mathbf{x}_s \right\rangle = 0$. ■

We conclude that the minimax consistent approximation $\hat{\mathbf{x}}$ lies within the subspace $L = P_{\mathcal{W}}(\mathcal{S})$. Note that in the special case $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$, the mapping $P_{\mathcal{W}}(\mathcal{S})$ is onto, *i.e.,* $L = \mathcal{W}$. Not surprisingly, for this known setup, the minimax consistent approximation (15) takes the well known form $\hat{\mathbf{x}} = P_{\mathcal{W},\mathcal{S}^{\perp}}\mathbf{x}_s$.

V. Example

We simulate an example where a five seconds interval of an ECG signal is sampled using $N = 360$ sampling functions of the form $\mathbf{s}_n(t) = \mathrm{rect}(t - nT)$, $n = 0, \ldots, N-1$. We defined $T = 1/72$ [sec] to be the shift interval, and $\mathrm{rect}(t)$ is an NT periodic integrator with support on $[0, T]$. Formally, $\mathrm{rect}(t) = \chi_{[0,T]}(t) * \sum_k \delta(t - kNT)$ where χ stands for the characteristic function, $*$ is the convolution operator and δ is the Dirac delta function. Thus, the sampling functions look like a periodic sequence of integrators.

The reconstruction vectors are $\mathbf{w}_n(t) = \mathbf{s}_n(t - T/2)$, $n = 0, \ldots, N-1$, such that these are also integrators, which are shifted by half the shift interval with respect to the sampling functions. It is not hard to show that for this example $\mathcal{W} \cap \mathcal{S}^{\perp} \neq \{0\}$. For instance, $\sum_{n=0}^{N-1}(-1)^n \mathbf{w}_n(t)$ is a nonzero function within $\mathcal{W} \cap \mathcal{S}^{\perp}$. In Fig. 2 we plot a section of the input $\mathbf{x}$ (dots), the minimax consistent approximation $F^{\dagger}\mathbf{x}_s = P_{L,\mathcal{S}^{\perp}}\mathbf{x}_s$ (solid line) and some consistent solution (dotted) which was arbitrarily chosen from the set of all consistent functions. All approximations produce the same sample sequence. However, as can be seen from the figure, the minimax consistent solution is also very close to the input ECG signal.

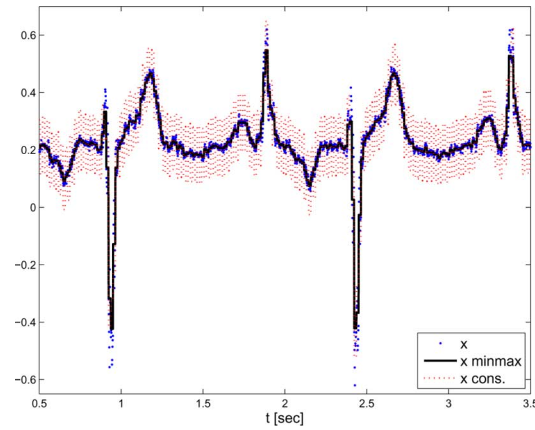

Fig. 2. Example of an input ECG signal (dots) and its consistent approximations. Though all approximations produce the same samples, the minimax consistent approximation (solid line) is also close to the input signal.

VI. Conclusion

In this letter we derived conditions for obtaining consistent reconstructions within a subspace. The geometrical form of all consistent solutions was identified, and the minimax center among all consistent solutions was obtained. A connection to previous work was also established, by identifying the subspace L of the reconstruction space $\mathcal{W}$, where the minimax consistent solution lies.

References

[1] M. Unser and A. Aldroubi, “A general sampling theory for nonideal acquisition devices,” *IEEE Trans. Signal Process.*, vol. 42, pp. 2915–2925, Nov. 1994.
[2] M. Unser, “Splines: A perfect fit for signal and image processing,” *IEEE Signal Process. Mag.*, pp. 22–38, Nov. 1999.
[3] Y. C. Eldar, “Sampling and reconstruction in arbitrary spaces and oblique dual frame vectors,” *J. Fourier Anal. Appl.*, vol. 1, no. 9, pp. 77–96, Jan. 2003.
[4] Y. C. Eldar and T. Michaeli, “Beyond bandlimited sampling,” *IEEE Signal Process. Mag.*, vol. 26, no. 3, pp. 48–68, May 2009.
[5] Y. C. Eldar and T. Werther, “General framework for consistent sampling in Hilbert spaces,” *Int. J. Wavelets, Multiresolution, Inform. Process.*, vol. 3, no. 3, pp. 347–359, Sep. 2005.
[6] A. Hirabayashi and M. Unser, “Consistent sampling and signal recovery,” *IEEE Trans. Signal Process.*, vol. 55, no. 8, pp. 4104–4115, Aug. 2007.
[7] Y. C. Eldar and T. G. Dvorkind, “A minimum squared-error framework for generalized sampling,” *IEEE Trans. Signal Process.*, vol. 54, pp. 2155–2167, Jun. 2006.
[8] W. S. Tang, “Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert space,” *Proc. Amer. Math. Soc.*, vol. 128, no. 2, pp. 463–473, 2000.
[9] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*. : Wiley, 1974.
[10] O. Christensen, *An Introduction to Frames and Riesz Bases*. Boston, MA: Birkhäuser, 2003.
[11] M. Milanese and R. Tempo, “Optimal algorithms theory for robust estimation and prediction,” *IEEE Trans. Automat. Contr.*, vol. AC-30, no. 8, pp. 730–738, Aug. 1985.
[12] M. Casini, “Set-Membership Estimation: An Advanced Tool for System Identification,” Ph.D. thesis, Univ. Degli Studi Di Siena, Dip. Ing. Dell’informazione, Siena, Italy, 2002.