

Quantum detection with uncertain states

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We address the problem of distinguishing among a finite collection of quantum states, when the states are not entirely known. For completely specified states, necessary and sufficient conditions on a quantum measurement minimizing the probability of a detection error have been derived. In this work, we assume that each of the states in our collection is a mixture of a known state and an unknown state. We investigate two criteria for optimality. The first is minimization of the worst-case probability of a detection error. For the second we assume a probability distribution on the unknown states, and minimize the expected probability of a detection error. We find that under both criteria, the optimal detectors are equivalent to the optimal detectors of an effective ensemble. In the worst case, the effective ensemble is comprised of the known states with altered prior probabilities, and in the average case it is made up of altered states with the original prior probabilities.

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I. INTRODUCTION

Quantum detection refers to the retrieval of classical information encoded in a quantum-mechanical medium. Representing the information as one of m possible messages, it is assumed that this medium has been prepared in a quantum state drawn from a collection of m known states, each associated with one of the messages. The medium is then subjected to a quantum measurement, in order to determine the prepared state. If the quantum states are not mutually orthogonal, then no measurement will distinguish perfectly between them. One, then, seeks a measurement scheme (detector) that optimally discriminates between the states in some sense. A popular criterion of optimality is minimization of the probability of a detection error.

Possible applications for distinguishing between quantum states are digital communication via a quantum channel, or the output module of a quantum computer [1]. In theoretical quantum computation, the possible outcomes of a calculation are normally mutually orthogonal, making the discrimination between the results trivial. In this paper, however, we address questions of imperfections in the setup, making the results relevant to the implementation of working quantum computers.

We consider an ensemble of quantum states, consisting of m density operators $\{\rho_{ij}\}_{i=1}^m$ on an n -dimensional complex Hilbert space \mathcal{H} , with prior probabilities $\{p_{ij}\}_{i=1}^m$. A density operator ρ is a positive semidefinite (PSD) Hermitian operator with $\text{Tr}(\rho)=1$; we write $\rho \geq 0$ to indicate that ρ is PSD. For our *measurement*, we consider general positive operator-valued measures [2,3] consisting of m PSD Hermitian operators $\{\Pi_{ij}\}_{i=1}^m$ that form a resolution on the identity on \mathcal{H} .

For a completely specified state set $\{\rho_{ij}\}_{i=1}^m$, necessary and sufficient conditions for an optimal measurement that minimizes the probability of a detection error have been derived [4–6]. Explicit solutions to the problem are known in some

particular cases [7–11], including ensembles obeying a large class of symmetries [12].

For arbitrary state sets, the problem of finding the optimal measurement can be cast as a semidefinite program (SDP) [6], which is a tractable convex optimization problem [13]. By exploiting the many well-known algorithms for solving SDPs [14,15], the optimal measurement can be computed very efficiently in polynomial time within any desired accuracy.

As with most physical systems, typically, one does not have full knowledge of the parameters. When applying the measurement, the states ρ_i are often unknown to a certain extent, whether due to degradation (noise, decoherence [16]) in the quantum medium, or to imperfect preparation. In this paper we investigate the effects of uncertainty in the states ρ_i on the optimal measurement.

To model the uncertainty we assume that each state ρ_i is a mixture of a known state and an unknown state,

$$\rho_i \triangleq q_i \rho_i^0 + (1 - q_i) \rho_i^1, \quad (1)$$

where the states ρ_i^0 and $0 \leq q_i \leq 1$ are known, and the operators ρ_i^1 are completely unspecified, except for being valid quantum states. The parameters q_i serve as a bound on the amount of mixing of each state.

A different detection strategy, known as *unambiguous detection* [17–19], is to design a measurement of order $m+1$, where the extra answer stands for an inconclusive result. If the measurement returns an answer, then it is correct with probability 1. The goal is to design the measurement so that the probability of an inconclusive result is small. When the states that are to be detected are uncertain as in (1), ensuring perfect detection of a state is impossible. For this reason we choose not to pursue this strategy.

Detection of uncertain states has so far been addressed in the special case where the quantum medium is the free-space channel and the known states ρ_i^0 are coherent states, i.e., pure states $|\alpha_i\rangle$, each characterized by a complex number α_i [20]. Vilnrotter and Lau [21] model thermal noise as a probability distribution over a finite subset $\{\alpha_j\}$ of the complex plane,

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and then find the optimal measurement for states mixed according to this distribution, i.e., $\rho_i^{\text{eff}} = \sum_j q_j |\alpha_i + \alpha_j\rangle\langle\alpha_i + \alpha_j|$. Concha and Poor [22] use Ohya's model [23] of quantum channels to model thermal noise in a multiaccess quantum free-space channel. Both works rely heavily on the simple parametrization of coherent states. This type of parametric averaging may not suit all quantum systems. In addition, the method we propose in this work assumes far less knowledge (only the probability q_0 associated with $\alpha_j=0$).

The measurement minimizing the probability of a detection error depends, in general, on the states ρ_i . Therefore, if the states are not known exactly, then the optimal measurement cannot be determined. Here, we develop two approaches to detection in the presence of state uncertainty. The first strategy is motivated by the recent theory of robust optimization [24–26], in which the worst-case solution is optimized. Adapting this method to our particular context, we consider maximizing the smallest possible probability of correct detection, over all quantum states of the form (1). Robust semidefinite programming has already been introduced to the domain of quantum information in the context of entanglement witnesses [27]. Our second strategy is to define a probability distribution over the region of uncertainty, and then maximize the average probability of correct detection. This strategy is conceptually similar to those used in [21,22], but does not rely on a particular quantum system.

In Sec. II we present the problem in detail and state some known results. Section III is an analysis of the worst-case approach to minimal detection error, and in Sec. IV we address the question of optimal detection on average. In both cases we find that the optimal measurement for the uncertain ensemble is equivalent to an optimal measurement for an "effective ensemble." The effective ensemble for worst-case detection is comprised of the known states ρ_i^0 with altered prior probabilities, whereby a bias toward the states that are more certain is introduced. In the average case, it is made up of the states $q_i\rho_i^0 + [(1-q_i)/n]I$, with the original prior probabilities. We show explicitly that averaging over the region of uncertainty is equivalent to choosing $\rho_i^1 = (1/n)I$, where I is the identity on \mathcal{H} . The quantum state $(1/n)I$, known as the *maximally mixed* state, regularly serves in quantum mechanics to represent a complete lack of knowledge.

For both strategies we address the special case where the uncertainty bounds are uniform in i ($q_i=q$). We find that for high values of q the worst-case optimal measurement coincides with the optimal nominal measurement. Also, in the equiprobable case, the optimal average measurement is identical with the optimal nominal measurement for any $q>0$.

Section V contains the results and analysis of numerical simulations of several examples. We compare the characteristics and performance of the two approaches.

II. FORMULATION OF THE DETECTION SETUP

Assume that a quantum channel is prepared in a quantum state drawn from a finite collection of quantum states. The quantum states are represented by a set of m PSD Hermitian density operators $\{\rho_i\}_{i=1}^m$ on an n -dimensional complex

Hilbert space \mathcal{H} . The states ρ_i , however, are not entirely known at the receiver, whose state of knowledge is defined by (1). The receiver performs a quantum measurement Π , comprising m PSD Hermitian operators $\Pi = \{\Pi_i\}_{i=1}^m$ on \mathcal{H} , in order to determine which of the messages was sent.

We assume without loss of generality that the eigenvectors of the known density operators $\{\rho_i\}_{i=1}^m$ span¹ \mathcal{H} [in this case, the eigenvectors of $q_i\rho_i^0 + (1-q_i)\rho_i^1$ also span \mathcal{H}]. Under this assumption, the measurement operators Π_i must satisfy

$$\Pi_i \geq 0, \quad \sum_{i=1}^m \Pi_i = I, \quad (2)$$

where I is the identity on \mathcal{H} , in order to be a valid measurement. We shall denote the set of all m -order positive operator-valued measures $\Pi = \{\Pi_i\}_{i=1}^m$ as

$$B \triangleq \left\{ \Pi \mid \Pi_i \geq 0, \quad \sum_{i=1}^m \Pi_i = I \right\}. \quad (3)$$

Given that the transmitted state is ρ_j , the probability of correctly detecting the state using measurement $\Pi = \{\Pi_i\}_{i=1}^m$ is $\text{Tr}(\Pi_j\rho_j)$. Therefore, the probability of correct detection is given by

$$P_d = \sum_{i=1}^m p_i \text{Tr}(\Pi_i\rho_i), \quad (4)$$

where $p_i > 0$ is the *a priori* probability of ρ_i , with $\sum_i p_i = 1$.

When the states are known, we may seek the measurement $\Pi \in B$ that minimizes the probability of detection error, or equivalently, maximizes the probability of correct detection. This can be expressed in the form of the optimization problem

$$\begin{aligned} & \max_{\Pi_i} \left\{ \sum_{i=1}^m p_i \text{Tr}(\Pi_i\rho_i) \right\} \\ & \text{s.t.} \begin{cases} \Pi_i \geq 0, & \text{(a)} \\ \sum_{i=1}^m \Pi_i = I. & \text{(b)} \end{cases} \end{aligned} \quad (5)$$

If $q_i=1$ for all i , so that ρ_i is completely specified, then it was shown in [4–6] that a measurement $\{\hat{\Pi}_i\}_{i=1}^m \in B$ solves (5) if and only if there exists an operator \hat{U} such that for all $1 \leq i \leq m$

¹Otherwise, we can transform the problem to a problem equivalent to the one considered in this paper, by reformulating the problem on the subspace spanned by the eigenvectors of $\{\rho_i\}_{i=1}^m$.

$$\begin{aligned} \hat{U} &\geq p_i \rho_i, \\ (\hat{U} - p_i \rho_i) \hat{\Pi}_i &= 0 \end{aligned} \quad (6)$$

(by the notation $X \geq Y$ we mean that $X - Y$ is PSD). Throughout this paper we shall denote the measurement that solves problem (5) by $\Pi^{\text{nom}} = \hat{\Pi}(\rho_i, p_i)$.

In general, the optimal measurement will depend on the states ρ_i . Because in our formulation they are unknown, different criteria for optimality must be defined. Before doing so, we present a measurement that is independent of the states ρ_i , and will thus provide a lower bound on the optimal probability of correct detection, regardless of the criterion used. The following measurement is dependent solely on the prior probability distribution: for all i choose

$$\bar{\Pi}_i = \begin{cases} \frac{1}{m_{\max}} I, & p_i = p_{\max}, \\ 0, & p_i < p_{\max}, \end{cases} \quad (7)$$

where $m_{\max} \leq m$ stands for the number of states with prior probability p_{\max} . Using this measurement, the probability of correct detection is

$$\bar{P}_d = \sum_{i=1}^m p_i \text{Tr}(\bar{\Pi}_i \rho_i) = \sum_{j=1}^{m_{\max}} \frac{p_{\max}}{m_{\max}} \text{Tr}(\rho_j) = p_{\max}. \quad (8)$$

This detector is effectively an unbiased guess from the subset of messages with maximal prior probability.

Thus, probability of correct detection equal to p_{\max} can always be achieved. We seek a measurement that under conditions of uncertainty can perform better. In the following two sections, we derive necessary and sufficient conditions for optimal measurements, using two different criteria. These criteria refer to the probability of correct detection, while specifying a certain point in the region of uncertainty. In Sec. III we present the optimal worst-case measurement, and in Sec. IV we propose an optimal expected measurement under the assumption of a probability distribution for the unknown states ρ_i^1 .

III. OPTIMAL WORST-CASE DETECTION

In a worst-case or *robust* approach, we first find the point in the region of uncertainty that would, for a given measurement, yield the poorest outcome. We then solve the ‘‘original’’ optimization problem for this point. This criterion serves to assure that the probability of correct detection obtained using the optimal detector will not be lower than a certain value (the optimal value). An uncertainty model such as ours, that assumes very little prior knowledge (only the bounds q_i), can be regarded as possessing a worst-case quality—adding prior knowledge will surely improve the performance of the optimal measurement. This observation makes this specific criterion especially interesting.

Using the uncertainty model (1) and problem (5), the worst-case measurement is the solution to

$$\begin{aligned} \max_{\Pi \in B} \min_{\rho_i^1} & \left\{ \sum_{i=1}^m p_i \text{Tr}[\Pi_i (q_i \rho_i^0 + (1 - q_i) \rho_i^1)] \right\} \\ \text{s.t.} & \begin{cases} \rho_i^1 \geq 0, & \text{(a)} \\ \text{Tr}(\rho_i^1) = 1, & \text{(b)} \end{cases} \end{aligned} \quad (9)$$

where the constraints represent valid measurements, and the region of uncertainty. We shall denote the optimal worst-case probability of correct detection as $P_d^{\text{wc}}(\vec{q})$.

We begin by proving our first result, stated in Theorem 1. We then explore in further detail the special case in which $q_i = q$ for all i .

Theorem 1. Let $\{\rho_i = q_i \rho_i^0 + (1 - q_i) \rho_i^1\}_{i=1}^m$ be a set of quantum states, where ρ_i^0 and $0 \leq q_i \leq 1$ are known and the states ρ_i^1 are unknown. Each state has prior probability p_i . Denote $\eta(\vec{q}) \triangleq \sum_{i=1}^m p_i q_i$, and define ‘‘effective probabilities’’

$$\tilde{p}_i = \frac{p_i q_i}{\eta(\vec{q})}.$$

Denote the optimal measurement on the ‘‘effective ensemble’’ (the states $\{\rho_i^0\}_{i=1}^m$ with prior probabilities \tilde{p}_i)

$$\tilde{\Pi} = \hat{\Pi}(\rho_i^0, \tilde{p}_i) = \underset{\Pi \in B}{\text{argmax}} \left\{ \sum_{i=1}^m \tilde{p}_i \text{Tr}(\Pi_i \rho_i^0) \right\}$$

and the probability of correct detection achieved by $\tilde{\Pi}$ on the effective ensemble

$$\tilde{P}_d = \sum_{i=1}^m \tilde{p}_i \text{Tr}(\tilde{\Pi}_i \rho_i^0).$$

The quantum measurement $\Pi^{\text{wc}} = \{\Pi_i^{\text{wc}}\}_{i=1}^m$ that minimizes the worst-case probability of a detection error is

$$\Pi^{\text{wc}}(\vec{q}) = \begin{cases} \tilde{\Pi}, & \eta(\vec{q}) \tilde{P}_d > p_{\max}, \\ \bar{\Pi}, & \eta(\vec{q}) \tilde{P}_d \leq p_{\max}, \end{cases}$$

where $\bar{\Pi}$ is defined in Eq. (7).

The worst-case probability of correct detection is

$$P_d^{\text{wc}}(\vec{q}) = \begin{cases} \eta(\vec{q}) \tilde{P}_d, & \Pi^{\text{wc}} = \tilde{\Pi}, \\ p_{\max}, & \Pi^{\text{wc}} = \bar{\Pi}. \end{cases}$$

Proof. The internal minimization in problem (9) can be written as

$$\begin{aligned} \min_{\rho_i^1} & \left\{ \sum_{i=1}^m p_i [q_i \text{Tr}(\Pi_i \rho_i^0) + (1 - q_i) \text{Tr}(\Pi_i \rho_i^1)] \right\} \\ \text{s.t.} & \begin{cases} \rho_i^1 \geq 0, & \text{(a)} \\ \text{Tr}(\rho_i^1) = 1, & \text{(b)} \end{cases} \end{aligned} \quad (10)$$

where the parameters $\{\Pi_i\}_{i=1}^m$ are all PSD.

Since the objective function is dependent on ρ_i^1 only through the expression on the right, and since it is also separable in i (the objective is additive and the constraints independent), the optimization is reduced to solving m cases of the form

$$\begin{aligned} & \min_{\rho} \text{Tr}(\Pi\rho) & (11) \\ & \text{s.t.} \begin{cases} \rho \geq 0, & \text{(a)} \\ \text{Tr}(\rho) = 1. & \text{(b)} \end{cases} \end{aligned}$$

By writing ρ as a convex combination of pure states

$$\rho = \sum_{j=1}^n g_j |\mu_j\rangle\langle\mu_j| \quad (12)$$

the problem (11) can be recast as

$$\begin{aligned} & \min_{g_j, |\mu_j\rangle} \sum_{j=1}^n g_j \langle\mu_j|\Pi|\mu_j\rangle & (13) \\ & \text{s.t.} \begin{cases} \sum_{j=1}^m g_j = 1, & \text{(a)} \\ \langle\mu_j|\mu_j\rangle = 1. & \text{(b)} \end{cases} \end{aligned}$$

For each j , the minimal value of $\langle\mu_j|\Pi|\mu_j\rangle$ is the minimal eigenvalue of Π , (denoted λ_{Π}^{\min}) and is achieved for a state $|\hat{\mu}_j\rangle$, which lies in the corresponding eigenspace. The optimal value of (13) is therefore $\sum_j g_j \lambda_{\Pi}^{\min} = \lambda_{\Pi}^{\min}$ and is achieved for a state $\hat{\rho}$ whose range space lies entirely in the associated eigenspace.

Using this result in (10), the original problem (9) is equivalent to

$$P_d^{\text{wc}}(\vec{q}) = \max_{\Pi \in \mathcal{B}} \left\{ \sum_{i=1}^m p_i [q_i \text{Tr}(\Pi_i \rho_i^0) + (1 - q_i) \lambda_{\Pi_i}^{\min}] \right\}. \quad (14)$$

By utilizing the fact that the minimal eigenvalue of a Hermitian operator Π can be written as the solution to

$$\lambda_{\Pi}^{\min} = \max\{\lambda \in \mathcal{R} | \Pi \geq \lambda I\}, \quad (15)$$

problem (14) becomes

$$\begin{aligned} & P_d^{\text{wc}}(\vec{q}) = \max_{\Pi_i, \lambda_i} \left\{ \sum_{i=1}^m p_i [q_i \text{Tr}(\Pi_i \rho_i^0) + (1 - q_i) \lambda_i] \right\} & (16) \\ & \text{s.t.} \begin{cases} \Pi_i \geq \lambda_i I \geq 0, & \text{(a)} \\ \sum_{i=1}^m \Pi_i = I. & \text{(b)} \end{cases} \end{aligned}$$

The objective function in problem (16) is linear and the constraints are all linear matrix equalities and inequalities, making it a convex problem. For convex optimization problems which are strictly feasible (i.e., the feasibility set has a nonempty relative interior), necessary and sufficient conditions for optimality are given by the *Karush-Kuhn-Tucker* (KKT) conditions [13]. In Appendix I we derive the KKT

conditions for the specific problem (16): A measurement $\tilde{\Pi} \in \mathcal{B}$ with bounds $\tilde{\lambda}_i \geq 0$ are optimal if there exists a Hermitian operator \tilde{U} satisfying

$$\tilde{\Pi}_i \geq \tilde{\lambda}_i I,$$

$$\tilde{U} \geq p_i q_i \rho_i^0,$$

$$\text{Tr}(\tilde{U}) \geq p_i,$$

$$(\tilde{U} - p_i q_i \rho_i^0)(\tilde{\Pi}_i - \tilde{\lambda}_i I) = 0,$$

$$[\text{Tr}(\tilde{U}) - p_i] \tilde{\lambda}_i = 0. \quad (17)$$

Moreover, from Lagrange duality we know that the optimal values of $\tilde{\Pi}_i$, $\tilde{\lambda}_i$, and \tilde{U} obey the relation

$$P_d^{\text{wc}}(\vec{q}) = \text{Tr}(\tilde{U}). \quad (18)$$

From Eq. (18) we see that the aforementioned lower bound p_{\max} on the optimal probability of correct detection is manifested in the requirement $\text{Tr}(\tilde{U}) \geq p_{\max} \geq p_i$. In Eq. (7) we give a measurement that achieves $P_d = p_{\max}$, and we shall therefore continue our analysis of the worst case under the assumption $P_d^{\text{wc}}(\vec{q}) > p_{\max}$.

With this extra demand in place, the last necessary condition in (17) can only be satisfied if for all i , the eigenvalue bounds $\tilde{\lambda}_i$ equal zero. The necessary and sufficient conditions (17) reduce to (after ignoring the now redundant constraints)

$$\tilde{U} \geq p_i q_i \rho_i^0,$$

$$(\tilde{U} - p_i q_i \rho_i^0) \tilde{\Pi}_i = 0. \quad (19)$$

Denoting $\eta(\vec{q}) \triangleq \sum_i p_i q_i$, and

$$\tilde{p}_i = \frac{p_i q_i}{\eta(\vec{q})} \quad (20)$$

these conditions [with $(1/\eta)\tilde{U}$ in place of \tilde{U}] are identical to the necessary and sufficient conditions (6), for the known states ρ_i^0 with prior probabilities \tilde{p}_i . ■

The larger q_i is, the greater the ratio \tilde{p}_i/p_i between the effective prior probability and the real one. When assuming $P_d^{\text{wc}}(\vec{q}) > p_{\max}$, the optimal worst-case measurement Π^{wc} is biased toward detecting the states with less uncertainty.

Intuitively, with no knowledge at all about the states ρ_i^1 , all that can be inferred about the uncertain part of the ensemble relies on the prior probabilities. This is the reason that the uncertain ensemble is (in terms of optimal detection) equivalent to an ensemble comprised of the known states ρ_i^0 with altered prior probabilities.

In the extreme case where a state ρ_j is completely unknown ($q_j=0$), the optimal worst-case detector ignores it entirely (because $\tilde{p}_j=0$) and attempts to distinguish optimally between the remaining $m-1$ states.

The next two corollaries give the probability of correct detection when the known states ρ_i^0 are mutually orthogonal.

Corollary 1. If ρ_i^0 are mutually orthogonal, and for all $i, q_i > 0$, then the worst-case probability of correct detection is

$$P_d^{wc}(\vec{q}) = \max\{p_{\max}, \eta(\vec{q})\}.$$

Proof. Provided that the effective prior probabilities satisfy $\tilde{p}_i > 0$, when ρ_i^0 are mutually orthogonal they can always be correctly detected. Therefore $\tilde{P}_d = 1$, and the corollary follows immediately from Theorem 1. ■

Corollary 2. Denote by I_0 the index set of the states which are completely unknown, i.e., have $q_i = 0$. Denote $P_0 = P(i \in I_0) = \sum_{i \in I_0} p_i$ and the maximal prior probability of a state from this subset $p_0 = \max\{p_i | i \in I_0\}$.

If ρ_i^0 are mutually orthogonal, and $I_0 \neq \emptyset$, then the worst-case probability of correct detection is

$$P_d^{wc}(\vec{q}) = \max\{p_{\max}, \eta(\vec{q})(1 - P_0 + p_0)\}.$$

Proof. The states ρ_i^0 with $\tilde{p}_i > 0$ are mutually orthogonal, and can therefore be detected correctly. In other words, when $i \notin I_0$, the conditional probability of correct detection is $P(d | i \notin I_0) = 1$. If $i \in I_0$, then the state must be guessed from within this subset. An optimal guess achieves $P(d | i \in I_0) = p_0 / P_0$.

All in all, the optimal measurement on the effective ensemble achieves

$$\begin{aligned} \tilde{P}_d &= P(i \notin I_0)P(d | i \notin I_0) + P(i \in I_0)P(d | i \in I_0) \\ &= (1 - P_0) \cdot 1 + P_0 \frac{p_0}{P_0} = 1 - P_0 + p_0. \end{aligned} \quad (21)$$

The corollary follows immediately from Theorem 1. ■

Note that when there is one state with $q_i = 0$, we get $P_0 = p_0$. The completely unknown state can be guessed by default.

A. Worst-case detection with uniform uncertainty

We now consider the special case in which the mixing bounds q_i are uniform in i , i.e.,

$$\rho_i = q\rho_i^0 + (1 - q)\rho_i^1. \quad (22)$$

Corollary 3. Denote the optimal nominal measurement $\Pi^{\text{nom}} = \hat{\Pi}(\rho_i^0, p_i)$, and the nominal probability of correct detection P_d^{nom} .

When the uncertainty is uniform, as in Eq. (22), then the optimal worst-case measurement is

$$\Pi^{wc}(q) = \begin{cases} \tilde{\Pi}, & 0 \leq q \leq \frac{p_{\max}}{P_d^{\text{nom}}}, \\ \Pi^{\text{nom}}, & \frac{p_{\max}}{P_d^{\text{nom}}} < q \leq 1, \end{cases}$$

and the optimal probability of correct detection is

$$P_d^{wc}(q) = \max\{p_{\max}, qP_d^{\text{nom}}\}.$$

Proof. When the uncertainty is uniform for all i , the effective probabilities defined in Eq. (20) are $\tilde{p}_i = p_i$. The opti-

mal measurement on the effective ensemble is therefore the one that would have been optimal for the known states ρ_i^0 with the original prior probabilities p_i , had there not been any uncertainty, i.e., $\tilde{\Pi} = \Pi^{\text{nom}}$.

In addition,

$$\eta(q) = \sum_{i=1}^m p_i q = q. \quad (23)$$

The corollary then follows from Theorem 1. ■

Corollary 3 implies that under uniform uncertainty with a large value of q , the best course of action in terms of worst-case performance is to ignore the uncertainty altogether. Nonetheless, although it is achieved by the optimal nominal measurement, the probability of correct detection itself is affected by the uncertainty (see Sec. V).

The complete symmetry in the uncertainty (total lack of knowledge about ρ_i^1 and equal mixing) does not bias the optimal measurement in any way, thus leaving it fixed with change in q , until the threshold is reached.

IV. OPTIMAL AVERAGE DETECTION

Optimality in the worst case does not grant good performance throughout the region of uncertainty. Also, as seen in the previous section, the optimal worst-case measurement is sometimes quite pessimistic, altogether ignoring the input state in favor of a guess. An alternative course of action is to define a distribution of probability over the region of uncertainty, thus enabling us to find a measurement, which on average maximizes P_d .

Our model of uncertainty (1) assumes a complete lack of knowledge about the states ρ_i^1 . This suggests two attributes of the probability distribution we shall define: (1) The different unknown states are statistically independent of each other; (2) for each i , ρ_i^1 is distributed uniformly over the entire set of n -dimensional quantum states.

A pure random state is equivalent to rotating an arbitrary state using a random rotation $|\mu\rangle = U|\mu_0\rangle$. Thus, the probability distribution of a “uniformly distributed” random pure state can be defined using the uniform measure on the group of order n rotation operators $SU(n)$, the well known Haar measure [28,29]. There does not, however, seem to be any natural uniform measure on the set of mixed states [28] (i.e., it is not simple to provide rational arguments for the superiority of a given measure). Many attempts to define such a distribution rely on *product measures* [30,31], whereby a random diagonal operator G , distributed using a measure on the simplex of eigenvalues, is rotated using a random unitary operator U , distributed using the Haar measure,

$$\rho_{\text{rand}} = UGU^*. \quad (24)$$

Writing the unknown states ρ_i^1 in similar fashion

$$\rho_i^1 = U_i G_i U_i^*, \quad (25)$$

we shall assume that each of the operators U_i is a random rotation operator distributed according to the Haar measure. For all i , U_i are statistically independent. We assume nothing

about the operators G_i except that they are valid quantum states [$G_i \geq 0; \text{Tr}(G_i) = 1$].

Lemma 1. Let ρ be a quantum state that has undergone a random rotation U ,

$$\rho = UGU^*,$$

where U is distributed using the Haar measure. Then for any Hermitian operator Π , the expectation value of $\text{Tr}(\Pi\rho)$ is

$$\langle \text{Tr}(\Pi\rho) \rangle = \frac{1}{n} \text{Tr}(\Pi).$$

The proof of Lemma 1 is given in Appendix II.

For a pure state ρ this result can be intuitively understood. The direction of the normalized vector is isotropically distributed, and so the average measurement result of any operator on it is the average of the possible outcomes (eigenvalues of the operator). Lemma 1 states that this is also true for mixed states.

We now wish to find the detector that maximizes the average probability of correct detection under the above probabilistic model, i.e., we wish to find

$$\max_{\Pi \in B} \left\{ E_{p_i} \left(\sum_{i=1}^m p_i \text{Tr}[\Pi_i(q_i\rho_i^0 + (1-q_i)\rho_i^1)] \right) \right\}. \quad (26)$$

The main result of this section is summarized in Theorem 2.

Theorem 2. Given an ensemble of m quantum states $\{\rho_i = q_i\rho_i^0 + (1-q_i)U_iG_iU_i^*\}_{i=1}^m$, where ρ_i^0 and $0 \leq q_i \leq 1$ are known, and U_i are statistically independent random unitary matrices distributed using the Haar measure, with prior probabilities p_i , obtaining the quantum measurement Π that minimizes the average probability of a detection error is equivalent to obtaining the optimal measurement for the ensemble $\{q_i\rho_i^0 + (1-q_i)/nI\}_{i=1}^m$ with prior probabilities p_i .

Proof. Using Lemma 1, the average probability of correct detection is given by

$$\begin{aligned} \langle P_d(\vec{q}) \rangle &= \sum_{i=1}^m p_i (q_i \text{Tr}(\Pi_i \rho_i^0) + (1-q_i) \langle \text{Tr}(\Pi_i \rho_i^1) \rangle) \\ &= \sum_{i=1}^m p_i \left(q_i \text{Tr}(\Pi_i \rho_i^0) + \frac{1-q_i}{n} \text{Tr}(\Pi_i) \right) \\ &= \sum_{i=1}^m p_i \text{Tr} \left[\Pi_i \left(q_i \rho_i^0 + \frac{1-q_i}{n} I \right) \right]. \end{aligned} \quad (27)$$

Maximizing Eq. (27) requires finding the optimal detector designed for states of the form $\rho_i = q_i\rho_i^0 + [(1-q_i)/n]I$ with prior probabilities p_i . Using our established notation, $\Pi^{\text{avg}} = \hat{\Pi}(q_i\rho_i^0 + [(1-q_i)/n]I, p_i)$. ■

Stating Theorem 2 in other words, maximizing the average probability of correct detection involves replacing the uncertainty with maximally mixed states. The maximization can be done by directly solving the SDP (5), or by utilizing one of the closed form solutions, if applicable to the new states $q_i\rho_i^0 + [(1-q_i)/n]I$.

The effective ensemble used in producing the optimal average measurement has the original prior probabilities. Therefore, the performance of the outcome of the optimization is bounded below by the performance of the measurement in (7). This implies, that the average performance is bounded below by p_{max} , but says nothing about the performance of this measurement in the worst case (see numerical results in Sec. V A).

A. Average detection with uniform uncertainty

When $q_i = q$, the optimization problem that must be solved in order to find the optimal average detector is

$$\max_{\Pi \in B} \left\{ q \sum_{i=1}^m p_i \text{Tr}(\Pi_i \rho_i^0) + \frac{1-q}{n} \sum_{i=1}^m p_i \text{Tr}(\Pi_i) \right\}. \quad (28)$$

Writing the measurement operators as $\Pi_i = \sigma_i \Psi_i$, where $\text{Tr}(\Psi_i) = 1$ and $\sigma_i = \text{Tr}(\Pi_i)$, we can regard Ψ_i as the “geometry” of the measurement operator, and σ_i as the “relative

TABLE I. Summary of the optimal measurements using different criteria.

Optimization criterion	Different mixing bounds q_i		Equal mixing bounds ($q_i = q$)	
	Measurement	Optimal P_d	Measurement	Optimal P_d
Max P_d in nominal case	$\Pi^{\text{nom}} \triangleq \hat{\Pi}(\rho_i^0, p_i)$	$P_d^{\text{nom}} \triangleq \sum_{i=1}^m p_i \text{Tr}(\Pi_i^{\text{nom}} \rho_i^0)$		Similar
Max P_d in worst case	$\hat{\Pi}(\rho_i^0, \tilde{p}_i)$	$\eta(\vec{q}) \sum_{i=1}^m \tilde{p}_i \text{Tr}(\hat{\Pi}_i \rho_i^0)$	Π^{nom}	$q P_d^{\text{nom}}$
	or ^a $\bar{\Pi}$	P_{max}	or $\bar{\Pi}$	P_{max}
Max P_d on average	$\Pi^{\text{avg}} \triangleq \hat{\Pi}\left(q_i \rho_i^0 + \frac{1-q_i}{n} I, p_i\right)$	$P_d^{\text{avg}}(\vec{q})^b$	Π^{avg}	$P_d^{\text{avg}}(\vec{q})$
			if $\forall i p_i = (1/m)$	Π^{nom} $q P_d^{\text{nom}} + \frac{1-q}{nm}$

^aOne must choose between the two measurements, according to P_d achieved by each.

^b $P_d^{\text{avg}}(\vec{q}) \triangleq \sum_{i=1}^m p_i \text{Tr}[\Pi_i^{\text{avg}}(q_i \rho_i^0 + ((1-q_i)/n)I)]$.

importance” of each operator. Note that the posteriori probability of detecting the i th message,

$$P(i) = \sum_{j=1}^m p_j \text{Tr}(\Pi_i \rho_j) = \sigma_i \text{Tr}(\Psi_i \sum_j p_j \rho_j), \quad (29)$$

is proportional to σ_i .

Stating problem (28) in these new terms, one gets

$$\max_{\Psi_i, \sigma_i} \left\{ q \sum_{i=1}^m p_i \sigma_i \text{Tr}(\Psi_i \rho_i^0) + \frac{1-q}{n} \sum_{i=1}^m p_i \sigma_i \right\} \quad (30)$$

$$\text{s.t.} \begin{cases} \Psi_i \geq 0, & \text{Tr}(\Psi_i) = 1, & \text{(a)} \\ \sum_{i=1}^m \sigma_i \Psi_i = I. & & \text{(b)} \end{cases}$$

The geometry of the measurement Ψ_i is determined only through the left-hand expression. As q grows smaller (more uncertainty), more importance in determining the optimal measurement is given to the prior probability distribution, with less regard to the states ρ_i^0 themselves.

When the different messages are equiprobable $p_i=1/m$, the right-hand term in (30) becomes a constant [with the value $(1-q)/mn$]. The optimization problem reduces to finding the optimal measurement in the nominal case. Therefore, for all $q>0$ the optimal average measurement is identical to the optimal nominal measurement. Again we find that the optimal course of action is to simply ignore the uncertainty when designing a measurement.

Numerical solutions of (5) for the states $q\rho_i^0 + [(1-q)/n]I$ reveal that when the prior probabilities p_i are not equal, many ensembles exhibit a certain value of q , above which the optimal average measurement is equal to the optimal nominal measurement. This property, however, is not universal.

V. NUMERICAL EXAMPLES AND DISCUSSION

Table I contains a summary of the results obtained in the previous sections. We have defined three criteria of optimality—nominal, worst case, and average. Throughout the following, we denote the three corresponding optimal measurements as Π^{nom} , Π^{wc} , and Π^{av} . We also refer to the measurement $\bar{\Pi}$, defined in (7). The effective prior probabilities \tilde{p}_i were defined in (20). In this section we aim to demonstrate the characteristics of the different measurements we have defined, and the relations between them, via the solutions for a specific ensemble.

The optimal measurement for each criterion was computed by explicitly solving the optimization problems (5) and (16) for the relevant cases. The computation was done using the SEDUMI toolbox in MATLAB.

A. A three-state system with uniform uncertainty

We examine an ensemble comprised of the pure states $\rho_i^0 = |u_i\rangle\langle u_i|$ in a three-dimensional Hilbert space,

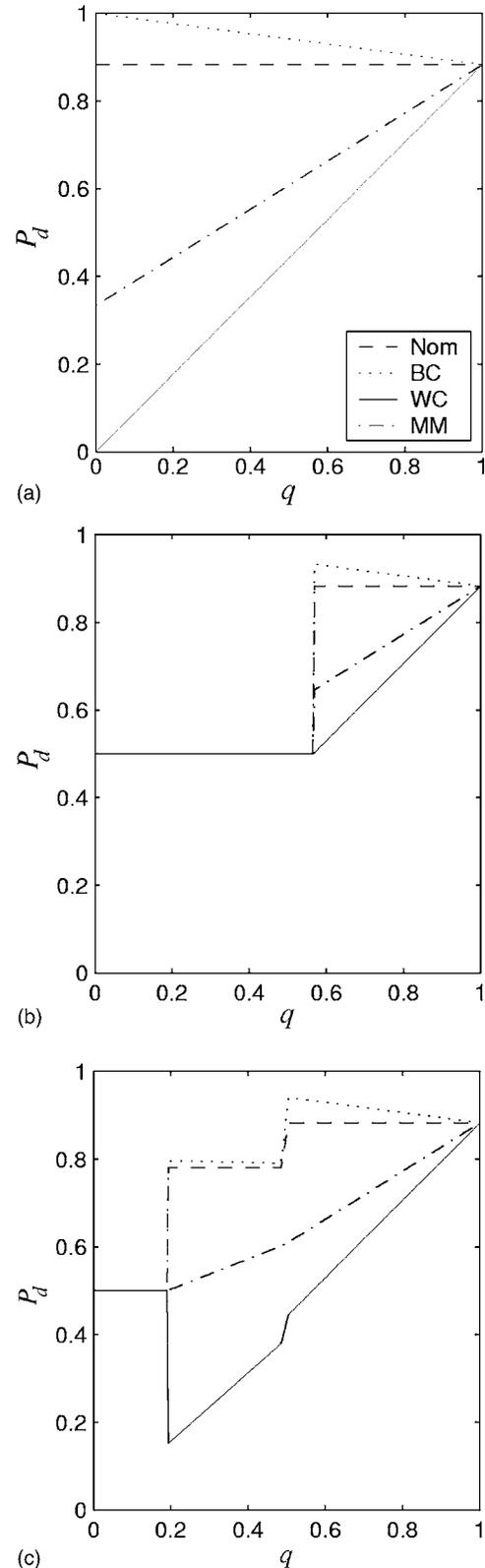


FIG. 1. Probabilities of correct detection using the different optimal measurements as a function of q , for the ensemble defined in (31): (a) optimal nominal measurement; (b) optimal worst-case measurement; (c) optimal average measurement. The different lines in each graph are different manifestations of the unknown states ρ_i^1 , where the abbreviations are defined in the text.

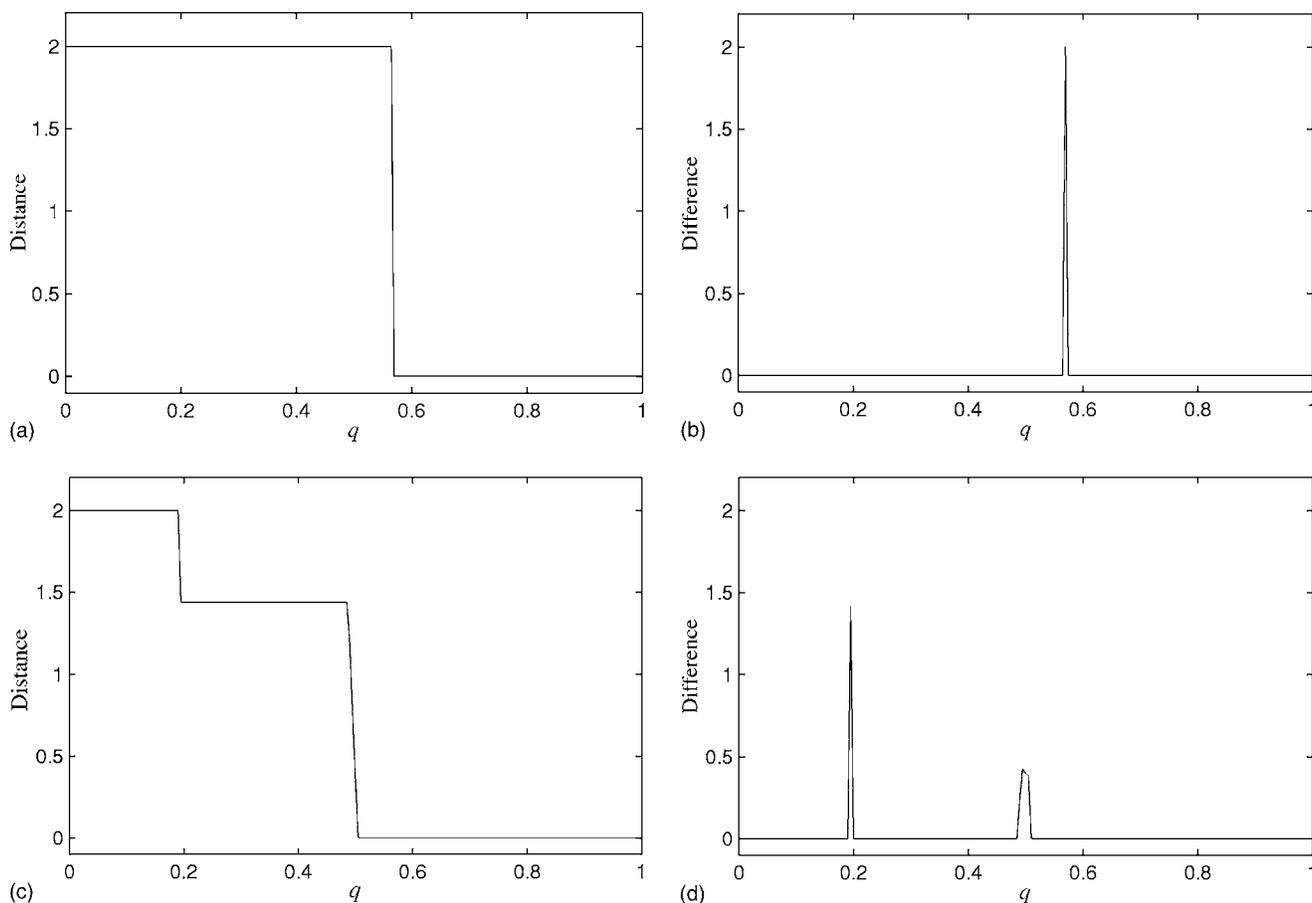


FIG. 2. Measurement distances and differences as a function of q , for the ensemble defined in (31): (a) distance between Π^{wc} and Π^{nom} ; (b) change in Π^{wc} ; (c) distance between Π^{av} and Π^{nom} ; (d) change in Π^{av} .

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \\ 0 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad (31)$$

with prior probabilities

$$p_1 = 0.2, \quad p_2 = 0.3, \quad p_3 = 0.5, \quad (32)$$

under conditions of uniform uncertainty ($q_i = q$).

Figure 1 shows the probabilities of correct detection using the different measurements as a function of q . The results referred to as “BC” and “WC” stand for best case and worst case respectively.² Also shown are the results for the nominal states, that is, $\rho_i^1 = \rho_i^0$, and the results for $\rho_i^1 = (1/n)I$ (denoted “MM”).

The optimal nominal measurement Π^{nom} is independent of q . Therefore the corresponding probabilities of correct detection, which are given by expressions of the form

$$qP_d^{nom} + (1 - q)P(\Pi^{nom}, \rho_i^1(\Pi^{nom})), \quad (33)$$

behave linearly in q .

As expected from Corollary 3, the optimal worst-case measurement Π^{wc} shows two distinct regions of behavior. One can verify the result by noting that for $q \leq p_{max}/P_d^{nom}$ the optimal probability of correct detection is $P_d(q) = p_{max}$ regardless of the choice of ρ_i^1 (in particular the worst and best cases are equal), whereas for $q > p_{max}/P_d^{nom}$ the probabilities coincide with those obtained by Π^{nom} .

The rightmost plot in Fig. 1 shows the results obtained using the optimal average measurement Π^{avg} . For high uncertainty (low q) this measurement also does not improve on the lower bound p_{max} . Using the same argument as above, we conclude that in this region of q $\Pi^{avg} = \bar{\Pi}$. Note that contrary to Π^{wc} , the lower bound measurement does not appear explicitly in solution to the problem of optimal average detection.

For optimal average detection, the transition to $\Pi^{avg} = \bar{\Pi}$ occurs at a lower value of q compared to Π^{wc} . An interpretation is that Π^{avg} is a less pessimistic measurement, relying on the input under conditions of uncertainty where Π^{wc} already regresses to guessing.

An important feature is that when q is high enough so that Π^{avg} begins to be dependent on the states themselves (and

²The states that achieve these results are measurement specific, e.g., for a given value of q , the unknown states ρ_i^1 , which would generate the best result using the nominal measurement, are not necessarily the ones that would do so for the worst-case measurement.

not only on the prior probabilities), although the objective P_d^{avg} improves monotonically, the worst-case probability of correct detection does not. In particular, there is a region of q where $P_d^{\text{wc}} < p_{\text{max}}$.

This illustrates the fact that the results of each measurement are highly dependent on the location of ρ_i^1 in the region of uncertainty. The optimal average solution may have a very bad worst case, and the optimal worst-case solution may lead to poor detection on average. In general, the designer of a specific setup must make similar calculations and use cost-benefit considerations in order to choose between the available ‘‘optimal’’ options. Hence, the choice of measurement that will be eventually used is dependent on the specific application at hand (the intended use of the apparatus, the known states ρ_i^0 and probabilities p_i , and the mixing bounds q_i).

Further insight can be revealed by examining the distances between measurements. Denoting $\Pi = [\Pi_1 \Pi_2 \cdots \Pi_m]$ (the matrix whose columns are the measurement operators), we use the Frobenius norm to define the *measurement distance*

$$D(\Pi, \Psi) = \|\Pi - \Psi\|_F \quad (34)$$

and the *measurement difference*

$$\delta\Pi(q) = \|\Pi(q) - \Pi(q - \delta q)\|_F. \quad (35)$$

The measurement difference can be thought of as a gradient of the distance from a reference measurement.

The left side of Figure 2 shows $D(\Pi^{\text{wc}}, \Pi^{\text{nom}})$ and $\delta\Pi^{\text{wc}}(q)$. The step size used in the calculations is $\delta q = 0.005$. Because Π^{nom} is not a function of q , any change in the distance is due to change in Π^{wc} . One can clearly see the discontinuous change between the region $q \leq p_{\text{max}}/P_d^{\text{nom}}$, where $\Pi^{\text{wc}} = \bar{\Pi}$, and the region of low uncertainty where $\Pi^{\text{wc}} = \Pi^{\text{nom}}$. The rightmost plots show $D(\Pi^{\text{avg}}, \Pi^{\text{nom}})$ and $\delta\Pi^{\text{avg}}(q)$. One can see that Π^{avg} exhibits both continuous and discontinuous change. Moreover, in this example, for high values of q , we find that $D(\Pi^{\text{avg}}, \Pi^{\text{nom}}) = 0$. This is characteristic of many ensembles, although as stated above, is not universal.

VI. CONCLUSION

We considered the discrimination of quantum states, drawn from a finite set with known prior probabilities, where the states themselves are not entirely known. We derived two sets of necessary and sufficient conditions for the optimality of a quantum measurement in discriminating between the states, the first in the sense of minimal worst-case probability of a detection error, the second in the sense of minimal average probability of detection error. In both cases, the uncertainty is manifested as optimally discriminating among an effective ensemble. We found that under our model, when the uncertainty is of uniform magnitude for all states, one can, in many cases, ignore it altogether.

Possible avenues for further work are looking into structured uncertainty—where the unknown states ρ_i^1 are known to some extent (for example their range space is restricted to

a certain subspace), and leakage between the possible states, i.e., the unknown states are a mixture of the known states $\rho_i^1 = \sum_j h_j \rho_j^0$.

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APPENDIX A: KKT CONDITIONS FOR THE WORST-CASE PROBLEM

The Lagrangian [13] of problem (16) is

$$L = - \sum_{i=1}^m p_i \text{Tr}[(q\Pi_i + (1-q)\lambda_i I)\rho_i^0] - \sum_{i=1}^m w_i \lambda_i - \sum_{i=1}^m \text{Tr}[Z_i(\Pi_i - \lambda_i I)] + \text{Tr}\left[U\left(I - \sum_{i=1}^m \Pi_i\right)\right], \quad (A1)$$

where $w_i \geq 0$, $Z_i \geq 0$, and U are the Lagrange multipliers.

The KKT necessary and sufficient conditions for optimality are given by

$$\sum_{i=1}^m \Pi_i = I, \quad \Pi_i \geq \lambda_i I \geq 0, \quad (A2)$$

$$Z_i \geq 0, \quad w_i \geq 0, \quad (A3)$$

$$\frac{\partial L}{\partial \Pi_i} = -p_i q_i \rho_i^0 - Z_i + U = 0, \quad (A4)$$

$$\frac{\partial L}{\partial \lambda_i} = -p_i(1-q_i) + \text{Tr}(Z_i) - w_i = 0, \quad (A5)$$

$$Z_i(\lambda_i I - \Pi_i) = 0, \quad (A6)$$

$$w_i \lambda_i = 0. \quad (A7)$$

From Eqs. (A4) and (A5) we find that at the optimum

$$Z_i = U - p_i q_i \rho_i^0, \quad (A8)$$

$$w_i = \text{Tr}(U) - p_i. \quad (A9)$$

Using these relations, the KKT conditions can be recast in the form (17).

APPENDIX B: PROOF OF LEMMA 1

Given a random pure state $|\mu\rangle$, with uniform distribution over the unit sphere, and an arbitrary orthogonal basis $\{|\pi_k\rangle\}_{k=1}^n$, the n probabilities

$$\sigma_k \triangleq |\langle \pi_k | \mu \rangle|^2 \quad (B1)$$

form a random vector in the $(n-1)$ -dimensional simplex B , defined by the conditions $\sigma_k \geq 0$ for all k and $\sum_k \sigma_k = 1$. The distribution of $\vec{\sigma}$ is given by (see [28,32])

$$\sigma_k = \frac{y_k}{\sum_{k=1}^n y_k}, \quad (\text{B2})$$

where $\{y_k\}_{k=1}^n$ are independent random variables obeying an exponential distribution with parameter 1.

Due to the fact that B , the domain in which $\vec{\sigma}$ is distributed, is a convex set, the expectation value of $\vec{\sigma}$ must lie in B , i.e., $\langle \vec{\sigma} \rangle \in B$. The distribution of $\vec{\sigma}$ is, of course, symmetrical with respect to exchange of any of its coordinates ($\sigma_k \leftrightarrow \sigma_{k'}$), and then so must be the average. The only such symmetrical point in B is $(1/n, \dots, 1/n)$, and so

$$\langle \sigma_k \rangle = \frac{1}{n}. \quad (\text{B3})$$

Given a quantum state that has undergone a random rotation U ,

$$\rho = UGU^*, \quad (\text{B4})$$

we can assume without loss of generality that the state G is diagonal, thereby permitting us to rewrite ρ in the form

$$\rho = \sum_{j=1}^n g_j |\mu_j\rangle\langle\mu_j| \quad (\text{B5})$$

with $\sum_{j=1}^n g_j = 1$ and $\langle \mu_j | \mu_{j'} \rangle = \delta_{j,j'}$. We now express $|\mu_j\rangle$ using their harmonic expansions in the eigenvectors of Π ,

$$|\mu_j\rangle = \sum_{k=1}^n \alpha_{jk} |\pi_k\rangle, \quad (\text{B6})$$

where $|\pi_k\rangle$ is the eigenvector of Π corresponding to eigenvalue λ_k . Substituting Eq. (B6) in Eq. (B5) we get

$$\rho = \sum_{j=1}^n g_j \sum_{k=1}^n \alpha_{jk} |\pi_k\rangle \sum_{k'=1}^n \alpha_{jk'}^* \langle \pi_{k'} | = \sum_{j=1}^n \sum_{k,k'=1}^n g_j \alpha_{jk} \alpha_{jk'}^* |\pi_k\rangle\langle\pi_{k'}| \quad (\text{B7})$$

and

$$\begin{aligned} \text{Tr}(\Pi\rho) &= \sum_{j=1}^n \sum_{k,k'=1}^n g_j \alpha_{jk} \alpha_{jk'}^* \text{Tr}(\Pi |\pi_k\rangle\langle\pi_{k'}|) \\ &= \sum_{j=1}^n \sum_{k,k'=1}^n g_j \alpha_{jk} \alpha_{jk'}^* \lambda_k \underbrace{\langle \pi_{k'} | \pi_k \rangle}_{\delta_{k',k}} \\ &= \sum_{j=1}^n \sum_{k=1}^n g_j |\alpha_{jk}|^2 \lambda_k. \end{aligned} \quad (\text{B8})$$

Although $|\mu_j\rangle$ are not probabilistically independent (they are mutually orthogonal), their marginal distributions are all uniform. The expectation values of the squared modulus are [from Eq. (B3)]

$$\langle |\alpha_{jk}|^2 \rangle = \langle |\langle \pi_k | \mu_j \rangle|^2 \rangle = \frac{1}{n}, \quad (\text{B9})$$

which in turn leads to

$$\langle \text{Tr}(\Pi\rho) \rangle = \sum_{j=1}^n \sum_{k=1}^n \frac{g_j \lambda_k}{n} = \frac{1}{n} \sum_{j=1}^n g_j \sum_{k=1}^n \lambda_k = \frac{1}{n} \text{Tr}(\Pi). \quad (\text{B10})$$

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