

# Optimal Detection of Symmetric Mixed Quantum States

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**Abstract**—We develop a sufficient condition for the least-squares measurement (LSM), or the square-root measurement, to minimize the probability of a detection error when distinguishing between a collection of *mixed* quantum states. Using this condition we derive the optimal measurement for state sets with a broad class of symmetries.

We first consider geometrically uniform (GU) state sets with a possibly non-Abelian generating group, and show that if the generator satisfies a weighted norm constraint, then the LSM is optimal. In particular, for pure-state GU ensembles, the LSM is shown to be optimal. For arbitrary GU state sets we show that the optimal measurement operators are GU with generator that can be computed very efficiently in polynomial time, within any desired accuracy.

We then consider compound GU (CGU) state sets which consist of subsets that are GU. When the generators satisfy a certain constraint, the LSM is again optimal. For arbitrary CGU state sets, the optimal measurement operators are shown to be CGU with generators that can be computed efficiently in polynomial time.

**Index Terms**—Compound geometrically uniform (CGU) quantum states, geometrically uniform quantum states, least-squares measurement (LSM), mixed quantum states, quantum detection, semidefinite programming, square-root measurement.

## I. INTRODUCTION

**I**N a quantum detection problem, a transmitter conveys classical information to a receiver using a quantum-mechanical channel. Each message corresponds to a preparation of the quantum channel in an associated quantum state represented by a density operator, drawn from a collection of known states. To detect the information, the receiver subjects the channel to a quantum measurement. Our problem is to construct a measurement that minimizes the probability of a detection error.

We consider a quantum state ensemble consisting of  $m$  density operators  $\{\rho_i, 1 \leq i \leq m\}$  with prior probabilities  $\{p_i > 0, 1 \leq i \leq m\}$ , acting on an  $n$ -dimensional complex Hilbert space  $\mathcal{H}$ . A density operator  $\rho$  is a positive semidefinite (PSD) Hermitian operator with  $\text{Tr}(\rho) = 1$ ; we write  $\rho \geq 0$  to indicate  $\rho$  is PSD. A mixed-state ensemble is one in which at least one of

the density operators  $\rho_i$  has rank larger than one. A pure-state ensemble is one in which each density operator  $\rho_i$  is a rank-one projector  $|\phi_i\rangle\langle\phi_i|$ , where the vectors  $|\phi_i\rangle$ , though evidently normalized to unit length, are not necessarily orthogonal.

For our *measurement* we consider general positive operator-valued measures [1], [2]. Necessary and sufficient conditions for an optimum measurement minimizing the probability of a detection error have been derived [3]–[5]. However, in general, obtaining a closed-form analytical expression for the optimal measurement directly from these conditions is a difficult and unsolved problem. Iterative algorithms minimizing the probability of a detection error have been proposed in [5], [6].

We note that recently, several other approaches to quantum detection have emerged, including unambiguous quantum detection [7]–[13], and mixed quantum state detection [12], [14], [15]. Here, we focus on strategies that maximize the probability of correct detection.

There are some particular cases in which the solution to the quantum detection problem is known explicitly [1], [16]–[19]. Ban *et al.* [18] derive the solution for a pure-state ensemble consisting of density operators  $\rho_i = |\phi_i\rangle\langle\phi_i|$ , where the vectors  $|\phi_i\rangle$  form a cyclic set, i.e., the vectors are generated by the action of a cyclic group of unitary matrices on a single generating vector. The optimal measurement for this case coincides with the least-squares measurement (LSM) [19], also known as the square-root measurement [20], [21]. Eldar and Forney [19] derive the optimal measurement for a pure-state ensemble in which the vectors  $|\phi_i\rangle$  have a strong symmetry property called geometric uniformity. In this case, the vectors  $|\phi_i\rangle$  are defined by the action of a finite Abelian group of unitary matrices on a single generating vector; the optimal measurement again coincides with the LSM. Note that a cyclic state set is a special case of a geometrically uniform state set. Barnett [22] considers the case of distinguishing between multiply symmetric pure states, which are generated by the action of two cyclic groups of unitary matrices on a single generating vector. When the groups satisfy a certain constraint, the optimal measurement is shown to coincide with the LSM.

The LSM has many desirable properties [18]–[21], [23]–[25] and has, therefore, been proposed as a detection measurement in many settings (see, e.g., [26]–[28]). It has also been realized experimentally, see, e.g., [29], [30]. In Section III, we derive a sufficient condition on the density operators for the LSM to minimize the probability of a detection error, when distinguishing between *mixed* states. For rank-one ensembles we show that the LSM minimizes the probability of a detection error if the probability of correctly detecting each of the states using the LSM is the same, regardless of the state transmitted.

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In Section IV, we consider geometrically uniform (GU) state sets defined over a finite group of unitary matrices. In contrast to [19], the GU state sets we consider are not constrained to be rank-one state sets but rather can be mixed-state sets, and the unitary group is not constrained to be Abelian. We obtain a convenient characterization of the LSM and show that the LSM operators have the same symmetries as the original state set. We then show that for such GU state sets, the probability of correctly detecting each of the states using the LSM is the same, so that for rank-one ensembles, the LSM minimizes the probability of a detection error. For an arbitrary mixed GU ensemble, the optimal measurement operators are shown to be GU with the same generating group, and can be computed very efficiently in polynomial time. Furthermore, under a certain constraint on the generator, the LSM again minimizes the probability of a detection error.

In Section V, we consider the case in which the state set is generated by a group of unitary matrices using *multiple* generators. Such a collection of states is referred to as a compound GU (CGU) state set [31]. We obtain a convenient characterization of the LSM for CGU state sets, and show that the LSM vectors are themselves CGU. When the probability of correctly detecting each of the *generators* using the LSM is the same, we show that the probability of correctly detecting each of the states using the LSM is the same. Therefore, for rank-one CGU ensembles with this property, the LSM minimizes the probability of a detection error. An interesting class of CGU state sets results when the set of generating vectors is itself GU, which we refer to as CGU state sets with GU generators. In the case in which the generating vectors are GU and generated by a group that commutes with the total density operator  $\sum_i \rho_i$ , we show that the LSM vectors are also CGU with GU generators, so that they are generated by a *single* generating vector. For such state sets, the probability of correctly detecting each of the states using the LSM is the same, so that for rank-one ensembles, the LSM again minimizes the probability of a detection error. An example of such a state set is when the generating group of the GU state sets commutes up to a phase factor with the CGU group. We note that the symmetric states studied by Barnett in [22] are a special case of the CGU state sets with commuting GU generators, in which both the GU group and the CGU group are chosen as cyclic groups. Finally, we show that for arbitrary CGU state sets, the measurement operators minimizing the probability of a detection error are also CGU, and we propose an efficient algorithm for computing the optimal generators.

Before proceeding to the detailed development, in Section II, we present our problem and summarize results from [5] pertaining to the conditions on the optimal measurement operators.

## II. OPTIMAL DETECTION OF QUANTUM STATES

Assume that a quantum channel is prepared in a quantum state drawn from a collection of given states, represented by density operators  $\{\rho_i, 1 \leq i \leq m\}$  in an  $m$ -dimensional complex Hilbert space  $\mathcal{H}$ . We assume, without loss of generality, that the eigenvectors of the density operators  $\rho_i, 1 \leq i \leq m$ , collec-

tively span<sup>1</sup>  $\mathcal{H}$ . Since  $\rho_i$  is Hermitian and PSD, we can express  $\rho_i$  as  $\rho_i = \phi_i \phi_i^*$  for some matrix  $\phi_i$ , e.g., via the Cholesky or eigendecomposition of  $\rho_i$  [32]. We refer to  $\phi_i$  as a *factor* of  $\rho_i$ . Note that the choice of  $\phi_i$  is not unique; if  $\phi_i$  is a factor of  $\rho_i$ , then any matrix of the form  $\tilde{\phi}_i = \phi_i Q_i$ , where  $Q_i$  is an arbitrary matrix satisfying  $Q_i Q_i^* = I$ , is also a factor of  $\rho_i$ .

At the receiver, the constructed measurement comprises  $m$  PSD Hermitian measurement operators  $\{\Pi_i, 1 \leq i \leq m\}$  on  $\mathcal{H}$  that satisfy  $\sum_{i=1}^m \Pi_i = I$ , where  $I$  is the identity operator on  $\mathcal{H}$ . We thus seek the measurement operators  $\{\Pi_i, 1 \leq i \leq m\}$  satisfying

$$\begin{aligned} \Pi_i &\geq 0, & 1 \leq i \leq m \\ \sum_{i=1}^m \Pi_i &= I \end{aligned} \quad (1)$$

that minimize the probability of a detection error, or equivalently, maximize the probability of correct detection. Given that the transmitted state is  $\rho_j$ , the probability of correctly detecting the state using measurement operators  $\{\Pi_i, 1 \leq i \leq m\}$  is  $\text{Tr}(\rho_j \Pi_j)$ . Therefore, the probability of correct detection is given by

$$P_d = \sum_{i=1}^m p_i \text{Tr}(\rho_i \Pi_i) \quad (2)$$

where  $p_i > 0$  is the prior probability of  $\rho_i$ , with  $\sum_i p_i = 1$ .

It was shown in [4], [5] that a set of measurement operators  $\{\hat{\Pi}_i, 1 \leq i \leq m\}$  maximizes  $P_d$  if and only if there exists a Hermitian  $\hat{X}$  satisfying

$$\hat{X} \geq p_i \rho_i, \quad 1 \leq i \leq m \quad (3)$$

(i.e.,  $\hat{X} - p_i \rho_i \geq 0$ ) such that

$$\left(\hat{X} - p_i \rho_i\right) \hat{\Pi}_i = 0, \quad 1 \leq i \leq m. \quad (4)$$

The matrix  $\hat{X}$  can be determined as the solution to the dual problem

$$\min_{X \in \mathcal{B}} \text{Tr}(X) \quad (5)$$

where  $\mathcal{B}$  is the set of Hermitian operators on  $\mathcal{H}$ , subject to

$$X \geq p_i \rho_i, \quad 1 \leq i \leq m. \quad (6)$$

Except in some particular cases [1], [16]–[19], obtaining a closed-form analytical expression for the optimal measurement operators directly from these necessary and sufficient conditions for optimality is a difficult and unsolved problem. Since (5) is a (convex) semidefinite programming problem [33]–[35], there are very efficient methods for solving it. In particular, the optimal matrix  $\hat{X}$  minimizing  $\text{Tr}(X)$  subject to (6) can be computed in Matlab using the linear matrix inequality (LMI) Toolbox. A convenient interface for using the LMI toolbox is the Matlab package<sup>2</sup> IQC $\beta$  (see [5] for further details). Once

<sup>1</sup>Otherwise, we can transform the problem to a problem equivalent to the one considered in this paper by reformulating the problem on the subspace spanned by the eigenvectors of  $\{\rho_i, 1 \leq i \leq m\}$ .

<sup>2</sup>This software was created by A. Megretski, C.-Y. Kao, U. Jönsson, and A. Rantzer and is available at: <http://web.mit.edu/ameg/www/>.

we determine  $\hat{X}$ , the optimal measurement operators  $\hat{\Pi}_i$  can be computed using (4) and (1), as described in [5].

A suboptimal measurement that has been employed as a detection measurement in many practical applications and has many desirable properties is the LSM [19], [25]. Using the necessary and sufficient conditions (1), (3), and (4), in Section III we derive a general condition under which the LSM is optimal, i.e., it minimizes the probability of a detection error when distinguishing between possibly mixed quantum states. In Sections IV and V, we consider some special cases of mixed- and pure-state sets for which the LSM is optimal, and derive explicit formulas for the optimal measurement operators.

### III. THE LSM AND THE OPTIMAL MEASUREMENT

The LSM corresponding to a set of density operators  $\{\rho_i = \phi_i \phi_i^*, 1 \leq i \leq m\}$  with eigenvectors that collectively span  $\mathcal{H}$  and prior probabilities  $\{p_i, 1 \leq i \leq m\}$  consists of the measurement operators  $\{\Sigma_i = \mu_i \mu_i^*, 1 \leq i \leq m\}$  where [19], [25]

$$\mu_i = (\Psi\Psi^*)^{-1/2} \psi_i = T\psi_i \quad (7)$$

with

$$T \triangleq (\Psi\Psi^*)^{-1/2}. \quad (8)$$

Here  $\Psi$  is the matrix of (block) columns  $\psi_i = \sqrt{p_i} \phi_i$ ,  $(\cdot)^{1/2}$  is the unique Hermitian square root of the corresponding matrix, and  $(\cdot)^{-1/2}$  is the inverse of this square root. Note that since the eigenvectors of the  $\{\rho_i\}$  collectively span  $\mathcal{H}$ , the columns of the  $\{\psi_i\}$  also together span  $\mathcal{H}$ , so  $\Psi\Psi^*$  is invertible. From (7)

$$\begin{aligned} \sum_{i=1}^m \mu_i \mu_i^* &= (\Psi\Psi^*)^{-1/2} \left( \sum_{i=1}^m \psi_i \psi_i^* \right) (\Psi\Psi^*)^{-1/2} \\ &= (\Psi\Psi^*)^{-1/2} \Psi\Psi^* (\Psi\Psi^*)^{-1/2} \\ &= I \end{aligned} \quad (9)$$

so that the LSM operators defined by (7) satisfy (1). In the case in which the prior probabilities are all equal

$$\mu_i = (\Phi\Phi^*)^{-1/2} \phi_i = M\phi_i \quad (10)$$

with

$$M \triangleq (\Phi\Phi^*)^{-1/2} \quad (11)$$

where  $\Phi$  is the matrix of (block) columns  $\phi_i$ .

Since the factors  $\phi_i$  are not unique, the LSM factors  $\mu_i$  are also not unique. In particular, if  $\mu_i$  are the LSM factors corresponding to  $\phi_i$ , then the LSM factors corresponding to  $\tilde{\phi}_i = \phi_i Q_i$  with  $Q_i Q_i^* = I$  are  $\tilde{\mu}_i = \mu_i Q_i$ . Therefore, although the LSM factors are not unique, the LSM operators  $\Sigma_i = \mu_i \mu_i^*$  are unique.

The LSM corresponding to a pure-state ensemble  $|\phi_i\rangle$  consists of the measurement vectors  $|\mu_i\rangle = T|\psi_i\rangle$ , where  $|\psi_i\rangle = \sqrt{p_i} |\phi_i\rangle$ . It was shown in [19] that for rank-one ensembles, the LSM vectors  $|\mu_i\rangle$  minimize the sum of the squared norms of the error vectors  $|e_i\rangle = |\mu_i\rangle - |\psi_i\rangle$ , so that they are the measurement vectors that satisfy (1) and are closest in a squared-error sense to the weighted state vectors  $|\psi_i\rangle = \sqrt{p_i} |\phi_i\rangle$ . In the case in

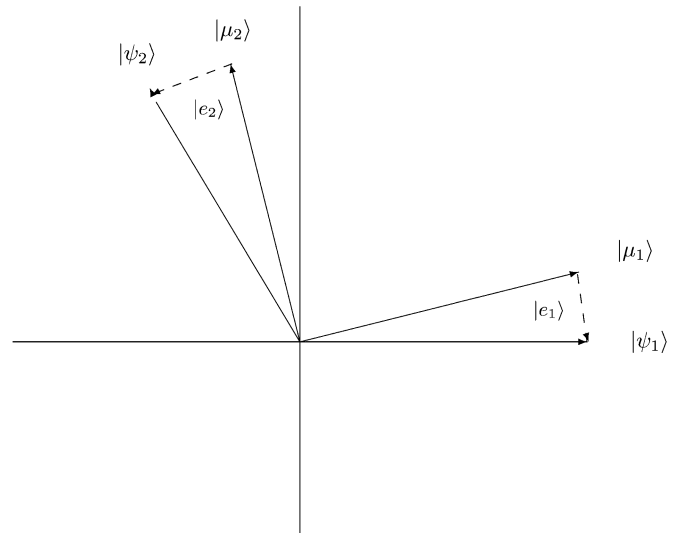


Fig. 1. Example of the LSM. Since the vectors  $|\psi_i\rangle$  are linearly independent, the LSM vectors  $|\mu_i\rangle$  are orthonormal and minimize  $\sum_i \langle e_i | e_i \rangle$  where  $|e_i\rangle = |\psi_i\rangle - |\mu_i\rangle$ .

which the vectors  $|\phi_i\rangle$  are linearly independent so that  $n = m$ , (9) implies that the vectors  $|\mu_i\rangle$  must be orthonormal, so that the LSM vectors are the closest orthonormal vectors to the vectors  $|\psi_i\rangle$  in a least-squares sense, as illustrated in Fig. 1. A similar result was obtained for the LSM factor  $\mu_i$  corresponding to a mixed-state ensemble with factors  $\phi_i$  [24].

The LSM is equivalent to the square-root measurement [18], [20], [21], [26]–[28], and has many desirable properties. Its construction is relatively simple; it can be determined directly from the given collection of states; it minimizes the probability of a detection error for pure-state ensembles that exhibit certain symmetries [18], [19]; it is “pretty good” when the states to be distinguished are equally likely and almost orthogonal [20]; it achieves a probability of error within a factor of two of the optimal probability of error [23]; and it is asymptotically optimal [21], [25]. Because of these properties, the LSM has been proposed as a detection measurement in many applications (see, e.g., [26]–[28]).

It turns out that in many cases of practical interest the LSM is optimal, i.e., it minimizes the probability of a detection error. From the necessary and sufficient conditions for optimality discussed in Section II it follows that the LSM minimizes the probability of a detection error if and only if the measurement operators  $\hat{\Pi}_i = \mu_i \mu_i^*$  defined by (7) satisfy (4) for some Hermitian  $\hat{X}$  satisfying (6). A sufficient condition for optimality of the LSM is given in the following theorem.

*Theorem 1:* Let  $\{\rho_i = \phi_i \phi_i^*, 1 \leq i \leq m\}$  denote a collection of quantum states with prior probabilities  $\{p_i, 1 \leq i \leq m\}$ . Let  $\{\Sigma_i = \mu_i \mu_i^*, 1 \leq i \leq m\}$  with  $\{\mu_i = T\psi_i, 1 \leq i \leq m\}$  denote the LSM operators corresponding to  $\{\psi_i = \sqrt{p_i} \phi_i, 1 \leq i \leq m\}$ , where  $T = (\Psi\Psi^*)^{-1/2}$  and  $\Psi$  is the matrix with block columns  $\psi_i$ . Then the LSM minimizes the probability of a detection error if, for each  $i$ ,  $\mu_i^* \psi_i = \psi_i^* T \psi_i = \alpha I$ , where  $\alpha$  is a constant independent of  $i$ .

(It is easily verified that the condition  $\psi_i^* T \psi_i = \alpha I$  does not depend on the choice of factor  $\phi_i$ .) A similar result for the spe-

cial case in which the density operators  $\rho_i$  are rank-one operators of the form  $\rho_i = |\phi_i\rangle\langle\phi_i|$  and the vectors  $|\phi_i\rangle$  are linearly independent was derived in [26].

*Proof:* We shall show that for a set of states  $\rho_i = \phi_i\phi_i^*$  with prior probabilities  $p_i$ , if  $\mu_i^*\psi_i = \alpha I$ , where  $\mu_i = (\Psi\Psi^*)^{-1/2}\psi_i = T\psi_i$  are the LSM factors and  $\psi_i = \sqrt{p_i}\phi_i$ , then there exists an Hermitian  $\hat{X}$  such that

$$\begin{aligned} \hat{X} &\geq \psi_i\psi_i^*, & 1 \leq i \leq m \\ (\hat{X} - \psi_i\psi_i^*)\mu_i\mu_i^* &= 0, & 1 \leq i \leq m. \end{aligned} \quad (12)$$

Let  $\hat{X} = \alpha T^{-1}$ . Since  $\alpha I = \psi_j^*T\psi_j = \psi_j^*T^{1/2}T^{1/2}\psi_j$ , it follows that the eigenvalues of  $\psi_j^*T^{1/2}T^{1/2}\psi_j$  are all equal to  $\alpha$ . Therefore, each eigenvalue of  $T^{1/2}\psi_j\psi_j^*T^{1/2}$  can take on the value  $\alpha$  or 0 which implies that

$$\alpha I \geq T^{1/2}\psi_j\psi_j^*T^{1/2}. \quad (13)$$

Right and left multiplying both sides of (13) by  $T^{-1/2}$  we have

$$\alpha T^{-1} \geq \psi_j\psi_j^* \quad (14)$$

which verifies the first condition in (12).

Next

$$(\hat{X} - \psi_i\psi_i^*)\mu_i = \alpha T^{-1}T\psi_i - \alpha\psi_i = 0 \quad (15)$$

so that the second condition in (12) is also satisfied.  $\square$

If the state  $\rho_i = \phi_i\phi_i^*$  is transmitted with prior probability  $p_i$ , then the probability of correctly detecting the state using measurement operators  $\Sigma_i = \mu_i\mu_i^*$  is

$$p_i \text{Tr}(\mu_i^*\phi_i\phi_i^*\mu_i) = \text{Tr}(\mu_i^*\psi_i\psi_i^*\mu_i).$$

It follows that if the condition for optimality of Theorem 1 is met, so that  $\mu_i^*\psi_i = \alpha I$ , then the probability of correctly detecting each of the states  $\rho_i$  using the LSM is  $\alpha^2$ , independent of  $i$ .

For a pure-state ensemble consisting of states  $|\phi_i\rangle$  with prior probabilities  $p_i$ , the probability of correct detection of the  $i$ th state is given by  $|\langle\mu_i|\psi_i\rangle|^2$ . Since  $\langle\mu_i|\psi_i\rangle = \langle\psi_i|T|\psi_i\rangle > 0$  for any set of weighted vectors  $|\psi_i\rangle$ ,  $\langle\mu_i|\psi_i\rangle$  is constant for all  $i$  if and only if  $|\langle\mu_i|\psi_i\rangle|^2$  is constant for all  $i$ . Therefore, we may state the condition in Theorem 1 for pure-state ensembles as the following corollary.

*Corollary 2:* The LSM is optimal for a pure-state set  $|\phi_i\rangle$  with prior probabilities  $p_i$  if the probability of detecting each one of the states using the LSM vectors is the same, regardless of the specific state chosen.

In the remainder of the paper, we use Theorem 1 to derive the optimal measurement for mixed- and pure-state sets with certain symmetry properties. The symmetry properties we consider are quite general, covering many cases of practical interest, including most cases where explicit solutions have been reported in the literature.

#### IV. GEOMETRICALLY UNIFORM STATE SETS

In this section, we consider the case in which the density operators  $\rho_i$  are defined over a (not necessarily Abelian) group of unitary matrices and are generated by a single generating matrix. Such a state set is called GU [36]. We first obtain a convenient characterization of the LSM in this case and then show that under a certain constraint on the generator, the LSM minimizes the probability of a detection error. In particular, for pure-state ensembles the LSM minimizes the probability of a detection error.

Let  $\mathcal{G} = \{U_i, 1 \leq i \leq m\}$  be a finite group of  $m$  unitary matrices  $U_i$ . That is,  $\mathcal{G}$  contains the identity matrix  $I$ ; if  $\mathcal{G}$  contains  $U_i$ , then it also contains its inverse  $U_i^{-1} = U_i^*$ ; and the product  $U_iU_j$  of any two elements of  $\mathcal{G}$  is in  $\mathcal{G}$  [37].

The state set generated by  $\mathcal{G}$  using a single generating operator  $\rho$  is the set  $\mathcal{S} = \{\rho_i = U_i\rho U_i^*, U_i \in \mathcal{G}\}$ . The group  $\mathcal{G}$  will be called the *generating group* of  $\mathcal{S}$ . For concreteness, we assume that  $U_1 = I$  so that  $\rho_1 = \rho$ . Such a state set has strong symmetry properties and is called GU. For consistency with the symmetry of  $\mathcal{S}$ , we will assume equiprobable prior probabilities on the elements of  $\mathcal{S}$ .

If the state set  $\{\rho_i, 1 \leq i \leq m\}$  is GU, then we can always choose factors  $\phi_i$  of  $\rho_i$  such that  $\{\phi_i = U_i\phi, U_i \in \mathcal{G}\}$  where  $\phi$  is a factor of  $\rho$ , so that the factors  $\phi_i$  are also GU with generator  $\phi$ . In the remainder of this section, we explicitly assume that the factors are chosen to be GU.

We note that in [19] a GU state set was defined for the case of rank-one ensembles. Furthermore, the generating group was assumed to be *Abelian*.

In Section IV-A, we derive the LSM operators for GU state sets and show that the LSM operators are also GU with the same generating group. We will see that this implies that when using the LSM, the probability of correct detection of each of the states in a GU state set is the same regardless of the particular state chosen. From Corollary 2 it then follows that for pure-state ensembles, the LSM is optimal.

##### A. The LSM for GU States

To derive the LSM for a GU state set with generating group  $\mathcal{G}$ , we first show that  $\Phi\Phi^*$  commutes with each of the matrices  $U_j \in \mathcal{G}$ . Indeed, expressing  $\Phi\Phi^*$  as

$$\Phi\Phi^* = \sum_{i=1}^m \phi_i\phi_i^* = \sum_{i=1}^m U_i\phi\phi^*U_i^* \quad (16)$$

we have that for all  $j$

$$\begin{aligned} \Phi\Phi^*U_j &= \sum_{i=1}^m U_i\phi\phi^*U_i^*U_j \\ &= U_j \sum_{i=1}^m U_j^*U_i\phi\phi^*U_i^*U_j \\ &= U_j \sum_{k=1}^m U_k\phi\phi^*U_k \\ &= U_j\Phi\Phi^* \end{aligned} \quad (17)$$

since  $\{U_k = U_j^*U_i, 1 \leq i \leq m\}$  is just a permutation of  $\mathcal{G}$ .

Because  $\Phi\Phi^*$  commutes with  $U_j$ , it follows that  $M = (\Phi\Phi^*)^{-1/2}$  also commutes with  $U_j$  for all  $j$ . Thus, from (10), the LSM operators are  $\Sigma_i = \mu_i\mu_i^*$  with

$$\mu_i = M\phi_i = MU_i\phi = U_iM\phi = U_i\mu \quad (18)$$

where

$$\mu = M\phi = (\Phi\Phi^*)^{-1/2}\phi. \quad (19)$$

It follows that the LSM factors  $\mu_i$  are also GU with generating group  $\mathcal{G}$  and generator  $\mu$  given by (19). Therefore, to compute the LSM factors for a GU state set, all we need is to compute the generator  $\mu$ . The remaining measurement factors are then obtained by applying the group  $\mathcal{G}$  to  $\mu$ .

A similar result was developed in [19] for rank-one ensembles in the case in which the group  $\mathcal{G}$  is Abelian, using the Fourier transform defined over  $\mathcal{G}$ .

### B. Optimality of the LSM

We have seen that for a GU state set  $\{\rho_i = \phi_i\phi_i^*, 1 \leq i \leq m\}$  with equal prior probabilities  $1/m$  and generating group  $\mathcal{G} = \{U_i, 1 \leq i \leq m\}$ , the LSM operators  $\{\Sigma_i = \mu_i\mu_i^*, 1 \leq i \leq m\}$  are also GU with generating group  $\mathcal{G}$ . Therefore,

$$\mu_i^*\psi_i = \frac{1}{\sqrt{m}}\mu_i^*\phi_i = \frac{1}{\sqrt{m}}\mu^*U_i^*U_i\phi = \frac{1}{\sqrt{m}}\mu^*\phi \quad (20)$$

where  $\phi$  and  $\mu$  are the generators of the state factors and the LSM factors, respectively. It follows that the probability of correct detection of each one of the states  $\rho_i$  using the LSM is the same, regardless of the state transmitted. This then implies, from Corollary 2, that for pure-state GU ensembles the LSM is optimal. For a mixed-state ensemble, if the generator  $\phi$  satisfies

$$\mu^*\phi = \phi^*(\Phi\Phi^*)^{-1/2}\phi = \alpha I \quad (21)$$

for some  $\alpha$ , then, from Theorem 1, the LSM minimizes the probability of a detection error.

Note that the condition  $\mu^*\phi = \alpha I$  does not depend on the choice of generator  $\phi$ . Indeed, if  $\tilde{\phi} = \phi Q$  is another factor of  $\rho$ , then from (19) the generator of the LSM factors is  $\tilde{\mu} = \mu Q$  so that  $\tilde{\mu}^*\tilde{\phi} = \alpha I$  if and only if  $\mu^*\phi = \alpha I$ .

### C. Optimal Measurement for Arbitrary GU State Sets

If the generator  $\phi$  does not satisfy (21), then the LSM is no longer guaranteed to be optimal. Nonetheless, as we now show, the optimal measurement operators that minimize the probability of a detection error are GU with generating group  $\mathcal{G}$ . The corresponding generator can be computed very efficiently in polynomial time.

Suppose one set of optimal measurement operators that maximize

$$J(\{\Pi_i\}) = \sum_{i=1}^m \text{Tr}(\rho_i\Pi_i) \quad (22)$$

is  $\{\hat{\Pi}_i\}$ , and let

$$\hat{J} = J(\{\hat{\Pi}_i\}) = \sum_{i=1}^m \text{Tr}(\rho_i\hat{\Pi}_i).$$

Let  $r(j, i)$  be the mapping from  $\mathcal{I} \times \mathcal{I}$  to  $\mathcal{I}$  with  $\mathcal{I} = \{1, \dots, m\}$ , defined by  $r(j, i) = k$  if  $U_j^*U_i = U_k$ . Then the measurement operators  $\{\hat{\Pi}_i^{(j)} = U_j\hat{\Pi}_{r(j,i)}U_j^*, 1 \leq i \leq m\}$  for any  $1 \leq j \leq m$  are also optimal. Indeed, it is easy to see that these operators satisfy (1) and (using the fact that  $\rho_i = U_i\rho U_i^*$  for some generator  $\rho$ ) that

$$\begin{aligned} J(\{\hat{\Pi}_i^{(j)}\}) &= \sum_{i=1}^m \text{Tr}(\rho U_i^* U_j \hat{\Pi}_{r(j,i)} U_j^* U_i) \\ &= \sum_{k=1}^m \text{Tr}(\rho U_k^* \hat{\Pi}_k U_k) \\ &= \sum_{i=1}^m \text{Tr}(\rho_i \hat{\Pi}_i) \\ &= \hat{J}. \end{aligned} \quad (23)$$

Since the measurement operators

$$\{\hat{\Pi}_i^{(j)} = U_j\hat{\Pi}_{r(j,i)}U_j^*, 1 \leq i \leq m\}$$

are optimal for any  $j$ , it follows immediately that the measurement operators

$$\left\{ \bar{\Pi}_i = (1/m) \sum_{j=1}^m U_j \hat{\Pi}_{r(j,i)} U_j^*, 1 \leq i \leq m \right\}$$

are also optimal. Indeed, it is immediate that  $\bar{\Pi}_i$  satisfy (1). In addition,  $J(\{\bar{\Pi}_i\}) = J(\{\hat{\Pi}_i^{(j)}\}) = \hat{J}$ . Now

$$\begin{aligned} \bar{\Pi}_i &= \frac{1}{m} \sum_{j=1}^m U_j \hat{\Pi}_{r(j,i)} U_j^* \\ &= \frac{1}{m} \sum_{k=1}^m U_i U_k^* \hat{\Pi}_k U_k U_i^* \\ &= U_i \left( \frac{1}{m} \sum_{k=1}^m U_k^* \hat{\Pi}_k U_k \right) U_i^* \\ &= U_i \hat{\Pi} U_i^* \end{aligned} \quad (24)$$

where  $\hat{\Pi} = (1/m) \sum_{k=1}^m U_k^* \hat{\Pi}_k U_k$ .

We, therefore, conclude that the optimal measurement operators can always be chosen to be GU with the same generating group  $\mathcal{G}$  as the original state set. Thus, to find the optimal measurement operators all we need is to find the optimal generator  $\hat{\Pi}$ . The remaining operators are obtained by applying the group  $\mathcal{G}$  to  $\hat{\Pi}$ .

The optimal measurement operators satisfy  $\Pi_i = U_i \Pi U_i^*$  and  $\rho_i = U_i \rho U_i^*$ ,  $\text{Tr}(\rho_i \Pi_i) = \text{Tr}(\rho \Pi)$  so that the problem (2) reduces to the maximization problem

$$\max_{\Pi \in \mathcal{B}} \text{Tr}(\rho \Pi) \quad (25)$$

where  $\mathcal{B}$  is the set of Hermitian operators on  $\mathcal{H}$ , subject to the constraints

$$\begin{aligned} \Pi &\geq 0 \\ \sum_{i=1}^m U_i \Pi U_i^* &= I. \end{aligned} \quad (26)$$

Since this problem is a (convex) semidefinite programming problem, the optimal  $\Pi$  can be computed very efficiently in polynomial time within any desired accuracy [33]–[35], for example, using the LMI Toolbox on Matlab. Note that the problem of (25) and (26) has  $n^2$  real unknowns and two constraints, in contrast with the original maximization problem (1) which has  $mn^2$  real unknowns and  $m + 1$  constraints.

We summarize our results regarding GU state sets in the following theorem.

*Theorem 3 (GU State Sets):* Let

$$\mathcal{S} = \{\rho_i = U_i \rho U_i^*, U_i \in \mathcal{G}\}$$

be a GU state set generated by a finite group  $\mathcal{G}$  of unitary matrices, where  $\rho = \phi\phi^*$  is an arbitrary generator, and let  $\Phi$  be the matrix of columns  $\phi_i = U_i\phi$ . Then the LSM is given by the measurement operators  $\Sigma_i = \mu_i \mu_i^*$  with

$$\mu_i = U_i \mu$$

where

$$\mu = (\Phi\Phi^*)^{-1/2}\phi.$$

The LSM has the following properties.

- 1) The measurement operators are GU with generating group  $\mathcal{G}$ .
- 2) The probability of correctly detecting each of the states  $\rho_i$  using the LSM is the same.
- 3) If  $\mu^*\phi = \phi^*(\Phi\Phi^*)^{-1/2}\phi = \alpha I$ , then the LSM minimizes the probability of a detection error. In particular, if  $\phi = |\phi\rangle$  is a vector so that the state set is a rank-one ensemble, then the LSM minimizes the probability of a detection error.

For an arbitrary generator  $\phi$  the optimal measurement operators that minimize the probability of a detection error are also GU with generating group  $\mathcal{G}$  and generator  $\Pi$  that maximizes  $\text{Tr}(\rho\Pi)$  subject to  $\Pi \geq 0$  and  $\sum_{i=1}^m U_i \Pi U_i^* = I$ .

## V. COMPOUND GEOMETRICALLY UNIFORM (CGU) STATE SETS

In this section, we consider state sets which consist of subsets that are GU, and are therefore referred to as CGU [31]. As we show, the LSM operators are also CGU so that they can be computed using a *set* of generators. Under a certain condition on the generators, we also show that the optimal measurement associated with a CGU state set is equal to the LSM. For arbitrary CGU state sets, the optimal measurement is no longer equal to the LSM. Nonetheless, as we show, the optimal measurement operators are also CGU and we derive an efficient computational method for finding the optimal generators.

A CGU state set is defined as a set of density operators

$$\mathcal{S} = \{\rho_{ik} = \phi_{ik}\phi_{ik}^*, 1 \leq i \leq l, 1 \leq k \leq r\}$$

such that  $\rho_{ik} = U_i \rho_k U_i^*$ , where the matrices  $\{U_i, 1 \leq i \leq l\}$  are unitary and form a group  $\mathcal{G}$ , and the operators  $\{\rho_k, 1 \leq k \leq r\}$  are the generators. For concreteness, we assume that

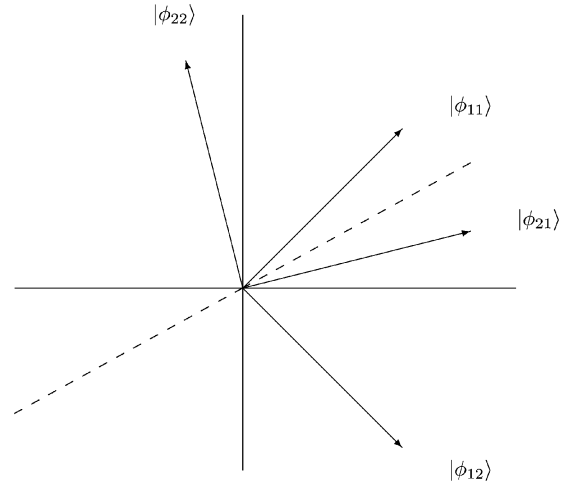


Fig. 2. A compound geometrically uniform pure-state set. The state sets  $\mathcal{S}_1 = \{|\phi_{11}\rangle, |\phi_{21}\rangle\}$  and  $\mathcal{S}_2 = \{|\phi_{12}\rangle, |\phi_{22}\rangle\}$  are both GU with the same generating group; both sets are invariant under a reflection about the dashed line. However, the combined set  $\mathcal{S} = \{|\phi_{11}\rangle, |\phi_{21}\rangle, |\phi_{12}\rangle, |\phi_{22}\rangle\}$  is no longer GU.

$U_1 = I$  so that  $\rho_{1k} = \rho_k$ . We also assume equiprobable prior probabilities on the elements of  $\mathcal{S}$ .

If the state set  $\{\rho_{ik}, 1 \leq i \leq l, 1 \leq k \leq r\}$  is CGU, then we can always choose factors  $\phi_{ik}$  of  $\rho_{ik}$  such that  $\{\phi_{ik} = U_i \phi_k, 1 \leq i \leq l\}$  where  $\phi_k$  is a factor of  $\rho_k$ , so that the factors  $\phi_{ik}$  are also CGU with generators  $\{\phi_k, 1 \leq k \leq r\}$ . In the remainder of this section we explicitly assume that the factors are chosen to be CGU.

A CGU state set is in general not GU. However, for every  $k$ , the matrices  $\{\phi_{ik}, 1 \leq i \leq l\}$  and the operators  $\{\rho_{ik}, 1 \leq i \leq l\}$  are GU with generating group  $\mathcal{G}$ .

### A. Example of a Compound Geometrically Uniform State Set

An example of a CGU state set is illustrated in Fig. 2. In this example, the state set is  $\{\rho_{ik} = |\phi_{ik}\rangle\langle\phi_{ik}|, 1 \leq i, k \leq 2\}$  where  $\{|\phi_{ik}\rangle = U_i |\phi_k\rangle, U_i \in \mathcal{G}\}$ ,  $\mathcal{G} = \{I_2, U\}$  with

$$U = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \quad (27)$$

and the generating vectors are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (28)$$

The matrix  $U$  represents a reflection about the dashed line in Fig. 2. Thus, the vector  $|\phi_{21}\rangle$  is obtained by reflecting the generator  $|\phi_{11}\rangle$  about this line, and, similarly, the vector  $|\phi_{22}\rangle$  is obtained by reflecting the generator  $|\phi_{12}\rangle$  about this line.

As can be seen from the figure, the state set is not GU. In particular, there is no isometry that transforms  $|\phi_{11}\rangle$  into  $|\phi_{12}\rangle$  while leaving the set invariant. However, the sets  $\mathcal{S}_1 = \{|\phi_{11}\rangle, |\phi_{21}\rangle\}$  and  $\mathcal{S}_2 = \{|\phi_{12}\rangle, |\phi_{22}\rangle\}$  are both GU with generating group  $\mathcal{G}$ .

### B. The LSM for CGU State Sets

We now derive the LSM for a CGU state set with equal prior probabilities. Let  $\Phi$  denote the matrix of columns  $\phi_{ik}$ . Using a similar argument to that used in (17) (see also [31]), we can

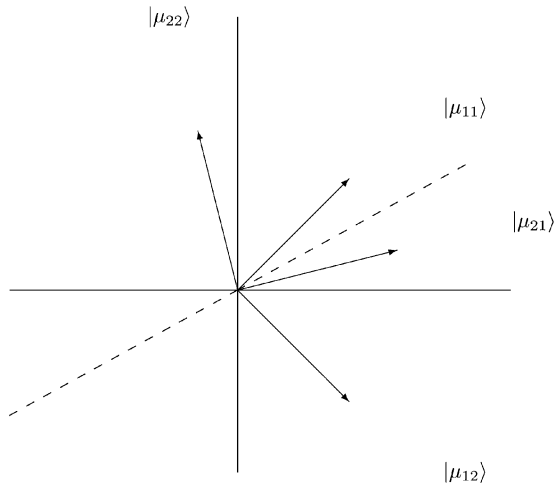


Fig. 3. The LSM vectors associated with the compound geometrically uniform state set of Fig. 2. As can be seen from the figure, the measurement vectors have the same symmetries as the original state set.

show that for a CGU state set with generating group  $\mathcal{G}$ ,  $\Phi\Phi^*$  commutes with each of the matrices  $U_i \in \mathcal{G}$ . If  $\Phi\Phi^*$  commutes with  $U_j$ , then  $M$  also commutes with  $U_j$  for all  $j$ . Thus, the LSM operators are  $\Sigma_{ik} = \mu_{ik}\mu_{ik}^*$  with

$$\mu_{ik} = M\phi_{ik} = MU_i\phi_k = U_iM\phi_k = U_i\mu_k \quad (29)$$

where

$$\mu_k = M\phi_k = (\Phi\Phi^*)^{-1/2}\phi_k. \quad (30)$$

Therefore, the LSM factors are also CGU with generating group  $\mathcal{G}$  and generators  $\mu_k$  given by (30). To compute the LSM factors, all we need is to compute the generators  $\mu_k$ . The remaining measurement factors are then obtained by applying the group  $\mathcal{G}$  to each of the generators.

For the CGU state set of Fig. 2 we have that

$$\Phi\Phi^* = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (31)$$

Therefore, the LSM vectors are

$$\{|\mu_{ik}\rangle = U_i|\mu_k\rangle, U_i \in \mathcal{G}, 1 \leq i, k \leq 2\}$$

where  $\mathcal{G} = \{I_2, U\}$  with  $U$  given by (27), and from (30) the generating vectors are

$$|\mu_1\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |\mu_2\rangle = \frac{1}{\sqrt{2}}|\phi_2\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (32)$$

The LSM vectors are depicted in Fig. 3. The vectors have the same symmetries as the state set of Fig. 2, with different generating vectors. Since in this example the generating vectors satisfy  $|\mu_k\rangle = (1/\sqrt{2})|\phi_k\rangle$ , we have that  $|\mu_{ik}\rangle = (1/\sqrt{2})|\phi_{ik}\rangle$  for  $1 \leq i, k \leq 2$ .

### C. CGU State Sets With GU Generators

A special class of CGU state sets is CGU state sets with GU generators in which the generators  $\{\rho_k = \phi_k\phi_k^*, 1 \leq k \leq r\}$  and the factors  $\phi_k$  are themselves GU. Specifically,  $\{\phi_k = V_k\phi\}$

for some generator  $\phi$ , where the matrices  $\{V_k, 1 \leq k \leq r\}$  are unitary, and form a group  $\mathcal{Q}$ .

We now consider the special case in which  $V_t$  commutes with  $\Phi\Phi^*$ . We refer to the resulting set as CGU with commuting GU generators. For example, if  $U_p$  and  $V_t$  commute up to a phase factor for all  $t$  and  $p$  so that  $U_pV_t = V_tU_p e^{j\theta(p,t)}$  where  $\theta(p,t)$  is an arbitrary phase function that may depend on the indexes  $p$  and  $t$ , then  $V_t$  commutes with  $\Phi\Phi^*$  [31]. (In the special case in which  $\theta = 0$  so that  $U_iV_k = V_kU_i$  for all  $i, k$ , the resulting state set is GU [31]). A special case of a CGU set with commuting GU generators is the multiply symmetric state set, defined by Barnett [22], in which both  $\mathcal{G}$  and  $\mathcal{Q}$  are cyclic groups.

For a CGU group with commuting GU generators, the LSM factors  $\mu_{ik}$  are

$$\mu_{ik} = M\phi_{ik} = MU_iV_k\phi = U_iV_kM\phi = U_iV_k\bar{\mu} \quad (33)$$

where  $\bar{\mu} = M\phi$ . Thus, even though the state set is not in general GU, the LSM factors can be computed using a single generator.

Alternatively, we can express  $\mu_{ik}$  as  $\mu_{ik} = U_i\mu_k$ , where the generators  $\mu_k$  are given by

$$\mu_k = V_k\bar{\mu}. \quad (34)$$

From (34) it follows that the generators  $\mu_k$  are GU with generating group  $\mathcal{Q} = \{V_k, 1 \leq k \leq r\}$  and generator  $\bar{\mu}$ .

We conclude that for a CGU state set with commuting GU generators and generating group  $\mathcal{Q}$ , the LSM vectors are also CGU with commuting GU generators and generating group  $\mathcal{Q}$ .

### D. Example of a CGU State Set With Commuting GU Generators

We now consider an example of a CGU state set with commuting GU generators. Consider the group  $\mathcal{G}$  of  $l$  unitary matrices on  $\mathbb{C}^l$  where  $U_i = Z^i, 1 \leq i \leq l$ , and  $Z$  is the matrix defined by

$$Z \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{l-1} \\ x_l \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_l \\ x_1 \end{bmatrix}. \quad (35)$$

Let  $\mathcal{Q}$  be the group of  $r = l$  unitary matrices  $V_k = B^k, 1 \leq k \leq l$ , where  $B$  is the diagonal matrix with diagonal elements  $e^{j2\pi s/l}, 0 \leq s \leq l-1$ . We can immediately verify that for this choice of  $\mathcal{Q}$  and  $\mathcal{G}, U_iV_k = V_kU_i e^{j2\pi/l}$ . We, therefore, conclude that the LSM operators corresponding to the CGU state set  $\mathcal{S} = \{\rho_{ik} = \phi_{ik}\phi_{ik}^*\}$  with  $\{\phi_{ik} = U_i\phi_k, U_i \in \mathcal{G}\}$  and  $\{\phi_k = V_k\phi, V_k \in \mathcal{Q}\}$  for some generator  $\phi$ , are also CGU with commuting GU generators and can, therefore, be generated by a single generator.

As a special case, suppose that  $l = 2$  so that  $\mathcal{G}$  consists of the matrices  $U_1 = I_2$  and  $U_2 = Z$  where

$$Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (36)$$

and  $\mathcal{Q}$  consists of the matrices  $V_1 = I_2$  and  $V_2 = B$  where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (37)$$

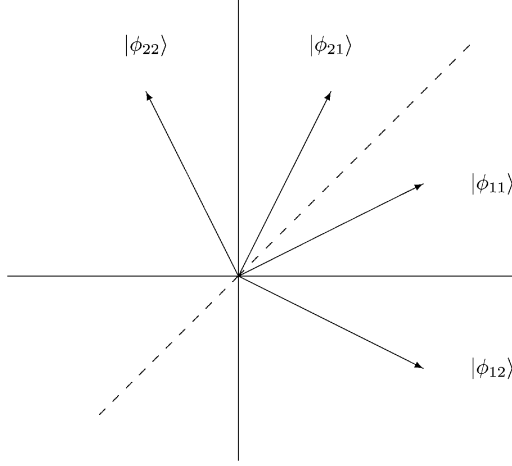


Fig. 4. A compound geometrically uniform state set with commuting GU generators. The state sets  $\mathcal{S}_1 = \{|\phi_{11}\rangle, |\phi_{21}\rangle\}$  and  $\mathcal{S}_2 = \{|\phi_{12}\rangle, |\phi_{22}\rangle\}$  are both GU with the same generating group; both sets are invariant under a reflection about the dashed line. The set of generators  $\{|\phi_{11}\rangle, |\phi_{12}\rangle\}$  is GU and is invariant under a reflection about the  $x$ -axis. The combined set  $\mathcal{S} = \{|\phi_{11}\rangle, |\phi_{21}\rangle, |\phi_{12}\rangle, |\phi_{22}\rangle\}$  is no longer GU. Nonetheless, the LSM vectors are generated by a single generating vector and are given by  $|\mu_{ik}\rangle = (1/\sqrt{2})|\phi_{ik}\rangle$ .

Let the state set be  $\mathcal{S} = \{|\phi_{ik}\rangle = U_i V_k |\phi\rangle, 1 \leq i, k \leq 2\}$ , where  $\phi = [\beta_1 \ \beta_2]^*$ . Since  $|\phi\rangle$  must be normalized,  $\beta_1^2 + \beta_2^2 = 1$ . Then

$$\begin{aligned} |\phi_{11}\rangle &= \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, & |\phi_{21}\rangle &= \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} \\ |\phi_{12}\rangle &= \begin{bmatrix} \beta_1 \\ -\beta_2 \end{bmatrix}, & |\phi_{22}\rangle &= \begin{bmatrix} -\beta_2 \\ \beta_1 \end{bmatrix}. \end{aligned} \quad (38)$$

The LSM vectors are given by  $\{|\mu_{ik}\rangle = U_i V_k |\bar{\mu}\rangle, 1 \leq i, k \leq 2\}$ , where  $|\bar{\mu}\rangle = (\Phi\Phi^*)^{-1/2}|\phi\rangle$ , and

$$\Phi = \begin{bmatrix} \beta_1 & \beta_2 & \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 & -\beta_2 & \beta_1 \end{bmatrix}. \quad (39)$$

We can immediately verify that

$$\Phi\Phi^* = \begin{bmatrix} 2(\beta_1^2 + \beta_2^2) & 0 \\ 0 & 2(\beta_1^2 + \beta_2^2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad (40)$$

Thus, the LSM vectors are

$$|\mu_{ik}\rangle = \frac{1}{\sqrt{2}}|\phi_{ik}\rangle. \quad (41)$$

In Fig. 4, we plot the state vectors given by (38) for the case in which  $|\phi\rangle = (1/\sqrt{5})[2 \ 1]^*$ . As can be seen from the figure, the state set is not GU. In particular, there is no isometry that transforms  $|\phi_{11}\rangle$  into  $|\phi_{12}\rangle$  while leaving the set invariant. Nonetheless, we have seen that the LSM vectors can be generated by a single generating vector  $|\mu\rangle = (1/\sqrt{2})|\phi\rangle$ .

Note, that in the example of Section V-A, the CGU state set also has GU generators. Specifically, the set of generators  $\{|\phi_1\rangle, |\phi_2\rangle\}$  with  $|\phi_1\rangle = |\phi_{11}\rangle$  and  $|\phi_2\rangle = |\phi_{12}\rangle$  is invariant under a reflection about the  $x$ -axis:  $|\phi_2\rangle = B|\phi_1\rangle$  where  $B$  is given by (37). However, the group  $\mathcal{Q} = \{I_2, B\}$  of generators does not commute up to a phase with the generating group  $\mathcal{G} = \{I_2, U\}$ , where  $U$  is given by (27) and represents a reflection about the dashed line in Fig. 2. This can be verified graphically from Fig. 2. Suppose we apply  $B$  to  $|\phi_{11}\rangle$  and then apply  $U$ . Then the resulting vector is equal to  $|\phi_{22}\rangle$ . If, on the other

hand, we first apply  $U$  to  $|\phi_{11}\rangle$  and then apply  $B$ , then the resulting vector is the reflection of  $|\phi_{21}\rangle$  about the  $x$ -axis, which is not related to  $|\phi_{22}\rangle$  by a phase factor.

Now, consider the state set in Fig. 4. In this case,  $\mathcal{Q} = \{I_2, B\}$  and  $\mathcal{G} = \{I_2, Z\}$  where  $B$  represents a reflection about the  $x$ -axis and  $Z$  represents a reflection about the dashed line in Fig. 4. We can immediately verify from the figure that applying  $Z$  and then  $B$  to any vector in the set results in a vector that is equal up to a minus sign to the vector that results from first applying  $B$  and then  $Z$ . For example, applying  $B$  to  $|\phi_{11}\rangle$  and then applying  $Z$  results in  $|\phi_{22}\rangle$ . If, on the other hand, we first apply  $Z$  to  $|\phi_{11}\rangle$  and then apply  $B$ , then the resulting vector is the reflection of  $|\phi_{21}\rangle$  about the  $x$ -axis, which is equal to  $-|\phi_{22}\rangle$ .

### E. Optimality of the LSM

We have seen in Section V-D that the LSM operators corresponding to a CGU state set with generating group  $\mathcal{G} = \{U_i, 1 \leq i \leq l\}$  are also CGU with the same generating group. In particular, for each  $k$ , the sets  $\mathcal{S}'_k = \{\mu_{ik}, 1 \leq i \leq l\}$  and  $\mathcal{S}_k = \{\phi_{ik}, 1 \leq i \leq l\}$  are both GU with generating group  $\mathcal{G}$ . Therefore,

$$\mu_{ik}^* \phi_{ik} = \mu_k^* U_i^* U_i \phi_k = \mu_k^* \phi_k \quad (42)$$

which implies that the probability of correctly detecting each of the states in  $\mathcal{S}_k$  using the LSM is the same. It follows from Theorem 1 that if

$$\mu_k^* \phi_k = \phi_k^* M \phi_k = \alpha I, \quad 1 \leq k \leq r \quad (43)$$

then the LSM minimizes the probability of a detection error. Note that the condition  $\mu_k^* \phi_k = \alpha I$  does not depend on the choice of generator  $\phi_k$ .

In Section V-C, we showed that the LSM operators corresponding to a CGU state set with GU generators  $\{\phi_k = V_k \phi\}$  where  $V_k \in \mathcal{Q}$  and  $\mathcal{G}$  and  $\mathcal{Q}$  commute with  $\Phi\Phi^*$ , are also CGU with GU generators generated by the same group  $\mathcal{Q}$  and some generator  $\bar{\mu}$ . Therefore, for all  $k$

$$\mu_k^* \phi_k = \bar{\mu}^* V_k^* V_k \phi = \bar{\mu}^* \phi \quad (44)$$

so that the probability of correctly detecting each of the states  $\phi_{ik}$  is the same. If in addition

$$\bar{\mu}^* \phi = \phi^* M \phi = \alpha I \quad (45)$$

then combining (42), (44), and (45) with Theorem 1 we conclude that the LSM minimizes the probability of a detection error. In particular, for a rank-one ensemble,  $\bar{\mu}^* \phi$  is a scalar so that (45) is always satisfied. Therefore, for a rank-one CGU state set with commuting GU generators, the LSM minimizes the probability of a detection error. This generalizes the previous result of Barnett [22].

### F. Optimal Measurement for Arbitrary CGU State Sets

If the generators  $\phi_k$  do not satisfy (43), then the LSM is no longer guaranteed to be optimal. Nonetheless, as we now show, the optimal measurement operators that minimize the probability of a detection error are CGU with generating group  $\mathcal{G}$ . The corresponding generators can be computed very efficiently in polynomial time within any desired accuracy.



Suppose that one set of optimal measurement operators that maximize

$$J(\{\Pi_{ik}\}) = \sum_{i=1}^l \sum_{k=1}^r \text{Tr}(\rho_{ik}\Pi_{ik}) \quad (46)$$

are  $\hat{\Pi}_{ik}$ , and let

$$\hat{J} = J(\{\hat{\Pi}_{ik}\}) = \sum_{i=1}^l \sum_{k=1}^r \text{Tr}(\rho_{ik}\hat{\Pi}_{ik}).$$

Let  $r(j, i)$  be the mapping from  $\mathcal{I} \times \mathcal{I}$  to  $\mathcal{I}$  with  $\mathcal{I} = \{1, \dots, l\}$ , defined by  $r(j, i) = s$  if  $U_j^* U_i = U_s$ . Then the measurement operators  $\hat{\Pi}_{ik}^{(j)} = U_j \hat{\Pi}_{r(j,i)k} U_j^*$  for any  $1 \leq j \leq l$  are also optimal. It is easy to see that these operators satisfy (1) and that using the fact that  $\rho_{ik} = U_i \rho_k U_i^*$  for some generators  $\rho_k$

$$\begin{aligned} J(\{\hat{\Pi}_{ik}^{(j)}\}) &= \sum_{i=1}^l \sum_{k=1}^r \text{Tr}(\rho_k U_i^* U_j \hat{\Pi}_{r(j,i)k} U_j^* U_i) \\ &= \sum_{s=1}^l \sum_{k=1}^r \text{Tr}(\rho_k U_s^* \hat{\Pi}_{sk} U_s) \\ &= \sum_{i=1}^l \sum_{k=1}^r \text{Tr}(\rho_{ik} \hat{\Pi}_{ik}) = \hat{J}. \end{aligned} \quad (47)$$

Since the measurement operators

$$\{\hat{\Pi}_{ik}^{(j)} = U_j \hat{\Pi}_{r(j,i)k} U_j^*, 1 \leq i \leq l, 1 \leq k \leq r\}$$

are optimal for any  $j$ , it follows immediately that the measurement operators

$$\left\{ \bar{\Pi}_{ik} = (1/l) \sum_{j=1}^l U_j \hat{\Pi}_{r(j,i)k} U_j^*, 1 \leq i \leq l, 1 \leq k \leq r \right\}$$

are also optimal, since  $\bar{\Pi}_{ik}$  satisfy (1) and

$$J(\{\bar{\Pi}_{ik}\}) = J(\{\hat{\Pi}_{ik}^{(j)}\}) = \hat{J}.$$

Now

$$\begin{aligned} \bar{\Pi}_{ik} &= \frac{1}{l} \sum_{j=1}^l U_j \hat{\Pi}_{r(j,i)k} U_j^* \\ &= \frac{1}{l} \sum_{s=1}^l U_i U_s^* \hat{\Pi}_{sk} U_s U_i^* \\ &= U_i \left( \frac{1}{l} \sum_{s=1}^l U_s^* \hat{\Pi}_{sk} U_s \right) U_i^* \\ &= U_i \hat{\Pi}_k U_i^* \end{aligned} \quad (48)$$

where  $\hat{\Pi}_k = (1/l) \sum_{s=1}^l U_s^* \hat{\Pi}_{sk} U_s$ .

We, therefore, conclude that the optimal measurement operators can always be chosen to be CGU with the same generating group  $\mathcal{G}$  as the original state set. Thus, to find the optimal measurement operators all we need is to find the optimal generators  $\{\hat{\Pi}_k, 1 \leq k \leq r\}$ . The remaining operators are obtained by applying the group  $\mathcal{G}$  to each of the generators.

Since the optimal measurement operators satisfy  $\Pi_{ik} = U_i \Pi_k U_i^*$  and  $\rho_{ik} = U_i \rho_k U_i^*$ ,  $\text{Tr}(\rho_{ik} \Pi_{ik}) = \text{Tr}(\rho_k \Pi_k)$ , so that the problem (2) reduces to the maximization problem

$$\max_{\Pi_k \in \mathcal{B}} \sum_{k=1}^r \text{Tr}(\rho_k \Pi_k) \quad (49)$$

subject to the constraints

$$\begin{aligned} \Pi_k &\geq 0, \quad 1 \leq k \leq r \\ \sum_{i=1}^l \sum_{k=1}^r U_i \Pi_k U_i^* &= I. \end{aligned} \quad (50)$$

Since this problem is a (convex) semidefinite programming problem, the optimal generators  $\Pi_k$  can be computed very efficiently in polynomial time within any desired accuracy [33]–[35], for example, using the LMI Toolbox on Matlab. Note that the problem of (49) and (50) has  $rn^2$  real unknowns and  $r+1$  constraints, in contrast with the original maximization problem (2) and (1) which has  $lrn^2$  real unknowns and  $lr+1$  constraints.

We summarize our results regarding CGU state sets in the following theorem.

*Theorem 4 (CGU State Sets):* Let

$$\mathcal{S} = \{\rho_{ik} = U_i \rho_k U_i^*, U_i \in \mathcal{G}, 1 \leq k \leq r\}$$

be a CGU state set generated by a finite group  $\mathcal{G}$  of unitary matrices and generators  $\{\rho_k = \phi_k \phi_k^*, 1 \leq k \leq r\}$ , and let  $\Phi$  be the matrix of columns  $\phi_{ik} = U_i \phi_k$ . Then, the LSM is given by the measurement operators  $\Sigma_i = \mu_i \mu_i^*$  with

$$\mu_{ik} = U_i \mu_k$$

where

$$\mu_k = (\Phi \Phi^*)^{-1/2} \phi_k.$$

The LSM has the following properties.

- 1) The measurement operators are CGU with generating group  $\mathcal{G}$ .
- 2) The probability of correctly detecting each of the states  $\phi_{ik}$  for fixed  $k$  using the LSM is the same.
- 3) If  $\mu_k^* \phi_k = \phi_k^* (\Phi \Phi^*)^{-1/2} \phi_k = \alpha I$  for  $1 \leq k \leq r$ , then the LSM minimizes the probability of a detection error.

If, in addition, the generators  $\{\phi_k = V_k \phi, 1 \leq k \leq r\}$  are geometrically uniform with  $V_k \Phi \Phi^* = \Phi \Phi^* V_k$  for all  $k$ , then

- 1)  $\mu_{ik} = U_i V_k \bar{\mu}$  where  $\bar{\mu} = (\Phi \Phi^*)^{-1/2} \phi$  so that the LSM operators are CGU with geometrically uniform generators.
- 2) The probability of correctly detecting each of the states  $\phi_{ik}$  using the LSM is the same.
- 3) If  $\bar{\mu}^* \phi = \phi^* (\Phi \Phi^*)^{-1/2} \phi = \alpha I$ , then the LSM minimizes the probability of a detection error. In particular, if  $\phi = |\phi\rangle$  is a vector so that the state set is a rank-one ensemble, then the LSM minimizes the probability of a detection error.

For arbitrary CGU state sets, the optimal measurement operators that minimize the probability of a detection error are CGU with generating group  $\mathcal{G}$  and generators  $\Pi_k$  that

maximize  $\sum_{k=1}^r \text{Tr}(\rho_k \Pi_k)$  subject to  $\Pi_k \geq 0$ ,  $1 \leq k \leq r$  and  $\sum_{i,k} U_i \Pi_k U_i^* = I$ .

## VI. CONCLUSION

In this paper, we considered the optimal measurement operators that minimize the probability of a detection error when distinguishing between a collection of *mixed* quantum states. We first derived a general condition under which the LSM minimizes the probability of a detection error. We then considered state sets with a broad class of symmetry properties for which the LSM is optimal. Specifically, we showed that for GU state sets and for CGU state sets with generators that satisfy certain constraints, the LSM is optimal. We also showed that for arbitrary GU and CGU state sets, the optimal measurement operators have the same symmetries as the original state sets. Therefore, to compute the optimal measurement operators, we need only to compute the corresponding generators. As we showed, the generators can be computed very efficiently in polynomial time within any desired accuracy by solving a semidefinite programming problem.

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