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Oblique dual frames and shift-invariant spaces

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Abstract

Given a frame for a subspace $\mathcal{W}$ of a Hilbert space $\mathcal{H}$, we consider a class of oblique dual frame sequences. These dual frame sequences are not constrained to lie in $\mathcal{W}$. Our main focus is on shift-invariant frame sequences of the form $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ in subspaces of $L^2(\mathbb{R})$; for such frame sequences we are able to characterize the set of shift-invariant oblique dual Bessel sequences. Given frame sequences $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\{\phi_1(\cdot - k)\}_{k\in\mathbb{Z}}$, we present an easily verifiable condition implying that $\overline{\text{span}}\{\phi_1(\cdot - k)\}_{k\in\mathbb{Z}}$ contains a generator for a shift-invariant dual of $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$; in particular, the exact statement of this result implies the somewhat surprising fact that there is a unique conventional dual frame that is shift-invariant. As an application of our results we consider frame sequences generated by B-splines, and show how to construct oblique duals with prescribed regularity.

Keywords: Frames for subspaces; Oblique dual frame; Shift-invariant spaces

MSC: 42C15

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space, and suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for a subspace $\mathcal{W} \subseteq \mathcal{H}$, i.e., that $\{f_k\}_{k=1}^{\infty} \subset \mathcal{W}$ and that there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{W}.$$

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Further, let S be the frame operator for $\{f_k\}_{k=1}^{\infty}$, i.e.,

$$S:\mathcal{H}\to\mathcal{H},\quad Sf=\sum_{k=1}^{\infty}\langle f,f_k\rangle f_k.$$

Denoting the pseudo-inverse of S by $S^{\dagger}$, standard frame theory tells us that each $f\in\mathcal{W}$ has a representation

$$f=\sum_{k=1}^{\infty}\langle f,S^{\dagger}f_k\rangle f_k.$$

It is well known [14] how to characterize all Bessel sequences $\{g_k\}_{k=1}^{\infty}$ belonging to $\mathcal{W}$, such that

$$f=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k,\quad \forall f\in\mathcal{W}. \tag{1}$$

As observed by Li and Ogawa [18], (1) might hold under much weaker restrictions on $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$; in fact, it is not necessary that these sequences belong to $\mathcal{W}$, and it is not necessary that $\{f_k\}_{k=1}^{\infty}$ forms a frame. In case (1) holds for a given $\mathcal{W}$ and some sequences $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$ (satisfying certain Bessel conditions), $\{g_k\}_{k=1}^{\infty}$ is called a *pseudoframe* for $\mathcal{W}$ w.r.t. $\{f_k\}_{k=1}^{\infty}$.

In this paper we will concentrate on the case where $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{W}$ and $\{g_k\}_{k=1}^{\infty}$ is a frame for a closed subspace $\mathcal{V}$ of $\mathcal{H}$; we elaborate on the motivation for our special interest in this case at the end of this section. A frame $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{V}$ for which (1) holds is called an *oblique dual frame of* $\{f_k\}_{k=1}^{\infty}$ *on* $\mathcal{V}$; this concept was first introduced in [8,9] for finite-dimensional frames, and later generalized to the infinite-dimensional case in [11]. The terminology oblique dual originates from the relation of these frames with oblique projections, which we discuss in Section 3. As we discuss further below, the connection with oblique projections is what renders these class of frames particularly useful in the context of consistent sampling methods.

Our focus will be on frames in shift-invariant spaces; however, to put the results in perspective, we collect a few results concerning oblique duals for sequences $\{f_k\}_{k=1}^{\infty}$ in general Hilbert spaces in Section 3.

In Section 4, which is the central part of the paper, we specialize to shift-invariant frame sequences $\{\phi(\cdot-k)\}_{k\in\mathbb{Z}}$ in subspaces of $L^2(\mathbb{R})$ and characterize all oblique duals $\{\tilde{\phi}(\cdot-k)\}_{k\in\mathbb{Z}}$. We also develop an easily verifiable condition on a function ϕ_1 such that there is a dual shift-invariant frame sequence belonging to $\overline{\text{span}}\{\phi_1(\cdot-k)\}_{k\in\mathbb{Z}}$. In addition, we show that under the same direct sum condition we considered for arbitrary Hilbert spaces, there is a *unique* function generating such a dual. For, e.g., frames generated by a B-spline, we can easily use our criterion to obtain shift-invariant duals, generated by a function with prescribed regularity. This is possible, even for the B-spline $B_1=\chi_{[-1/2,1/2]}$, whose conventional dual is not even continuous.

Section 5 contains further examples of frame constructions via the results in Section 4. Finally, in Appendix A we give more information on the condition $\mathcal{H}=\mathcal{W}\oplus\mathcal{V}^{\perp}$ in general Hilbert spaces.

Before proceeding to the detailed development, in the next section we collect some basic results and definitions used throughout the paper.

To conclude this introduction, we motivate our study further and relate our work to previous results. There are a variety of contexts in which the oblique dual frames are useful. One application is in cases in which we are forced to do analysis and reconstruction in two different spaces. For example, we may be given samples of a signal $f \in \mathcal{W}$, where $\mathcal{W} \subset \mathcal{H}$, that can be described as inner products of the signal with a set of analysis vectors that span a subspace $\mathcal{V} \subset \mathcal{H}$. In this case, we cannot reconstruct f from the given samples using standard frame theory. However, using the concept of oblique dual frames allows us to reconstruct f perfectly from these samples when our general condition $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ is satisfied; see [8–10]. Furthermore, if $f \in \mathcal{H}$ does not lie entirely in $\mathcal{W}$, and we are constrained to use a particular set of reconstruction vectors (e.g., a particular reconstruction filter) that span $\mathcal{W}$, then the class of oblique duals are the unique frame vectors that result in a consistent reconstruction of f, namely a reconstruction that has the property that although, in general, is not equal to f, it nonetheless yields the same samples. There are also applications in which we may purposely choose the analysis space to be different than the synthesis space, since this allows us much more freedom in the design of the analysis frame. As we show in Section 4, if we (as in conventional frame theory) restrict the analysis and synthesis frame vectors to lie in the same shift-invariant space, then there is a unique dual frame that is shift-invariant, so that we have no freedom in choosing this frame. However, if we allow the analysis frame to lie in a different space, then there are infinitely many possibilities of frames that are shift-invariant, so that we have much more freedom in the design. Our example with duals of B-splines is a concrete case, where this is very useful.

The reader may argue that even more freedom would be obtained via pseudoframes. This is certainly true. For a bandlimited generator, a version of Theorem 4.1 can be found in [15]; however, for general functions $\phi \in L^2(\mathbb{R})$ it remains an open problem how to extend the results in Section 4 to the setting of pseudoframes.[1] Furthermore, in the context of sampling, consistency is of prime importance, which leads to the oblique dual frame vectors. Among the previous applications of pseudoframes, we mention that they have been used to define a type of generalized multiresolution analysis in [15]; a further study, aiming at construction of generalized frame sequences via unbounded operators, was reported in [17]. A related idea appears in the paper [1] by Aldroubi, in the context of average sampling in shift-invariant spaces.

Note that the discussion in [9], [11] focused on the minimal oblique dual frame, i.e., the oblique dual frame resulting in minimal-norm coefficients. Our approach here is more general, focusing on the flexibility we obtain compared to conventional frame theory.

2. Definitions and basic results

Given closed subspaces $\mathcal{W}$ and $\mathcal{V}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ (a direct sum, not necessarily orthogonal), the oblique projection of $\mathcal{H}$ on $\mathcal{W}$ along $\mathcal{V}^\perp$ is defined by

$$E_{\mathcal{W}\mathcal{V}^\perp}(w + v^\perp) = w, \quad w \in \mathcal{W},\ v^\perp \in \mathcal{V}^\perp.$$

The definition implies that $\mathcal{R}(E_{\mathcal{W}\mathcal{V}^\perp}) = \mathcal{W}$ and $\mathcal{N}(E_{\mathcal{W}\mathcal{V}^\perp}) = \mathcal{V}^\perp$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the range space and the null space, respectively, of the corresponding transformation. The orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{W}$ will be denoted by $P_\mathcal{W}$.

Note that ${E_{\mathcal{W}\mathcal{V}^\perp}}^2 = E_{\mathcal{W}\mathcal{V}^\perp}$. On the other hand, *any* projection P (i.e., a bounded linear operator on $\mathcal{H}$ for which $P^2 = P$) leads to a decomposition of $\mathcal{H}$; in fact, as proved in, e.g., [12, Proposition 38.4],

$$\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P).$$

1 This problem was solved by the authors shortly before the present article went to press.

That is, there is a one-to-one correspondence between the considered type of decomposition of $\mathcal{H}$ and projections on $\mathcal{H}$. Thus, our results obtained via the splitting assumption $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ could as well be formulated starting with a projection.

The assumption $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ will play a crucial role throughout the paper. Lemma 2.1 (below), proved by Tang [19, Theorem 2.3], deals with this condition, and relies on the concept of the angle between two subspaces. The angle from $\mathcal{V}$ to $\mathcal{W}$ is defined as the unique number $\theta(\mathcal{V},\mathcal{W}) \in [0, \pi/2]$ for which

$$\cos\theta(\mathcal{V},\mathcal{W}) = \inf_{f\in\mathcal{V},\, \|f\|=1} \|P_{\mathcal{W}} f\|.$$

Lemma 2.1. *Given closed subspaces $\mathcal{V}, \mathcal{W}$ of a separable Hilbert space $\mathcal{H}$, the following are equivalent*:

(i) $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$;
(ii) $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^\perp$;
(iii) $\cos\theta(\mathcal{V},\mathcal{W}) > 0$ *and* $\cos\theta(\mathcal{W},\mathcal{V}) > 0$.

Further comments concerning the condition $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ are in Appendix A. As a consequence of Lemma 2.1, the oblique projection $E_{\mathcal{V}\mathcal{W}^\perp}$ is also well defined in our setting. Straightforward calculation gives that the adjoint operator associated to the bounded operator $E_{\mathcal{W}\mathcal{V}^\perp}$ is

$$E_{\mathcal{W}\mathcal{V}^\perp}{}^* = E_{\mathcal{V}\mathcal{W}^\perp}. \tag{2}$$

In Section 4 we consider frames of translates. Defining the translation operator acting on functions in $L^2(\mathbb{R})$ by $T_k f(x) = f(x-k)$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$, these frames have the form $\{T_k\phi\}_{k\in\mathbb{Z}}$ for some $\phi \in L^2(\mathbb{R})$. For the sake of convenience, we will frequently refer to such a frame as being *shift-invariant*. Theorem 2.2 (below) states conditions on ϕ in order for $\{T_k\phi\}_{k\in\mathbb{Z}}$ to be a frame for its closed linear span ($\{T_k\phi\}_{k\in\mathbb{Z}}$ cannot be a frame for all of $L^2(\mathbb{R})$, cf. [7]). Before stating the theorem, we need some further definitions.

For $f \in L^1(\mathbb{R})$ we denote the Fourier transform by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i x\gamma}\,\mathrm{d}x.$$

As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$.

For a sequence $\mathbf{c} = \{c_k\} \in \ell^2$, we define the discrete-time Fourier transform as the function in $L^2(0,1)$ given by

$$\mathcal{F}\mathbf{c}(\gamma) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{-2\pi i k\gamma}.$$

Note that the discrete-time Fourier transform is 1-periodic.

The pre-frame operator T associated with a frame $\{T_k\phi\}_{k\in\mathbb{Z}}$ is given by

$$T : \ell^2 \to \mathcal{H}, \quad T\{c_k\} = \sum_{k\in\mathbb{Z}} c_k T_k\phi,$$

and its adjoint is given by

$$T^* : \mathcal{H} \to \ell^2, \quad T^* f = \big\{\langle f, T_k\phi\rangle\big\}_{k\in\mathbb{Z}}.$$

The discrete-time Fourier transform acting on the adjoint of the pre-frame operator is

$$\mathcal{F}T^* f = \sum_k \hat{f}(\cdot + k)\overline{\hat{\phi}(\cdot + k)}, \quad f \in L^2(\mathbb{R}). \tag{3}$$

The Fourier transform acting on the pre-frame operator is

$$\mathcal{F}T\{c_k\} = \mathcal{F}\sum_{k\in\mathbb{Z}} c_k T_k \phi = \hat{\phi}(\cdot)\mathcal{F}\mathbf{c}(\cdot), \quad \mathbf{c} \in \ell^2. \tag{4}$$

Given $\phi \in L^2(\mathbb{R})$, let

$$\Phi(\gamma) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\gamma + k)\right|^2,$$

and

$$\mathcal{N}(\Phi) = \left\{\gamma\colon \Phi(\gamma) = 0\right\}.$$

The theorem below is basically due to Benedetto and Li [2,3], with some technical assumptions removed by various authors (see [5] for details).

Theorem 2.2. *Let $\phi \in L^2(\mathbb{R})$. For any $A, B > 0$, the following characterizations hold*:

(i) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a Bessel sequence with bound B if and only if*

$$\Phi(\gamma) \leqslant B \quad \text{a.e. } \gamma \in [0, 1].$$

(ii) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is an orthonormal sequence if and only if*

$$\Phi(\gamma) = 1 \quad \text{a.e. } \gamma \in [0, 1].$$

(iii) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a Riesz sequence with bounds A, B if and only if*

$$A \leqslant \Phi(\gamma) \leqslant B \quad \text{a.e. } \gamma \in [0, 1].$$

(iv) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a frame sequence with bounds A, B if and only if*

$$A \leqslant \Phi(\gamma) \leqslant B \quad \text{a.e. on } \left\{\gamma\colon \Phi(\gamma) \neq 0\right\}.$$

In case $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence, we have

$$\mathcal{W} := \overline{\operatorname{span}}\{T_k\phi\}_{k\in\mathbb{Z}} = \left\{\sum_{k\in\mathbb{Z}} c_k T_k\phi\colon \{c_k\} \in \ell^2\right\}. \tag{5}$$

A space of this type is said to be *shift-invariant*.

3. The oblique dual frames on $\mathcal{V}$

As already mentioned, our focus is on frames in shift-invariant spaces. However, to put the results in perspective, we first collect a few results concerning sequences in general Hilbert spaces.

Throughout the section we will consider a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ belonging to a Hilbert space $\mathcal{H}$, and let

$$\mathcal{W} = \overline{\text{span}}\{f_k\}_{k=1}^{\infty}.$$

The Bessel assumption implies that the pre-frame operator associated with $\{f_k\}_{k=1}^{\infty}$ is bounded; we denote it by T.

Our first lemma relates our setup to classical frame theory. The actual content of the lemma is new; the idea of stating the lemma in this form comes from the paper [17].

Lemma 3.1. *Assume that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences in $\mathcal{H}$, and let $\mathcal{V} = \overline{\text{span}}\{g_k\}_{k=1}^{\infty}$. Assume that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. Then the following are equivalent*:

(i) $f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k$, $\forall f \in \mathcal{W}$;
(ii) $E_{\mathcal{W}\mathcal{V}^{\perp}} f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k$, $\forall f \in \mathcal{H}$;
(iii) $E_{\mathcal{V}\mathcal{W}^{\perp}} f = \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k$, $\forall f \in \mathcal{H}$;
(iv) $\langle E_{\mathcal{V}\mathcal{W}^{\perp}} f, g\rangle = \sum_{k=1}^{\infty} \langle f, f_k\rangle \langle g_k, g\rangle$, $\forall f, g \in \mathcal{H}$;
(v) $\langle E_{\mathcal{W}\mathcal{V}^{\perp}} f, g\rangle = \sum_{k=1}^{\infty} \langle f, g_k\rangle \langle f_k, g\rangle$, $\forall f, g \in \mathcal{H}$.

In case the equivalent conditions are satisfied, $\{g_k\}_{k=1}^{\infty}$ is an oblique dual frame of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$, and $\{f_k\}_{k=1}^{\infty}$ is an oblique dual frame of $\{g_k\}_{k=1}^{\infty}$ on $\mathcal{W}$. Furthermore, $\{f_k\}_{k=1}^{\infty}$ and $\{P_{\mathcal{W}} g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{W}$ (in the sense of classical frame theory), and $\{g_k\}_{k=1}^{\infty}$ and $\{P_{\mathcal{V}} f_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{V}$.

Proof. The equivalence between (i) and (ii) is clear. Now, let U denote the pre-frame operator associated with $\{g_k\}_{k=1}^{\infty}$. In terms of the pre-frame operators T, U, (ii) means that $TU^* = E_{\mathcal{W}\mathcal{V}^{\perp}}$; via (2), this is equivalent to

$$UT^* = E_{\mathcal{V}\mathcal{W}^{\perp}}, \tag{6}$$

which is identical to the statement in (iii). It is also clear that (iii) implies (iv) and that (ii) implies (v). To prove that (iv) implies (iii) we fix $f \in \mathcal{H}$ and note that $\sum_{k=1}^{\infty} \langle f, f_k\rangle g_k$ is well defined as an element in $\mathcal{H}$ because $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences. Now the assumption in (iv) shows that

$$\left\langle E_{\mathcal{V}\mathcal{W}^{\perp}} f - \sum_{k=1}^{\infty} \langle f, f_k\rangle g_k, g \right\rangle = 0, \quad \forall g \in \mathcal{H},$$

and (iii) follows. In the same way we can show that (v) implies (ii).

In case the equivalent conditions are satisfied, (iv) and the fact that $E_{\mathcal{V}\mathcal{W}^{\perp}} f = f$ for $f \in \mathcal{V}$ shows that

$$\|f\|^2 = \sum_{k=1}^{\infty} \langle f, f_k\rangle \langle g_k, f\rangle = \sum_{k=1}^{\infty} \langle f, P_{\mathcal{V}} f_k\rangle \langle g_k, f\rangle, \quad \forall f \in \mathcal{V}.$$

Using Cauchy–Schwarz' inequality and that $\{g_k\}_{k=1}^{\infty}$ (respectively $\{P_{\mathcal{V}} f_k\}_{k=1}^{\infty}$) is a Bessel sequence, we obtain that $\{P_{\mathcal{V}} f_k\}_{k=1}^{\infty}$ (respectively $\{g_k\}_{k=1}^{\infty}$) satisfies the lower frame condition for all $f \in \mathcal{V}$, i.e., both are frames for $\mathcal{V}$. That they are dual frames in the classical sense follows from (iii). The proof of $\{f_k\}_{k=1}^{\infty}$ and $\{P_{\mathcal{W}} g_k\}_{k=1}^{\infty}$ being frames for $\mathcal{W}$ is similar. Now the statement about the relevant frames being oblique duals follows from the definition. □

The following theorem characterizes the oblique dual frames on $\mathcal{V}$. After completing the paper we became aware that a more general result for pseudoframes has been obtained independently by Li and Ogawa [18].

Theorem 3.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a subspace $\mathcal{W} \subseteq \mathcal{H}$, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. Then the oblique dual frames of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ E_{\mathcal{V}\mathcal{W}^{\perp}} S^{\dagger} f_k + h_k - \sum_{j=1}^{\infty} \langle S^{\dagger} f_k, f_j \rangle h_j \right\}_{k=1}^{\infty}, \tag{7}$$

where $\{h_k\}_{k=1}^{\infty} \subset \mathcal{V}$ is a Bessel sequence.

For completeness, a proof of Theorem 3.2 is given in Appendix B.

Given the setup in Theorem 3.2 and considering $f \in \mathcal{H}$, it is proved in [11] that among all coefficients $\{c_k\}_{k=1}^{\infty} \in \ell^2$ for which

$$E_{\mathcal{W}\mathcal{V}^{\perp}} f = \sum_{k=1}^{\infty} c_k f_k,$$

the coefficients with minimal ℓ^2-norm are $\{c_k\}_{k=1}^{\infty} = \{\langle f, E_{\mathcal{V}\mathcal{W}^{\perp}} S^{\dagger} f_k \rangle\}_{k=1}^{\infty}$; the sequence $\{E_{\mathcal{V}\mathcal{W}^{\perp}} S^{\dagger} f_k\}_{k=1}^{\infty}$ is the oblique dual frame resulting from Theorem 3.2 if we choose $h = 0$. In contrast to the present paper, [9,11] concentrate on this particular dual.

In case we have obtained the reconstruction formula (1) for any Bessel sequence $\{g_k\}_{k=1}^{\infty}$, i.e., an expansion of the pseudoframe type in [18], we now show how to find an oblique dual frame of $\{f_k\}_{k=1}^{\infty}$ on an arbitrary closed subspace $\mathcal{U}$ for which $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}^{\perp}$. The result shows that having reconstruction with respect to one family of analysis vectors (namely, $\{g_k\}_{k=1}^{\infty}$) immediately delivers a whole class of analysis vectors leading to reconstruction.

Proposition 3.3. *Assume that $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are Bessel sequences in $\mathcal{H}$, and that*

$$f = \sum_{k=1}^{\infty} \langle f, h_k \rangle f_k, \quad \forall f \in \mathcal{W}.$$

Let $\mathcal{U}$ be any closed subspace of $\mathcal{H}$ for which $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}^{\perp}$. Then $\{E_{\mathcal{U}\mathcal{W}^{\perp}} h_k\}_{k\in\mathbb{Z}}$ is an oblique dual frame of $\{f_k\}_{k\in\mathbb{Z}}$ on $\mathcal{U}$.

Proof. We have that

$$f = \sum_{k\in\mathbb{Z}} \langle E_{\mathcal{W}\mathcal{U}^{\perp}} f, h_k \rangle f_k = \sum_{k\in\mathbb{Z}} \langle f, E_{\mathcal{U}\mathcal{W}^{\perp}} h_k \rangle f_k, \quad \forall f \in \mathcal{W}.$$

In addition, for any $f \in \mathcal{U}^{\perp}$, $\langle f, E_{\mathcal{U}\mathcal{W}^{\perp}} h_k \rangle = 0$ so that $\sum_{k\in\mathbb{Z}} \langle f, E_{\mathcal{U}\mathcal{W}^{\perp}} h_k \rangle f_k = 0$. Thus,

$$E_{\mathcal{W}\mathcal{U}^{\perp}} f = \sum_{k\in\mathbb{Z}} \langle f, E_{\mathcal{U}\mathcal{W}^{\perp}} h_k \rangle f_k, \quad \forall f \in \mathcal{H}. \tag{8}$$

Denoting the pre-frame operators of $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ by T and V, respectively, we have from (8) that $TV^* = E_{\mathcal{W}\mathcal{U}^{\perp}}$, or equivalently,

$$VT^* = E_{\mathcal{U}\mathcal{W}^{\perp}}.$$

This can be expressed as

$$E_{\mathcal{U}\mathcal{W}^\perp} f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle E_{\mathcal{U}\mathcal{W}^\perp} h_k, \quad \forall f \in \mathcal{H};$$

in particular,

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle E_{\mathcal{U}\mathcal{W}^\perp} h_k, \quad \forall f \in \mathcal{U}.$$

For any $f \in \mathcal{U}$ we then have that

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \langle E_{\mathcal{U}\mathcal{W}^\perp} h_k, f\rangle. \tag{9}$$

Using Cauchy–Schwarz' inequality on (9) now yields that $\{E_{\mathcal{U}\mathcal{W}^\perp} h_k\}_{k=1}^\infty$ satisfies the lower frame condition on $\mathcal{U}$; thus, it is a frame for $\mathcal{U}$, and we obtain the conclusion. □

4. Frame sequences in shift-invariant spaces

In this section we consider frames in shift-invariant spaces. Assuming that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an overcomplete frame for $\mathcal{W}$ (see (5)), general frame theory tells us that there exist infinitely many different choices of sequences $\{g_k\}_{k\in\mathbb{Z}} \subset \mathcal{W}$ such that

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle T_k\phi, \quad \forall f \in \mathcal{W}. \tag{10}$$

However, Corollary 4.4 (below) states that if we want $\{g_k\}_{k\in\mathbb{Z}}$ to consist of integer translates of a single function, so that $g_k = T_k\tilde{\phi}$ for some $\tilde{\phi} \in \mathcal{W}$, then $\tilde{\phi}$ is *unique*. That is, standard frame theory does not give us any freedom in the choice of the dual if we want the shift-invariant structure. This motivates the rest of the results in this section: in fact, we prove that *infinitely* many choices of $\tilde{\phi}$ are possible if we do not require $\tilde{\phi}$ to belong to $\mathcal{W}$.

For $\phi \in L^2(\mathbb{R})$, we let

$$\mathcal{W} = \overline{\operatorname{span}}\{T_k\phi\}_{k\in\mathbb{Z}}, \tag{11}$$

and denote the orthogonal projection of $L^2(\mathbb{R})$ onto $\mathcal{W}$ by $P_{\mathcal{W}}$.

Given two Bessel sequences $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$, the following theorem provides a necessary and sufficient condition on the generators such that $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ is a dual of $\{T_k\phi\}_{k\in\mathbb{Z}}$.

Theorem 4.1. *Let $\phi, \tilde{\phi} \in L^2(\mathbb{R})$, and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are Bessel sequences. Then the following are equivalent*:

(i) $f = \sum_{k\in\mathbb{Z}} \langle f, T_k\tilde{\phi}\rangle T_k\phi, \forall f \in \mathcal{W}$;

(ii) $\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma + k)\overline{\hat{\tilde{\phi}}(\gamma + k)} = 1$ *a.e. on* $\{\gamma\colon \Phi(\gamma) \neq 0\}$.

If the conditions are satisfied, then $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{P_{\mathcal{W}}T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are dual frames for $\overline{\operatorname{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$.

Proof. First, consider an arbitrary function $f \in L^2(\mathbb{R})$ for which $\gamma \mapsto \sum_{k\in\mathbb{Z}} |\hat{f}(\gamma+k)|^2$ is bounded. Then Cauchy–Schwarz' inequality implies that

$$\left[\gamma \mapsto \sum_{k\in\mathbb{Z}} \hat{f}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)}\right] \in L^2(0,1).$$

Now observe that

$$\begin{aligned}\mathcal{F}\sum_{k\in\mathbb{Z}}\langle f, T_k\tilde{\phi}\rangle T_k\phi(\gamma) &= \sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}\hat{f}(\mu)\overline{\hat{\tilde{\phi}}(\mu)}e^{2\pi ik\mu}\,d\mu\,\hat{\phi}(\gamma)e^{-2\pi ik\gamma}\\ &= \sum_{k\in\mathbb{Z}}\int_0^1\sum_{n\in\mathbb{Z}}\hat{f}(\mu+n)\overline{\hat{\tilde{\phi}}(\mu+n)}e^{2\pi ik\mu}\,d\mu\,\hat{\phi}(\gamma)e^{-2\pi ik\gamma}\\ &= \hat{\phi}(\gamma)\sum_{n\in\mathbb{Z}}\hat{f}(\gamma+n)\overline{\hat{\tilde{\phi}}(\gamma+n)}. \end{aligned}\tag{12}$$

Assuming that (i) holds and letting $f=\phi$, it follows that:

$$\sum_{k\in\mathbb{Z}}\hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)} = 1 \quad \text{a.e. on } \{\gamma\colon \hat{\phi}(\gamma)\neq 0\}.$$

Using the above calculation with γ replaced by $\gamma+m$ for some $m\in\mathbb{Z}$ (and using the periodicity of $\gamma\mapsto\sum_{k\in\mathbb{Z}}\hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)}$) we even arrive at

$$\sum_{k\in\mathbb{Z}}\hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)} = 1 \quad \text{a.e. on } \{\gamma\colon \hat{\phi}(\gamma+m)\neq 0\},\ \forall m\in\mathbb{Z}.$$

This proves (ii). On the other hand, assuming (ii), our calculation (12) shows that for $m\in\mathbb{Z}$,

$$\begin{aligned}\mathcal{F}\sum_{k\in\mathbb{Z}}\langle T_m\phi, T_k\tilde{\phi}\rangle T_k\phi(\gamma) &= \hat{\phi}(\gamma)\sum_{n\in\mathbb{Z}}\mathcal{F}T_m\phi(\gamma+n)\overline{\hat{\tilde{\phi}}(\gamma+n)}\\ &= \hat{\phi}(\gamma)\sum_{n\in\mathbb{Z}}\hat{\phi}(\gamma+n)e^{-2\pi im(\gamma+n)}\overline{\hat{\tilde{\phi}}(\gamma+n)}\\ &= \hat{\phi}(\gamma)e^{-2\pi im\gamma} = \mathcal{F}T_m\phi(\gamma).\end{aligned}$$

Thus, (i) holds for all functions $T_m\phi$, $m\in\mathbb{Z}$, and hence for any finite linear combination of such functions. By continuity of the map $f\mapsto\sum_{k\in\mathbb{Z}}\langle f, T_k\tilde{\phi}\rangle T_k\phi$, (i) therefore holds for all $f\in\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$.

That $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{P_{\mathcal{W}}T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are dual frames for $\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$ follows by standard frame theory. □

The results in Section 3 have immediate consequences for frames of translates. In particular, Proposition 3.3 gives a principle for obtaining an oblique dual frame in a space $\mathcal{V}$ for which $L^2(\mathbb{R}) = \mathcal{W}\oplus\mathcal{V}^\perp$; we now prove that if $\mathcal{V}$ is shift-invariant, then this oblique dual frame is shift-invariant as well. In Proposition 4.8 we consider the condition $L^2(\mathbb{R}) = \mathcal{W}\oplus\mathcal{V}^\perp$ in more detail.

Corollary 4.2. *Assume that the setup and the equivalent conditions in Theorem* 4.1 *are satisfied. Then, given any closed subspace* $\mathcal{V}$ *such that* $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$*, the sequence* $\{E_{\mathcal{V}\mathcal{W}^\perp} T_k \tilde{\phi}\}_{k\in\mathbb{Z}}$ *is an oblique dual frame of* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *on* $\mathcal{V}$*; in case* $\mathcal{V}$ *is shift-invariant this sequence is shift-invariant, in fact,*

$$E_{\mathcal{V}\mathcal{W}^\perp} T_k = T_k E_{\mathcal{V}\mathcal{W}^\perp}.$$

Proof. That $\{E_{\mathcal{V}\mathcal{W}^\perp} T_k \tilde{\phi}\}_{k\in\mathbb{Z}}$ is an oblique dual frame of $\{T_k\phi\}_{k\in\mathbb{Z}}$ on $\mathcal{V}$ follows from Proposition 3.3; thus, all we have to prove is that shift-invariance of $\mathcal{V}$ implies that $E_{\mathcal{V}\mathcal{W}^\perp} T_k = T_k E_{\mathcal{V}\mathcal{W}^\perp}$. Now, given $f \in L^2(\mathbb{R})$, $f = v + w^\perp$ for some $v \in \mathcal{V}$, $w^\perp \in \mathcal{W}^\perp$,

$$T_k E_{\mathcal{V}\mathcal{W}^\perp} f = T_k v.$$

It is easy to see that since $\mathcal{W}$ is shift-invariant, then also $\mathcal{W}^\perp$ is shift-invariant. Thus,

$$E_{\mathcal{V}\mathcal{W}^\perp} T_k f = E_{\mathcal{V}\mathcal{W}^\perp} T_k v + E_{\mathcal{V}\mathcal{W}^\perp} T_k w^\perp = T_k v. \qquad \square$$

A related result (for pseudoframe decompositions, but with orthogonal projections instead of general projections) is stated in [18], as a step towards a construction of a class of shift-invariant duals.

Assuming that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence, we now search for conditions on a function ϕ_1 which imply that the subspace

$$\mathcal{V} := \overline{\text{span}}\{T_k\phi_1\}_{k\in\mathbb{Z}} \tag{13}$$

contains a function $\tilde{\phi}$ generating an oblique dual $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ of $\{T_k\phi\}_{k\in\mathbb{Z}}$. Theorem 4.3 below gives such conditions.

Theorem 4.3. *Let* $\phi, \phi_1 \in L^2(\mathbb{R})$*, and assume that* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *and* $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ *are frame sequences. If there exists a constant* $A > 0$ *such that*

$$\left|\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\overline{\hat{\phi}_1(\gamma+k)}\right| \geqslant A \quad \text{a.e. on } \{\gamma\colon \Phi(\gamma) \neq 0\}, \tag{14}$$

then the following holds:

(i) *There exists a function* $\tilde{\phi} \in \mathcal{V}$ *such that*

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\tilde{\phi}\rangle T_k\phi, \quad \forall f \in \overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}; \tag{15}$$

(ii) *One choice of* $\tilde{\phi} \in \mathcal{V}$ *satisfying* (15) *is given in the Fourier domain by*

$$\hat{\tilde{\phi}}(\gamma) = \begin{cases} \frac{\hat{\phi}_1(\gamma)}{\overline{\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\overline{\hat{\phi}_1(\gamma+k)}}} & \text{on } \{\gamma\colon \Phi(\gamma) \neq 0\}, \\ 0 \quad \text{on } \{\gamma\colon \Phi(\gamma) = 0\}. \end{cases}$$

(iii) *There is a unique function* $\tilde{\phi} \in \mathcal{V}$ *such that* (15) *is satisfied, if and only if*

$$\mathcal{N}(\Phi) = \mathcal{N}(\Phi_1);$$

if this condition is satisfied, $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{V}$ *and an oblique dual of* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *on* $\mathcal{V}$*.*

Proof. First, we note that the assumption of $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ being frame sequences implies that the sum in (14) is bounded above (use Cauchy–Schwarz' inequality and Theorem 2.2). The functions in $\mathcal{V}=\overline{\text{span}}\{T_k\phi_1\}_{k\in\mathbb{Z}}$ have the form $\tilde{\phi}=\sum_{k\in\mathbb{Z}}h_kT_k\phi_1$ for some $\{h_k\}\in\ell^2$, or, in the Fourier domain,

$$\hat{\tilde{\phi}}(\gamma)=H(\gamma)\hat{\phi}_1(\gamma) \tag{16}$$

for some $H\in L^2(0,1)$. The function $\tilde{\phi}$ satisfies the conditions in Theorem 4.1 if and only if

$$H(\gamma)\sum_{k\in\mathbb{Z}}\hat{\phi}(\gamma+k)\overline{\hat{\phi}_1(\gamma+k)}=1 \quad \text{on } \{\gamma\colon\ \Phi(\gamma)\neq 0\}.$$

Any function H satisfying this is bounded below and above on $\{\gamma\colon\ \Phi(\gamma)\neq 0\}$; extending it in an arbitrary way to a function in $L^2(0,1)$ yields a function $\tilde{\phi}\in\mathcal{V}$ satisfying (15).

The assumption (14) further implies that

$$\mathcal{N}(\Phi_1)\subseteq\mathcal{N}(\Phi).$$

If $\mathcal{N}(\Phi)=\mathcal{N}(\Phi_1)$, then (16) shows that $\hat{\tilde{\phi}}(\gamma)=0$ on $\{\gamma\colon\ \Phi(\gamma)=0\}$, no matter how H is chosen on this set. Thus, in this case there is a unique function $\tilde{\phi}$ fulfilling the requirements, namely, the one given in (16). On the other hand, if $\mathcal{N}(\Phi_1)$ is a proper subspace of $\mathcal{N}(\Phi)$, different choices of H on $\mathcal{N}(\Phi)\setminus\mathcal{N}(\Phi_1)$ will lead to different values for the function $\hat{\tilde{\phi}}(\gamma)$ because $\hat{\phi}_1(\gamma)\neq 0$ on $\mathcal{N}(\Phi)\setminus\mathcal{N}(\Phi_1)$; thus, in this case, there exist several choices of a function $\tilde{\phi}$ satisfying (15). □

In Proposition 4.8 we will show that the conditions leading to a unique oblique dual frame in Theorem 4.3 are equivalent to $L^2(\mathbb{R})=\mathcal{W}\oplus\mathcal{V}^{\perp}$, i.e., the condition we used in our analysis of oblique duals in general Hilbert spaces.

It follows from Theorem 4.3 that there is a unique function in $\mathcal{W}$ which generates a dual frame having the shift-invariant structure:

Corollary 4.4. *Let $\phi\in L^2(\mathbb{R})$ and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence. Then there is a unique function $\tilde{\phi}\in\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$ such that*

$$f=\sum_{k\in\mathbb{Z}}\langle f,T_k\tilde{\phi}\rangle T_k\phi,\quad \forall f\in\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}},$$

namely, $\tilde{\phi}=S^{\dagger}\phi$.

Corollary 4.4 is surprising in light of the fact that an overcomplete frame sequence $\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$ always has several dual frames belonging to $\mathcal{W}$; it shows that the additional wish of having shift-invariance removes the freedom. This is exactly where Theorem 4.3 comes in handy: it gives us some freedom back by allowing generators $\tilde{\phi}\notin\mathcal{W}$.

It is well known how to construct biorthogonal bases of compactly supported wavelets via B-splines. Using the pseudoframe approach, Li [16] has been able to find smoother duals, without increasing the support and the length of the associated filters.

Our approach immediately shows that for shift-invariant frames generated by B-splines we can find oblique duals of arbitrary smoothness, however, with increased support.

Example 4.5. The B-splines are defined inductively by $B_1 = \chi_{[-1/2,1/2]}$ and

$$B_{n+1}(x) = B_n * B_1(x), \quad n \in \mathbb{N}.$$

The Fourier transform of B_n is given by

$$\mathcal{F}B_n(\gamma) = \left(\frac{\sin \pi\gamma}{\pi\gamma}\right)^n.$$

It is well known that $\{T_k B_n\}_{k\in\mathbb{Z}}$ is a Riesz sequence for any $n \in \mathbb{N}$. For any $m \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \mathcal{F}B_n(\gamma+k)\overline{\mathcal{F}B_{n+2m}(\gamma+k)} = \sum_{k\in\mathbb{Z}} \left(\frac{\sin \pi(\gamma+k)}{\pi(\gamma+k)}\right)^{2(m+n)};$$

by the fact that $\{T_k B_{n+m}\}_{k\in\mathbb{Z}}$ is a Riesz sequence, the infimum of this function is strictly positive. Thus, by Theorem 4.3 there exists for any $m \in \mathbb{N}$ a unique function $\tilde{\phi} \in \overline{\text{span}}\{T_k B_{n+2m}\}_{k\in\mathbb{Z}}$, which generates an oblique dual frame of $\{T_k B_n\}_{k\in\mathbb{Z}}$. That is, for an arbitrary spline B_n, we can find an oblique dual frame, for which the generator has prescribed smoothness. In contrast, the classical dual of $\{T_k B_1\}_{k\in\mathbb{Z}}$ is generated by B_1 itself, which is not even continuous.

In Figs. 1 and 2 we plot the generator of the oblique dual frame of $\{T_k B_1\}_{k\in\mathbb{Z}}$ corresponding to $m = 1$ and $m = 3$, respectively.

In the remaining part of this section we investigate the condition $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$ versus (14). The reader may have observed that in the general theory, the condition $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$ played the major role, while (14) was used for frames of translates. Our purpose is to relate these conditions. Before we do so in Proposition 4.8, we need some preparation.

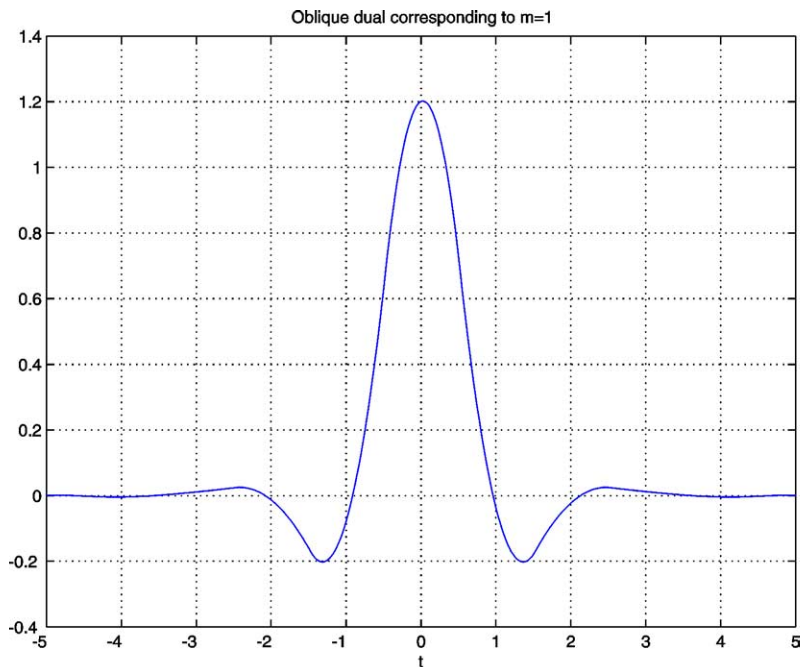

Fig. 1. The generator of the oblique dual of $\{T_k B_1\}_{k\in\mathbb{Z}}$ corresponding to $m = 1$.

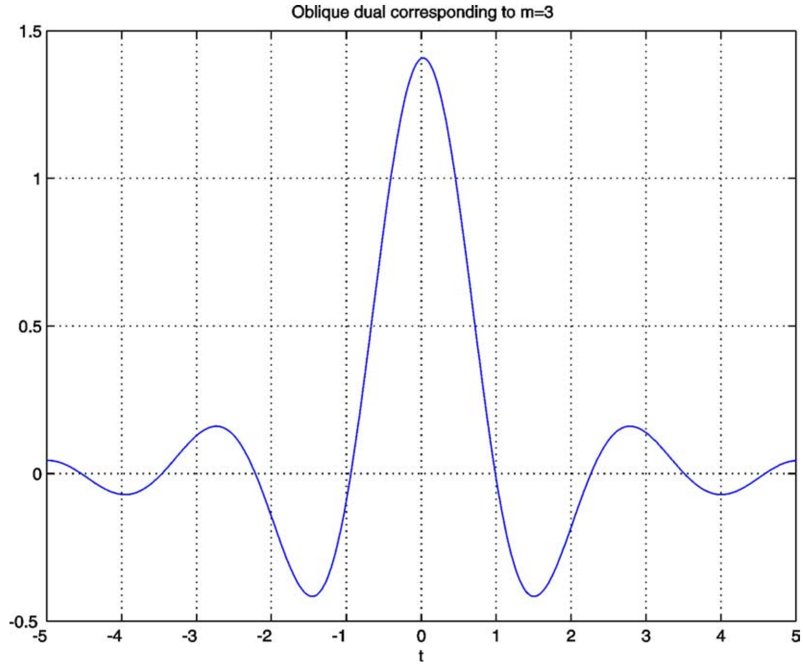

Fig. 2. The generator of the oblique dual of $\{T_k B_1\}_{k\in\mathbb{Z}}$ corresponding to $m = 3$.

As before, we denote the pre-frame operator for $\{T_k\phi\}_{k\in\mathbb{Z}}$ by T; whenever $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ is known to be a Bessel sequence, we denote its pre-frame operator by V. We continue to define the spaces $\mathcal{W}$, $\mathcal{V}$ as in (11) and (13).

Lemma 4.6. *Let $\phi, \phi_1 \in L^2(\mathbb{R})$, and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ are frame sequences. If $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$, then V^*T maps $\mathcal{R}(T^*)$ bijectively onto $\mathcal{R}(V^*)$.*

Proof. First, assume that $V^*T\mathbf{c} = 0$ for some $\mathbf{c} \in \mathcal{R}(T^*)$. Then

$$T\mathbf{c} \in \mathcal{W} \cap \mathcal{N}(V^*) = \mathcal{W} \cap \mathcal{R}(V)^\perp = \mathcal{W} \cap \mathcal{V}^\perp = \{0\},$$

so that

$$\mathbf{c} \in \mathcal{N}(T) \cap \mathcal{R}(T^*) = \mathcal{N}(T) \cap \mathcal{N}(T)^\perp = \{0\}.$$

Hence V^*T is injective. To prove that V^*T maps $\mathcal{R}(T^*)$ onto $\mathcal{R}(V^*)$, let $\mathbf{c} \in \mathcal{R}(V^*)$. Then there exists an element $h = w + v^\perp \in L^2(\mathbb{R})$, $w \in \mathcal{W}$, $v^\perp \in \mathcal{V}^\perp$, with $V^*h = V^*w = \mathbf{c}$. Since $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{W}$ we can find a sequence $\mathbf{d} \in \mathcal{R}(T^*)$ such that $T\mathbf{d} = w$. Hence, $V^*T\mathbf{d} = \mathbf{c}$ and the range of V^*T equals $\mathcal{R}(V^*)$. □

Lemma 4.7. *If $\phi \in L^2(\mathbb{R})$ and $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence, then*

$$\mathcal{R}(T^*) = \{\mathbf{c} \in \ell^2\colon\ \mathcal{F}\mathbf{c} = 0 \text{ on } \mathcal{N}(\Phi)\}.$$

Proof. If $\mathbf{c} \in \mathcal{R}(T^*)$, then $c_k = \langle y, T_k\phi\rangle$ for some $y \in L^2(\mathbb{R})$. By (3),

$$\mathcal{F}\mathbf{c}(\gamma) = \sum_{k\in\mathbb{Z}} \hat{y}(\gamma + k)\overline{\hat{\phi}(\gamma + k)}. \tag{17}$$

It is clear that $\mathcal{F}\mathbf{c} = 0$ on $\mathcal{N}(\Phi)$. Conversely, if $\mathbf{c} \in \ell^2$ and $\mathcal{F}\mathbf{c}(\gamma) = 0$ on $\{\gamma\colon \Phi(\gamma) = 0\}$, we can define $y \in L^2(\mathbb{R})$ by

$$\hat{y}(\gamma) = \begin{cases} \frac{\mathcal{F}\mathbf{c}(\gamma)\hat{\phi}(\gamma)}{\sum_{k\in\mathbb{Z}}|\hat{\phi}(\gamma+k)|^2} & \text{on } \{\gamma\colon \Phi(\gamma) \neq 0\}, \\ 0 \quad \text{on } \{\gamma\colon \Phi(\gamma) = 0\}, \end{cases}$$

then

$$\mathcal{F}T^*y(\gamma) = \sum_{k\in\mathbb{Z}} \hat{y}(\gamma + k)\overline{\hat{\phi}(\gamma + k)} = \mathcal{F}\mathbf{c}(\gamma),$$

i.e., $\mathbf{c} = T^*y \in \mathcal{R}(T^*)$. $\square$

Given two sequences $\mathbf{c}, \mathbf{d} \in \ell^2$, we define the convolution $\mathbf{c} * \mathbf{d}$ by

$$\mathbf{c} * \mathbf{d} = \left\{\sum_{m\in\mathbb{Z}} c_m d_{k-m}\right\}_{k\in\mathbb{Z}}.$$

Proposition 4.8. *Let $\phi, \phi_1 \in L^2(\mathbb{R})$, and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\phi_1\}_{k\in\mathbb{Z}}$ are frame sequences. Then the following are equivalent*:

(i) $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$;
(ii) $\mathcal{N}(\Phi) = \mathcal{N}(\Phi_1)$ *and there exists a constant* $A > 0$ *such that*

$$A \leqslant \left|\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma + k)\overline{\hat{\phi}_1(\gamma + k)}\right| \quad \text{on } \bigl\{\gamma\colon \Phi(\gamma) \neq 0\bigr\}. \tag{18}$$

Proof. First, we prove (i) $\Rightarrow$ (ii). Given $\mathbf{c}$, let $\mathbf{e} = V^*T\mathbf{c}$. Then

$$e_k = \left\langle\sum_{m\in\mathbb{Z}} c_m T_m\phi, T_k\phi_1\right\rangle = \sum_{m\in\mathbb{Z}} c_m\langle\phi, T_{k-m}\phi_1\rangle,$$

so that $\mathbf{e} = \mathbf{c} * \mathbf{d}$, where $d_k = \langle\phi, T_k\phi_1\rangle$. From (3),

$$\mathcal{F}\mathbf{d} = \sum_{k\in\mathbb{Z}} \hat{\phi}(\cdot + k)\overline{\hat{\phi}_1(\cdot + k)}.$$

Thus,

$$\mathcal{F}V^*T\mathbf{c} = \mathcal{F}\mathbf{c}\,\mathcal{F}\mathbf{d} = \mathcal{F}\mathbf{c}\sum_{k\in\mathbb{Z}} \hat{\phi}(\cdot + k)\overline{\hat{\phi}_1(\cdot + k)}. \tag{19}$$

From Lemma 4.6 we have that $\mathcal{F}V^*T\mathbf{c}$ maps $\mathcal{R}(T^*)$ bijectively onto $\mathcal{F}\mathcal{R}(V^*)$; now the characterization of $\mathcal{R}(T^*)$ (Lemma 4.7) shows that $\mathcal{N}(\Phi) \subseteq \mathcal{N}(\Phi_1)$. However, due to Lemma 2.1 we can interchange the roles of $\mathcal{W}$ and $\mathcal{V}$; thus we also have that $\mathcal{N}(\Phi_1) \subseteq \mathcal{N}(\Phi)$. The existence of the lower bound in (18) also follows from (19).

For the proof of (ii) $\Rightarrow$ (i), from Lemma 2.1, it is sufficient to show that

$$\cos\theta(\mathcal{W},\mathcal{V})>0 \quad\text{and}\quad \cos\theta(\mathcal{V},\mathcal{W})>0.$$

We first estimate

$$\cos\theta(\mathcal{V},\mathcal{W})=\inf_{f\in\mathcal{V},\ \|f\|=1}\|P_{\mathcal{W}}f\|.$$

Any $f\in\mathcal{V}$ can be expressed as $f=\sum_k c_k T_k\phi_1$ so that $\hat{f}(\gamma)=\mathcal{F}\mathbf{c}(\gamma)\hat{\phi}_1(\gamma)$. Assuming that $\|f\|=1$, and denoting the characteristic function for the complement of $\mathcal{N}(\Phi)$ by $\chi_{\mathcal{N}(\Phi)^c}$, we have that

$$\begin{aligned}1&=\int\left|\mathcal{F}\mathbf{c}(\gamma)\hat{\phi}_1(\gamma)\right|^2\mathrm{d}\gamma=\int_0^1\left|\mathcal{F}\mathbf{c}(\gamma)\right|^2\sum_k\left|\hat{\phi}_1(\gamma+k)\right|^2\mathrm{d}\gamma\\&=\int_0^1\chi_{\mathcal{N}(\Phi)^c}\left|\mathcal{F}\mathbf{c}(\gamma)\right|^2\sum_k\left|\hat{\phi}_1(\gamma+k)\right|^2\mathrm{d}\gamma.\end{aligned}$$

It is well known (see [4]) that

$$\mathcal{F}P_{\mathcal{W}}f=\begin{cases}\hat{\phi}(\cdot)\frac{\sum_k\hat{f}(\cdot+k)\overline{\hat{\phi}(\cdot+k)}}{\sum_k|\hat{\phi}(\cdot+k)|^2} & \text{outside } \mathcal{N}(\Phi),\\ 0 \quad \text{on } \mathcal{N}(\Phi).\end{cases}$$

Thus, outside $\mathcal{N}(\Phi)$,

$$\mathcal{F}P_{\mathcal{W}}f=\mathcal{F}\mathbf{c}(\cdot)\hat{\phi}(\cdot)\frac{\sum_k\hat{\phi}_1(\cdot+k)\overline{\hat{\phi}(\cdot+k)}}{\sum_k|\hat{\phi}(\cdot+k)|^2}.$$

It follows that

$$\begin{aligned}\|P_{\mathcal{W}}f\|^2=\|\mathcal{F}P_{\mathcal{W}}f\|^2&=\int_{\gamma\notin\mathcal{N}(\Phi)}\left|\mathcal{F}\mathbf{c}(\gamma)\hat{\phi}(\gamma)\right|^2\frac{|\sum_k\hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{|\sum_k|\hat{\phi}(\gamma+k)|^2|^2}\mathrm{d}\gamma\\&=\int_0^1\left|\chi_{\mathcal{N}(\Phi)^c}(\gamma)\mathcal{F}\mathbf{c}(\gamma)\right|^2\frac{|\sum_k\hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{\sum_k|\hat{\phi}(\gamma+k)|^2}\mathrm{d}\gamma.\end{aligned}$$

Outside $\mathcal{N}(\Phi)=\mathcal{N}(\Phi_1)$, the frame condition implies that Φ and Φ_1 are bounded above and below. Thus,

$$\begin{aligned}\|P_{\mathcal{W}}f\|^2&=\int_0^1\chi_{\mathcal{N}(\Phi)^c}(\gamma)\left|\mathcal{F}\mathbf{c}(\gamma)\right|^2\sum_k\left|\hat{\phi}_1(\gamma+k)\right|^2\frac{|\sum_k\hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{\sum_k|\hat{\phi}(\gamma+k)|^2\sum_k|\hat{\phi}_1(\gamma+k)|^2}\mathrm{d}\gamma\\&\geqslant \operatorname{ess}\inf_{\gamma\notin\mathcal{N}(\Phi)}\frac{|\sum_k\hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{\sum_k|\hat{\phi}(\gamma+k)|^2\sum_k|\hat{\phi}_1(\gamma+k)|^2}\int_0^1\left|\chi_{\mathcal{N}(\Phi)^c}(\gamma)\mathcal{F}\mathbf{c}(\gamma)\right|^2\sum_k\left|\hat{\phi}_1(\gamma+k)\right|^2\\&=\operatorname{ess}\inf_{\gamma\notin\mathcal{N}(\Phi)}\frac{|\sum_k\hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{\sum_k|\hat{\phi}(\gamma+k)|^2\sum_k|\hat{\phi}_1(\gamma+k)|^2},\end{aligned}$$

so that

$$\cos\theta(\mathcal{V},\mathcal{W}) \geqslant \operatorname{ess}\inf_{\gamma\notin\mathcal{N}(\Phi)} \frac{|\sum_k \hat{\phi}_1(\gamma+k)\overline{\hat{\phi}(\gamma+k)}|^2}{\sum_k |\hat{\phi}(\gamma+k)|^2 \sum_k |\hat{\phi}_1(\gamma+k)|^2}.$$

We conclude that $\cos\theta(\mathcal{V},\mathcal{W}) > 0$ if

$$\operatorname{ess}\inf_{\gamma\notin\mathcal{N}(\Phi)} \left|\sum_k \hat{\phi}_1(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)}\right|^2 > 0.$$

The fact that $\cos\theta(\mathcal{W},\mathcal{V}) > 0$ follows by symmetry. □

5. Further examples

Given a frame sequence $\{T_k\psi\}_{k\in\mathbb{Z}}$, Theorem 4.1 can be used to generate new pairs of frames and their oblique duals. Let us illustrate this with some examples.

Example 5.1. Assume that $\psi \in L^2(\mathbb{R})$ and that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame sequence. Let H, $\widetilde{H}$ be a pair of measurable 1-periodic functions, which are bounded and bounded below. Define $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ via

$$\hat{\phi}(\gamma) = H(\gamma)\hat{\psi}(\gamma), \qquad \hat{\tilde{\phi}}(\gamma) = \widetilde{H}(\gamma)\hat{\psi}(\gamma).$$

Then $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are frame sequences, spanning the same space as $\{T_k\psi\}_{k\in\mathbb{Z}}$; in fact,

$$\phi = \sum_{k\in\mathbb{Z}} a_k T_k\psi, \quad \tilde{\phi} = \sum_{k\in\mathbb{Z}} \tilde{a}_k T_k\psi \quad \text{for some } \{a_k\}, \{\tilde{a}_k\} \in \ell^2.$$

Since

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)} = H(\gamma)\overline{\widetilde{H}(\gamma)} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\gamma+k)\right|^2, \tag{20}$$

we see that (ii) in Theorem 4.1 is satisfied if

$$H(\gamma)\overline{\widetilde{H}(\gamma)} = \frac{1}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{\psi}(\gamma+k)|^2}} \quad \text{on } \left\{\gamma\colon \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\gamma+k)\right|^2 \neq 0\right\}.$$

This leads to several choices of a frame sequence $\{T_k\phi\}_{k\in\mathbb{Z}}$ and a corresponding oblique dual $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$. The special choice

$$H(\gamma) = \widetilde{H}(\gamma) = \frac{1}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{\psi}(\gamma+k)|^2}} \quad \text{on } \left\{\gamma\colon \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(\gamma+k)\right|^2 \neq 0\right\}$$

leads to the case where ϕ and $\tilde{\phi}$ are equal to the generator of the canonical tight frame associated with $\overline{\operatorname{span}}\{T_k\psi\}_{k\in\mathbb{Z}}$.

We can also generate frame sequences in other spaces than $\overline{\operatorname{span}}\{T_k\psi\}_{k\in\mathbb{Z}}$.

Example 5.2. With the assumptions in Example 5.1, define ϕ, $\tilde{\phi}$ by

$$\hat{\phi}(2\gamma) = H(\gamma)\hat{\psi}(\gamma), \qquad \hat{\tilde{\phi}}(2\gamma) = \widetilde{H}(\gamma)\hat{\psi}(\gamma).$$

Then

$$\phi(x) = \sum_{k\in\mathbb{Z}} a_k \psi(2x-k), \quad \tilde{\phi}(x) = \sum_{k\in\mathbb{Z}} \tilde{a}_k \psi(2x-k) \quad \text{for some } \{a_k\}, \{\tilde{a}_k\} \in \ell^2.$$

As before, $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are Bessel sequences, and

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)} &= \sum_{k\in\mathbb{Z}} H\left(\frac{\gamma+k}{2}\right)\hat{\psi}\left(\frac{\gamma+k}{2}\right)\overline{\widetilde{H}\left(\frac{\gamma+k}{2}\right)\hat{\psi}\left(\frac{\gamma+k}{2}\right)} \\ &= H\left(\frac{\gamma}{2}\right)\overline{\widetilde{H}\left(\frac{\gamma}{2}\right)}\sum_{k\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\gamma}{2}+k\right)\right|^2 \\ &\quad + H\left(\frac{\gamma}{2}+\frac{1}{2}\right)\overline{\widetilde{H}\left(\frac{\gamma}{2}+\frac{1}{2}\right)}\sum_{k\in\mathbb{Z}}\left|\hat{\psi}\left(\frac{\gamma}{2}+\frac{1}{2}+k\right)\right|^2. \end{aligned} \tag{21}$$

Again, it is easy to choose H, $\widetilde{H}$ satisfying (ii) in Theorem 4.1. This leads to frame expansions in $\overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$, which is now a subspace of

$$\overline{\text{span}}\{\psi(2\cdot -k)\}_{k\in\mathbb{Z}};$$

the oblique dual we construct this way belongs to $\overline{\text{span}}\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$, which is in general another subspace of $\overline{\text{span}}\{\psi(2\cdot -k)\}_{k\in\mathbb{Z}}$.

The frame decompositions obtained via Example 5.2 take place in subspaces of $\overline{\text{span}}\{\psi(2\cdot -k)\}_{k\in\mathbb{Z}}$ for the given function ψ; it might not be so easy to control which space we obtain the decomposition in. However, in an important special case, we obtain decompositions in the space $\overline{\text{span}}\{T_k\psi\}_{k\in\mathbb{Z}}$; namely, if $\psi = \phi$ is a function for which $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence and which satisfies a scaling equation

$$\hat{\phi}(2\gamma) = H(\gamma)\hat{\phi}(\gamma)$$

for a 1-periodic function H. The simplest example is as follows:

Example 5.3. Consider the translated B-spline $\phi = \chi_{[0,1]}$. Then

$$\hat{\phi}(\gamma) = \mathrm{e}^{-\pi i\gamma}\frac{\sin\pi\gamma}{\pi\gamma};$$

thus,

$$\hat{\phi}(2\gamma) = H(\gamma)\hat{\phi}(\gamma) \quad \text{with } H(\gamma) = \mathrm{e}^{-\pi i\gamma}\cos\pi\gamma.$$

Also, since $\{T_k\phi\}_{k\in\mathbb{Z}}$ forms an orthonormal sequence,

$$\Phi(\gamma) = \sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\gamma+k)\right|^2 = 1 \quad \text{a.e.}$$

Using the calculation (21) now leads to

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(\gamma+k)\overline{\hat{\tilde{\phi}}(\gamma+k)} = \mathrm{e}^{-\pi i\gamma/2}\cos\frac{\pi\gamma}{2}\overline{\widetilde{H}\left(\frac{\gamma}{2}\right)} + \mathrm{e}^{\pi i(\gamma+1)/2}\cos\left(\pi\frac{\gamma+1}{2}\right)\overline{\widetilde{H}\left(\frac{\gamma}{2}+\frac{1}{2}\right)}$$
$$= \mathrm{e}^{\pi i\gamma/2}\cos\frac{\pi\gamma}{2}\overline{\widetilde{H}\left(\frac{\gamma}{2}\right)} - i\mathrm{e}^{\pi i\gamma/2}\sin\frac{\pi\gamma}{2}\overline{\widetilde{H}\left(\frac{\gamma}{2}+\frac{1}{2}\right)}.$$

The choice $\widetilde{H} = H$ leads to $\tilde{\phi} = \phi$, which is the generator of the canonical dual of the orthonormal sequence $\{T_k\phi\}_{k\in\mathbb{Z}}$. However, the above calculation shows that other choices of $\widetilde{H}$ are possible, which make (ii) in Theorem 4.1 satisfied. They lead to oblique duals of $\{T_k\phi\}_{k\in\mathbb{Z}}$ having the form $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$, and these duals belong to the subspace

$$\left\{\sum_{k\in\mathbb{Z}} c_k\phi(2\cdot -k)\colon \{c_k\}\in\ell^2\right\}.$$

Similar examples are possible with splines of higher order; we leave the easy calculations to the reader.

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Appendix A. The condition $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$

Given closed subspaces $\mathcal{W}$, $\mathcal{V}$ of the Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$, it follows from Lemma 2.1 that

$$\mathcal{W}\cap\mathcal{V}^\perp = \{0\}, \qquad \mathcal{V}\cap\mathcal{W}^\perp = \{0\}. \tag{A.1}$$

On the other hand, (A.1) is not enough for $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ to hold, as the following example shows:

Example A.1. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}$, and let

$$\mathcal{V} = \overline{\operatorname{span}}\{e_{2k-1} - e_{2k}\}_{k=1}^\infty,$$
$$\mathcal{W} = \overline{\operatorname{span}}\{e_{2k} + e_{2k+1}\}_{k=1}^\infty.$$

Then

$$\mathcal{V}^\perp = \overline{\operatorname{span}}\{e_{2k-1} + e_{2k}\}_{k=1}^\infty;$$

clearly (A.1) is satisfied, but $\mathcal{H} \neq \mathcal{W} \oplus \mathcal{V}^\perp$ (otherwise $\{e_k + e_{k+1}\}_{k=1}^\infty$ would be a frame for $\mathcal{H}$, which is not the case, see Example 5.4.6 in [5]).

The condition for $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ can also be expressed using the *gap*. When $\mathcal{V} \neq \{0\}$, the gap from $\mathcal{V}$ to $\mathcal{W}$ is (cf. [13])

$$\delta(\mathcal{V},\mathcal{W}) := \sup_{f\in\mathcal{V},\ \|f\|=1}\ \inf_{g\in\mathcal{W}} \|f-g\|.$$

The lemma below relates the gap to the angle used so far.

Lemma A.2. *Let $\mathcal{V} \neq \{0\}$, $\mathcal{W}$ be closed subspaces of $\mathcal{H}$. Then*

$$\delta(\mathcal{V},\mathcal{W}) = \left\|(I-P_{\mathcal{W}})P_{\mathcal{V}}\right\| = \sin\theta(\mathcal{V},\mathcal{W}). \tag{A.2}$$

Proof. An elementary calculation (Lemma 2.1 in [6]) shows that $\delta(\mathcal{V},\mathcal{W}) = \|P_{\mathcal{W}^\perp}P_{\mathcal{V}}\|$; this proves the first equality in (A.2). Now, by definition,

$$\delta(\mathcal{V},\mathcal{W}) = \sup_{f\in\mathcal{V},\ \|f\|=1} \|f - P_{\mathcal{W}}f\| = \sup_{f\in\mathcal{V},\ \|f\|=1} \sqrt{\|f\|^2 - \|P_{\mathcal{W}}f\|^2} = \sin\theta(\mathcal{V},\mathcal{W}). \qquad \square$$

Expressed in terms of the gap, Lemma 2.1 states that if $\delta(\mathcal{V},\mathcal{W}) < 1$ and $\delta(\mathcal{W},\mathcal{V}) < 1$, then $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. A short direct proof of this fact goes as follows. Let $f \in \mathcal{H}$; then we can write

$$f = P_{\mathcal{W}}P_{\mathcal{V}}f + (I-P_{\mathcal{W}})P_{\mathcal{V}}f + (I-P_{\mathcal{V}})f.$$

The first and the third belong to $\mathcal{W}$ and $\mathcal{V}^\perp$, respectively; iterating the decomposition on the middle term leads to

$$\begin{aligned} f &= P_{\mathcal{W}}P_{\mathcal{V}}f + P_{\mathcal{W}}P_{\mathcal{V}}(I-P_{\mathcal{W}})P_{\mathcal{V}}f + \left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^2 f + (I-P_{\mathcal{V}})(I-P_{\mathcal{W}})P_{\mathcal{V}}f + (I-P_{\mathcal{V}})f \\ &= P_{\mathcal{W}}P_{\mathcal{V}}\sum_{n=0}^{N}\left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^n f + \left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^{N+1} f + (I-P_{\mathcal{V}})\sum_{n=0}^{N}\left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^n f. \end{aligned}$$

Using (A.2) and the assumption $\delta(\mathcal{V},\mathcal{W}) < 1$, we know that

$$\left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^{N+1} f \to 0 \quad \text{as } N \to \infty;$$

thus

$$f = P_{\mathcal{W}}P_{\mathcal{V}}\sum_{n=0}^{\infty}\left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^n f + (I-P_{\mathcal{V}})\sum_{n=0}^{\infty}\left[(I-P_{\mathcal{W}})P_{\mathcal{V}}\right]^n f \in \mathcal{W} + \mathcal{V}^\perp.$$

This demonstrates that $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$. To show that the sum is direct, it suffices to prove that $\mathcal{W} \cap \mathcal{V}^\perp = \{0\}$. To do so, note that if $f \in \mathcal{W} \cap \mathcal{V}^\perp$, then

$$f = (I-P_{\mathcal{V}})P_{\mathcal{W}}f = \cdots = \left[(I-P_{\mathcal{V}})P_{\mathcal{W}}\right]^N f;$$

now, letting $N \to \infty$ and using that $\delta(\mathcal{W},\mathcal{V}) < 1$ leads to $f = 0$ as desired.

Appendix B. Proof of Theorem 3.2

Let again T denote the pre-frame operator for the given frame $\{f_k\}_{k=1}^\infty$ for $\mathcal{W}$.

The proof of Theorem 3.2 is based on a series of lemmas. When (6) is satisfied for a bounded operator $U : \ell^2(\mathbb{N}) \to \mathcal{H}$, we say that U is a *left-inverse* of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$.

Lemma B.1. *Let $\{f_k\}_{k=1}^\infty$ be a frame for $\mathcal{W}$, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\{\delta_k\}_{k=1}^\infty$ be the canonical orthonormal basis for $\ell^2(\mathbb{N})$. The oblique dual frames for $\{f_k\}_{k=1}^\infty$ on $\mathcal{V}$ are precisely the families $\{g_k\}_{k=1}^\infty = \{V\delta_k\}_{k=1}^\infty$, where $V : \ell^2(\mathbb{N}) \to \mathcal{V}$ is a bounded left-inverse of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$.*

Proof. Assuming that $V:\ell^2(\mathbb{N})\to\mathcal{V}$ is a bounded operator with range equal to $\mathcal{V}$, it is well known that the sequence $\{g_k\}_{k=1}^\infty := \{V\delta_k\}_{k=1}^\infty$ is a frame for $\mathcal{V}$. Note that in terms of $\{\delta_k\}_{k=1}^\infty$,

$$T^*f = \big\{\langle f, f_k\rangle\big\}_{k=1}^\infty = \sum_{k=1}^\infty \langle f, f_k\rangle\delta_k;$$

thus, if V is a bounded left-inverse of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$, then for all $f\in\mathcal{H}$,

$$E_{\mathcal{V}\mathcal{W}^\perp} f = VT^*f = \sum_{k=1}^\infty \langle f, f_k\rangle g_k.$$

This shows that $\{g_k\}_{k=1}^\infty$ is an oblique dual frame of $\{f_k\}_{k=1}^\infty$ on $\mathcal{V}$. For the other implication, assume that $\{g_k\}_{k=1}^\infty$ is an oblique dual frame of $\{f_k\}_{k=1}^\infty$ on $\mathcal{V}$. Then the pre-frame operator U for $\{g_k\}_{k=1}^\infty$ satisfies the conditions: in fact, $\{g_k\}_{k=1}^\infty = \{U\delta_k\}_{k=1}^\infty$, and by Lemma 3.1, $UT^* = E_{\mathcal{V}\mathcal{W}^\perp}$. □

Lemma B.2. *Let $\{f_k\}_{k=1}^\infty$ be a frame for a subspace $\mathcal{W}\subset\mathcal{H}$, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H}=\mathcal{W}\oplus\mathcal{V}^\perp$. The bounded left-inverses of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$, with range equal to $\mathcal{V}$, are precisely the operators having the form $E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger T + W(I - T^*S^\dagger T)$, where $W:\ell^2(\mathbb{N})\to\mathcal{V}$ is a bounded operator, and I denotes the identity operator on $\ell^2(\mathbb{N})$.*

Proof. Note that $S^\dagger TT^* = P_\mathcal{W}$, $E_{\mathcal{V}\mathcal{W}^\perp}P_\mathcal{W} = E_{\mathcal{V}\mathcal{W}^\perp}$, and $T^*P_\mathcal{W} = T^*$ since $\mathcal{N}(T^*) = \mathcal{W}^\perp$; then straightforward calculation gives that an operator of the given form is a left-inverse of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$.

On the other hand, if $U:\ell^2(\mathbb{N})\to\mathcal{V}$ is a given left-inverse of T^* on $\mathcal{V}$ along $\mathcal{W}^\perp$, then by taking $W=U$,

$$E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger T + W(I - T^*S^\dagger T) = E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger T + U - UT^*S^\dagger T = U. \qquad \square$$

Proof of Theorem 3.2. By Lemmas B.1 and B.2 we can characterize the oblique dual frames on $\mathcal{V}$ as all families of the form

$$\{g_k\}_{k=1}^\infty = \big\{E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger T\delta_k + W(I - T^*S^\dagger T)\delta_k\big\}_{k=1}^\infty, \tag{B.1}$$

where $W:\ell^2(\mathbb{N})\to\mathcal{V}$ is a bounded operator, or, equivalently, an operator of the form $W\{c_j\}_{j=1}^\infty = \sum_{j=1}^\infty c_j h_j$, where $\{h_k\}_{k=1}^\infty\subset\mathcal{V}$ is a Bessel sequence. By inserting this expression for W in (B.1) we get

$$\begin{aligned}\{g_k\}_{k=1}^\infty &= \{E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger f_k + W\delta_k - WT^*S^\dagger T\delta_k\}_{k=1}^\infty \\ &= \left\{E_{\mathcal{V}\mathcal{W}^\perp}S^\dagger f_k + h_k - \sum_{j=1}^\infty \langle S^\dagger f_k, f_j\rangle h_j\right\}_{k=1}^\infty. \qquad \square\end{aligned}$$

References

[1] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, Appl. Comput. Harmon. Anal. 13 (2002) 151–161.

[2] J. Benedetto, S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998) 389–427.
[3] J. Benedetto, D. Walnut, Gabor frames for L^2 and related spaces, in: J. Benedetto, M. Frazier (Eds.), Wavelets: Mathematics and Applications, CRC Press, Boca Raton, FL, 1993, pp. 97–162.
[4] C. de Boor, R. DeVore, A. Ron, Approximation from shift-invariant subspaces of $L^2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994) 787–806.
[5] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, MA, 2003.
[6] O. Christensen, C. deFlicht, C. Lennard, Perturbation of frames for a subspace of a Hilbert space, Rocky Mountain J. Math. 30 (4) (2000) 1237–1249.
[7] O. Christensen, B. Deng, C. Heil, Density of Gabor frames, Appl. Comput. Harmon. Anal. 7 (1999) 292–304.
[8] Y.C. Eldar, A.V. Oppenheim, Nonredundant and redundant sampling with arbitrary sampling and reconstruction spaces, in: Proceedings of the 2001 Workshop on Sampling Theory and Applications (SampTA ’01), May 2001, pp. 229–234.
[9] Y.C. Eldar, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors, J. Fourier Anal. Appl. 9 (1) (2003) 77–96.
[10] Y.C. Eldar, Sampling without input constraints: Consistent reconstruction in arbitrary spaces, in: A. Zayed, J.J. Benedetto (Eds.), Sampling, Wavelets and Tomography, Birkhäuser, 2003.
[11] Y.C. Eldar, T. Werther, General framework for consistent sampling in Hilbert spaces, 2003, submitted for publication.
[12] H. Heuser, Functional Analysis, Wiley, New York, 1982.
[13] T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1976.
[14] S. Li, On general frame decompositions, Numer. Funct. Anal. Optim. 16 (9/10) (1995) 1181–1191.
[15] S. Li, A theory of generalized multiresolution structure and pseudoframes of translates, J. Fourier Anal. Appl. 7 (1) (2001) 23–40.
[16] S. Li, New biorthogonal wavelets and filterbanks via the theory of pseudoframes for subspaces, Preprint, 2003.
[17] S. Li, H. Ogawa, Pseudo-duals of frames with applications, Appl. Comput. Harmon. Anal. 11 (2) (2001) 289–304.
[18] S. Li, H. Ogawa, A theory for pseudoframes for subspaces with applications, Internat. J. Wavelets Multiresolut. Inform. Process., in press.
[19] W.S. Tang, Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces, Proc. Amer. Math. Soc. 128 (1999) 463–473.