Robust Deconvolution of Deterministic and Random Signals

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***Abstract*—We consider the problem of designing an estimation filter to recover a signal $x[n]$ convolved with a linear time-invariant (LTI) filter $h[n]$ and corrupted by additive noise. Our development treats the case in which the signal $x[n]$ is deterministic and the case in which it is a stationary random process. Both formulations take advantage of some *a priori* knowledge on the class of underlying signals. In the deterministic setting, the signal is assumed to have bounded (weighted) energy; in the stochastic setting, the power spectra of the signal and noise are bounded at each frequency. The difficulty encountered in these estimation problems is that the mean-squared error (MSE) at the output of the estimation filter depends on the problem unknowns and therefore cannot be minimized.**

Beginning with the deterministic setting, we develop a minimax MSE estimation filter that minimizes the worst case point-wise MSE between the true signal $x[n]$ and the estimated signal, over the class of bounded-norm inputs. We then establish that the MSE at the output of the minimax MSE filter is smaller than the MSE at the output of the conventional inverse filter, for all admissible signals. Next we treat the stochastic scenario, for which we propose a minimax regret estimation filter to deal with the power spectrum uncertainties. This filter is designed to minimize the worst case difference between the MSE in the presence of power spectrum uncertainties, and the MSE of the Wiener filter that knows the correct power spectra. The minimax regret filter takes the entire uncertainty interval into account, and as demonstrated through an example, can often lead to improved performance over traditional minimax MSE approaches for this problem.

***Index Terms*—Deconvolution, minimax mean-squared error (MSE), regret, spectral uncertainty, Wiener filtering.**

I. INTRODUCTION

Deconvolution is aimed at removing the impact of a system on an input signal. A classical formulation of this problem is to deconvolve a filtered, noisy signal assuming knowledge of the channel impulse response. This problem can be cast in the framework of estimation in a linear model in which the goal is to estimate an input signal $x[n]$ from corrupted observations $y[n]$ using a linear time invariant (LTI) estimation filter, where the signal is convolved with an LTI filter with impulse response $h[n]$, and corrupted by a stationary noise process $w[n]$.

A possible approach to designing the estimation filter is to minimize the mean-squared error (MSE) between the input signal $x[n]$ and the output of the estimation filter $\hat{x}[n]$. If the signal $x[n]$ is deterministic, then the MSE depends on $x[n]$ which is unknown; when $x[n]$ is a stationary random process, the MSE depends on the power spectrum of the signal, which may be subject to uncertainty. In both formulations of the problem the MSE depends on signal-related functions which in our setting are assumed to be unknown, and therefore cannot be minimized. Alternatives to MSE minimization must therefore be sought for designing the deconvolution filter.

In this correspondence, we address the filtering problem illustrated in Fig. 1 for both the deterministic and the stochastic settings. Our approach takes advantage of prior knowledge on the class of underlying signals and constraints the solution accordingly. In the deterministic setting, we assume that the signal has bounded (weighted) norm; in the stochastic setting, the power spectra of the signal and noise are bounded at each frequency. For both scenarios, we seek a robust filtering method which has relatively good performance over the class of input signals.

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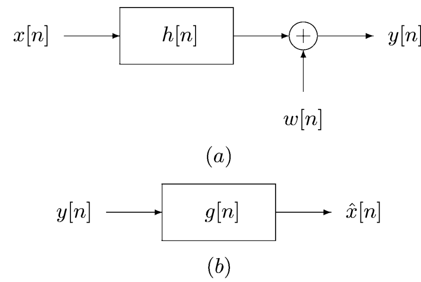

Fig. 1. Linear deconvolution problem.

One of the major techniques for designing robust systems with respect to modeling uncertainties is in the spirit of the minimax MSE approach, initiated by Huber [1], [2], in which the goal is to optimize the worst case performance. This approach has been applied to a variety of different signal processing and communication problems [3]–[5], as well as linear filtering problems in which the unknown input signal $x[n]$ is assumed to be random, but the statistics of $x[n]$ and the noise $w[n]$ are not completely specified [6]–[12]. Recently, the minimax MSE approach has also been applied to a finite-dimensional analog of the filtering problem of Fig. 1, in which the problem is to estimate a deterministic parameter vector from a finite set of noisy observations [13].

Extending the minimax MSE approach to our setting, in Section III, we develop an estimation filter that minimizes the worst case point-wise MSE over all norm-bounded inputs. The advantage of this strategy over the conventional inverse filtering method is demonstrated by proving that the point-wise MSE at the output of the minimax MSE filter is smaller than the MSE resulting from the least-squares inverse filter, for *all* bounded-norm input signals.

Similar minimax approaches have also been considered in the context of H_∞ estimation with deterministic input signals [14]–[16]. However, while our formulation treats the estimation error as a stochastic quantity and attempts to reduce its expectation, H_∞ methods in this setting consider the estimation error as a deterministic quantity that depends on the unknown signal and noise, and seek to minimize the maximum energy gain from the unknown parameters to the estimation error.

In contrast to the deterministic setting, in which the MSE depends on $x[n]$ and, therefore, typically, methods based on data error are employed, when $x[n]$ is a stationary random process, the MSE is signal independent; instead, it depends on the signal and noise power spectra. If these power spectra are known, then the MSE can be minimized directly: the optimal solution is the well-known Wiener filter [17]. However, if the power spectra are not completely specified, then the solution minimizing the MSE cannot be obtained in general. An interesting problem that has attracted considerable attention in the literature is that of designing robust Wiener filters that have reasonable performance over all possible power spectra, in some region of uncertainty. The predominant approach is to choose the filter that minimizes the worst case MSE over an appropriately chosen class of power spectra [6]–[10].

In Section IV, we consider the case in which $x[n]$ is a stationary random process, independent of the noise process $w[n]$, and the power spectra of the signal and noise conform to a band uncertainty model in which the signal and noise power spectra lie between frequency-dependent known lower and upper bounds. As we show in Section IV-A, for this model, the standard minimax MSE filter is a Wiener filter matched to the upper bound on the power spectra, and is therefore too conservative. It also does not take the complete uncertainty region or the impulse response of the filter into account, since it depends only on the upper bound of the uncertainty region.

Based on the estimation framework developed in [12], [18], [19] for the problem of estimating a finite-dimensional parameter vector from finitely many observations, we develop a competitive robust filter whose performance is uniformly close to that of the Wiener filter, for all possible values of the unknown power spectra. Specifically, in Section IV-B, we design a filter to minimize the worst case *regret*, which is the difference between the MSE of the filter, ignorant of the signal and noise power spectra, and the smallest attainable MSE with a filter that knows the power spectra. By considering the *difference* between the MSE and the optimal MSE rather than the MSE directly, we can counterbalance the conservative character of the minimax MSE approach for this problem.

Before proceeding to the detailed development, in Section II, we provide an overview of our problem and summarize the main results.

In the sequel, capital letters are used to denote the discrete-time Fourier transform, e.g., $H(\omega)$ denotes the Fourier transform of the sequence $h[n]$. The correlation sequence and power spectrum of a random process $x[n]$ are denoted by $R_x[n]$ and $S_x(\omega)$, respectively, so that $S_x(\omega)$ is the Fourier transform of $R_x[n]$. The complex conjugate is denoted by $(\cdot)^*$, and $\widehat{(\cdot)}$ denotes an estimated variable.

II. Problem Formulation

We consider the basic deconvolution problem of recovering a signal $x[n]$ from observations $y[n]$, where, as depicted in Fig. 1(a), the sequence $y[n]$ is a filtered, noisy version of $x[n]$

$$y[n] = h[n] * x[n] + w[n]. \tag{1}$$

Here $h[n]$ is a known filter with frequency response $H(\omega)$, and $w[n]$ is a second-order wide-sense-stationary noise process with zero-mean, correlation sequence $R_w[n]$, and power spectrum $S_w(\omega)$. Our objective is to design a linear estimator $\hat{x}[n] = g[n] * y[n]$ of $x[n]$, as depicted in Fig. 1(b), where $g[n]$ is the impulse response of the estimation filter.

Two formulations of the problem are treated: the deterministic setting, in which $x[n]$ is a fixed signal, and the stochastic setting, in which $x[n]$ is a zero-mean stationary random process independent of $w[n]$, with correlation sequence $R_x[n]$, where in this case $R_x[n]$ and $R_w[n]$ may not be known precisely.

To design an estimator $\hat{x}[n]$ of $x[n]$ that is close to $x[n]$, we may seek the filter $g[n]$ that minimizes the MSE. Since the noise is stationary, the total MSE

$$\sum_{n=-\infty}^{\infty} E\left\{|\hat{x}[n] - x[n]|^2\right\}$$

is unbounded; instead, we consider the point-wise MSE at some time index n_0, which is given by $E\left\{|\hat{x}[n_0] - x[n_0]|^2\right\}$. Our development of the filter minimizing the worst case point-wise MSE considers separately the case in which $x[n]$ is a deterministic signal, and the case in which $x[n]$ is a stationary random process.

A. Deterministic Signals

If $x[n]$ is deterministic then, as we show in Section III, the point-wise MSE of $\hat{x}[n]$ depends generally on $x[n]$, and therefore cannot be minimized directly. Furthermore, the MSE is a function of n_0, so that a different filter may be optimal for different values of n_0.

A common design strategy that does not depend on n_0 and $x[n]$, is to minimize the (weighted) least-squares error

$$\epsilon_{\text{LS}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_w^{-1}(\omega)\left|\widehat{Y}(\omega) - Y(\omega)\right|^2 d\omega \tag{2}$$

where $Y(\omega)$ and $\widehat{X}(\omega)$ are the Fourier transforms of the data $y[n]$ and the estimated signal $\hat{x}[n]$, respectively, and $\widehat{Y}(\omega) = H(\omega)\widehat{X}(\omega)$ is the estimated data. Assuming for simplicity that $H(\omega) \neq 0$ for all ω, the minimizing $\hat{x}[n]$ is given in the Fourier domain by

$$\widehat{X}(\omega) = \frac{1}{H(\omega)} Y(\omega) \tag{3}$$

and the resulting estimation filter, denoted by $G_{\text{INV}}(\omega)$, is the inverse filter

$$G_{\text{INV}}(\omega) = \frac{1}{H(\omega)}. \tag{4}$$

Although this approach is very simple, its drawback is that the inverse filter can lead to noise enhancement, resulting in large MSE values. Indeed, the least-squares error measures the data error between $y[n]$ and $\hat{y}[n]$ but it does not guarantee that $x[n]$ is close to $\hat{x}[n]$.

The least-squares approach is completely data driven. To improve its performance, we take advantage of some *a priori* knowledge on the class of input signals by assuming that the signal $x[n]$ has bounded (weighted) norm. Thus, we consider the class $\mathcal{T}$ of signals $x[n]$ defined by

$$\mathcal{T} = \left\{x[n] : \int_{-\pi}^{\pi} T(\omega)|X(\omega)|^2 d\omega \leq 2\pi L^2\right\} \tag{5}$$

for some weighting function $T(\omega) > 0$ and scalar $L > 0$. Throughout our derivations, we assume that L is given; however, the estimator we develop can also be applied to problems in which the norm-bound L is unknown, by first estimating it from the data; see [20]–[22].

The constraint set $\mathcal{T}$ can be incorporated into the least-squares design method by adding a regularization term to the data fitting error. The optimal solution, known as the Tikhonov regularizer, minimizes the least-squares error (2) subject to the constraint that $\hat{x}[n] \in \mathcal{T}$, and is given in the frequency domain by (6) at the bottom of the page. The constant $\lambda > 0$ depends on the data $y[n]$, and is chosen such that

$$\int_{-\pi}^{\pi} T(\omega)|\widehat{X}(\omega)|^2 d\omega = \int_{-\pi}^{\pi} T(\omega)|G_{\text{TIK}}(\omega)Y(\omega)|^2 d\omega = 2\pi L^2. \tag{7}$$

As can be seen from (6) and (7), the Tikhonov filter is a nonlinear filter, which does not have an explicit solution; the parameter λ does not have a closed form, but is rather determined as the solution to the nonlinear data-dependent equation in (7).

To improve the performance over the least-squares inverse filter, without requiring a data-dependent estimator, we develop an estimator that minimizes the worst case MSE over all signals $x[n] \in \mathcal{T}$. The minimax MSE filter is derived in Section III, and is a linear filter that has

$$G_{\text{TIK}}(\omega) = \begin{cases} H^{-1}(\omega), & \int_{\pi}^{\pi} T(\omega)|Y(\omega)H^{-1}(\omega)|^2 d\omega \leq 2\pi L^2 \\ \frac{\lambda T^{-1}(\omega)H^*(\omega)}{S_w(\omega)+\lambda T^{-1}(\omega)|H(\omega)|^2}, & \text{otherwise.} \end{cases} \tag{6}$$

the same form as $G_{\rm TIK}(\omega)$ given by (6), where the nonlinear parameter λ is replaced by the constant L^2. Thus, in contrast with $G_{\rm TIK}(\omega)$, the minimax MSE filter has an explicit closed-form solution, that is independent of the data. In Section III-B, we show that the MSE of the minimax MSE filter is smaller than that of the inverse filter, for all $x[n] \in \mathcal{T}$. Therefore, this filter is guaranteed to result in better MSE performance than the inverse filter at every time-instance n_0, assuming that the input has bounded norm.

B. Stochastic Signals

When $x[n]$ is a zero-mean, stationary random process independent of the noise $w[n]$, the error $\hat{x}[n] - x[n]$ is also a stationary process, so that the MSE $E\left\{|\hat{x}[n_0] - x[n_0]|^2\right\}$ does not depend on $x[n]$ or n_0. For a given estimation filter $g[n]$ with frequency response $G(\omega)$, the MSE in this case is given by [17], [14]

$$\begin{aligned} E\left\{|\hat{x}[n] - x[n]|^2\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|1 - G(\omega)H(\omega)|^2 S_x(\omega) + |G(\omega)|^2 S_w(\omega)\right)d\omega \\ &\triangleq E(G, S_x, S_w) \end{aligned} \quad (8)$$

where $S_x(\omega)$ and $S_w(\omega)$ are the power spectra of $x[n]$ and $w[n]$, respectively. If $S_x(\omega)$ and $S_w(\omega)$ are known, then the filter minimizing the MSE of (8) is the Wiener filter [17]

$$G_{\rm W}(\omega) = \frac{H^*(\omega)S_x(\omega)}{S_w(\omega) + S_x(\omega)|H(\omega)|^2}. \quad (9)$$

The smallest attainable MSE, which is equal to the MSE of the Wiener filter, is

$$E(G_{\rm W}, S_x, S_w) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S_w(\omega)S_x(\omega)}{S_w(\omega) + S_x(\omega)|H(\omega)|^2}d\omega. \quad (10)$$

Note that the Wiener filter has a similar form as the Tikhonov filter (6), where $\lambda(\omega)T^{-1}(\omega)$ in the Tikhonov filter is replaced by $S_x(\omega)$ in the Wiener filter.

In many practical applications, $S_x(\omega)$ and $S_w(\omega)$ may not be known precisely, in which case the Wiener filter of (9) cannot be implemented. One possible approach in this case is to use a Wiener filter matched to the estimated power spectra. However, if the true power spectra deviate from the ones assumed, then the performance of the Wiener filter may deteriorate considerably [9]. Therefore, there is a need for a robust Wiener filter whose performance is reasonably good across all possible power spectra, in the region of uncertainty.

To reflect the uncertainty in our knowledge of $S_x(\omega)$ and $S_w(\omega)$, we consider an uncertainty model, illustrated in Fig. 2, that resembles the "band model" which has been widely used in previous approaches to robust Wiener filtering [7], [9], [23], [24]. In this model, the power spectra of the signal and noise satisfy

$$\begin{aligned} l(\omega) \le S_x(\omega) \le u(\omega) \\ L(\omega) \le S_w(\omega) \le U(\omega) \end{aligned} \quad (11)$$

where the bounds $l(\omega)$, $u(\omega)$, $L(\omega)$, and $U(\omega)$ are known, and $l(\omega), L(\omega) \ge 0$. To ensure that $S_x(\omega)|H(\omega)|^2 + S_w(\omega)$ is invertible in the region of uncertainty, we further assume that

$$l(\omega) + L(\omega) > 0. \quad (12)$$

The model (11) is reasonable when the power spectra are estimated from the data. Specifically, suppose we estimate the signal power spectrum as $S_x^0(\omega)$. We may then assume that the true power spectrum $S_x(\omega)$ lies in an uncertainty level of length $2\epsilon(\omega)$ around $S_x^0(\omega)$, where $\epsilon(\omega) = (u(\omega) - l(\omega))/2$. The interval specified by $\epsilon(\omega)$ can be regarded as a confidence interval around our estimate $S_x^0(\omega)$ and may be chosen to be proportional to the standard deviation of the estimate $S_x^0(\omega)$. The same interpretation holds for the uncertainty interval around $S_w(\omega)$.

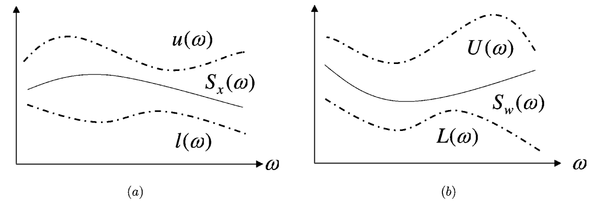

Fig. 2. Band uncertainty model. (a) Signal power spectrum. (b) Noise power spectrum.

Given an uncertainty interval, the most common approach for developing robust Wiener filters is to seek a filter that minimizes the worst case MSE in this region, as in the deterministic case [6]–[11]. However, as shown in Section IV-B, in the stochastic case the minimax MSE approach tends to be too conservative and often does not lead to satisfactory performance. Indeed, for the band uncertainty model of (11), the minimax MSE filter turns out to be a Wiener filter matched to the worst possible choice of power spectra, i.e., $S_x(\omega) = u(\omega)$ and $S_w(\omega) = U(\omega)$. Thus, the filter does not take the complete uncertainty interval into account, or the frequency response of the filter $h[n]$. To improve the performance over the minimax MSE approach under the model (11), we consider, in Section IV-B, a competitive approach, similar to that suggested in [12], [18] for a finite-dimensional analog of the Wiener filtering problem. Instead of minimizing the worst case MSE, we suggest minimizing the worst case regret with respect to the optimal linear filter without uncertainty, where the regret is defined as the difference between the MSE of a filter ignorant of the true power spectra, and the optimal MSE attainable using a filter that knows the true power spectra. The minimax regret filter is again a Wiener filter matched to a "least favorable" pair of power spectra, which depend explicitly on the uncertainty interval and on the frequency response of the filter. We then demonstrate through an example that the minimax regret approach can often lead to improved performance over traditional minimax MSE methods for this problem.

III. Minimax MSE Filter for Deterministic Signals

We first consider the deterministic setting, in which $x[n]$ is a deterministic signal. In Section III-A, we derive the minimax MSE filter, and in Section III-B, we analyze its performance.

A. Minimax MSE Filter

In this setting the MSE at time n_0 is given by

$$\begin{aligned} E\left\{(x[n_0] - \hat{x}[n_0])^2\right\} = |x[n_0] - (g[n]*h[n]*x[n])|_{n=n_0}|^2 \\ +E\left\{|(g[n]*w[n])|_{n=n_0}|^2\right\}. \end{aligned} \quad (13)$$

Using the relation

$$z[n_0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(\omega)e^{j\omega n_0}d\omega \quad (14)$$

for any signal $z[n]$ with Fourier transform $Z(\omega)$

$$\begin{aligned} |x[n_0] - (g[n]*h[n]*x[n])\,|_{n=n_0}|^2 \\ = \frac{1}{(2\pi)^2}\left|\int_{-\pi}^{\pi}(1 - G(\omega)H(\omega))X(\omega)e^{j\omega n_0}d\omega\right|^2. \end{aligned} \quad (15)$$

Next, since $w[n]$ is stationary, so is the sequence $b[n] = g[n]*w[n]$, so that

$$E\left\{b^2[n_0]\right\} = E\{b^2[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|G(\omega)|^2 S_w(\omega)d\omega. \quad (16)$$

Combining (15) with (16), we conclude that the MSE of (13) is given by

$$E\left\{|\hat{x}[n_0]-x[n]|^2\right\} = \frac{1}{(2\pi)^2}\left|\int_{-\pi}^{\pi}(1-G(\omega)H(\omega))X(\omega)e^{j\omega n_0}d\omega\right|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi}|G(\omega)|^2 S_w(\omega)d\omega. \tag{17}$$

Since the MSE (17) depends on $X(\omega)$ which is unknown, it cannot be minimized directly. Instead, we propose minimizing the worst case MSE over all possible (weighted) bounded energy signals $x[n] \in \mathcal{T}$, where $\mathcal{T}$ is defined by (5). Thus, we seek the filter $g[n]$ that is the solution to (18) at the bottom of the page. To derive the minimax filter we first consider the inner maximization problem. To this end we rely on the following lemma.

Lemma 1: Let $\mathcal{T}$ be the set of signals defined by (5). Then for any function $Z(\omega)$

$$\max_{X(\omega)\in\mathcal{T}}\left|\int_{-\pi}^{\pi}Z(\omega)X(\omega)e^{j\omega n_0}d\omega\right|^2 = 2\pi L^2\int_{-\pi}^{\pi}T^{-1}(\omega)|Z(\omega)|^2 d\omega.$$

Proof: To prove the lemma we note that

$$\begin{aligned}\left|\int_{-\pi}^{\pi}Z(\omega)X(\omega)e^{j\omega n_0}d\omega\right|^2 &\le \left|\int_{-\pi}^{\pi}T^{-1/2}|Z(\omega)|T^{1/2}(\omega)|X(\omega)|d\omega\right|^2 \\ &\le \int_{-\pi}^{\pi}T^{-1}(\omega)|Z(\omega)|^2 d\omega\int_{-\pi}^{\pi}T(\omega)|X(\omega)|^2 d\omega \\ &\le 2\pi L^2\int_{-\pi}^{\pi}T^{-1}(\omega)|Z(\omega)|^2 d\omega \end{aligned} \tag{19}$$

where the second inequality follows from applying Cauchy–Schwarz, and the last inequality holds for any $x[n] \in \mathcal{T}$. We have equality in (19) if

$$X(\omega) = \frac{\sqrt{2\pi}LT^{-1}(\omega)Z^*(\omega)e^{-j\omega n_0}}{\left(\int_{-\pi}^{\pi}T^{-1}(\omega)|Z(\omega)|^2 d\omega\right)^{\frac{1}{2}}} \tag{20}$$

which satisfies the constraint $\int_{-\pi}^{\pi}T(\omega)|X(\omega)|^2 d\omega \le 2\pi L^2$, completing the proof. □

From Lemma 1, we have that

$$\max_{X(\omega)\in\mathcal{T}}\left|\int_{-\pi}^{\pi}(1-G(\omega)H(\omega))X(\omega)e^{j\omega n_0}d\omega\right|^2 = 2\pi L^2\int_{-\pi}^{\pi}T^{-1}(\omega)|1-G(\omega)H(\omega)|^2 d\omega. \tag{21}$$

Substituting (21) into (18), our problem becomes

$$\min_{G(\omega)}\int_{-\pi}^{\pi}\left(|G(\omega)|^2 S_w(\omega) + L^2T^{-1}(\omega)|1-G(\omega)H(\omega)|^2\right)d\omega \tag{22}$$

or equivalently

$$\min_{G(\omega)}\left\{|G(\omega)|^2 S_w(\omega) + L^2T^{-1}(\omega)|1-G(\omega)H(\omega)|^2\right\}. \tag{23}$$

Since the objective of (23) is convex for each ω, the optimal filter can be found by setting the derivative to zero, which results in

$$G(\omega)\left(S_w(\omega) + L^2T^{-1}(\omega)|H(\omega)|^2\right) = L^2T^{-1}(\omega)H^*(\omega). \tag{24}$$

The optimal filter is then given by

$$G_{\mathrm{MX}}(\omega) = \frac{L^2T^{-1}(\omega)H^*(\omega)}{S_w(\omega) + L^2T^{-1}(\omega)|H(\omega)|^2}. \tag{25}$$

The minimax MSE filter has the same form as the Tikhonov filter of (6), where the nonlinear, data-dependent parameter λ is replaced by the constant L^2. Thus, in contrast with the Tikhonov filter that needs to be determined iteratively and is data dependent, the minimax MSE filter has an explicit, data-independent expression. Comparing (25) with (9), we can also interpret the minimax MSE filter as a Wiener filter matched to a signal power spectrum of $S_x(\omega) = L^2T^{-1}(\omega)$. It is also interesting to note that the optimal filter (25) does not depend on n_0 so that the same solution is optimal for any time index n_0.

B. MSE Analysis

We now compare the MSE performance of the minimax MSE filter $G_{\mathrm{MX}}(\omega)$ of (25) with that of the inverse filter $G_{\mathrm{INV}}(\omega)$ of (4).

If $S_w(\omega) = 0$ for all ω, then $G_{\mathrm{MX}}(\omega) = G_{\mathrm{INV}}(\omega)$. Therefore, we assume that $S_w(\omega) > 0$ over some interval of frequencies ω. With $G(\omega) = G_{\mathrm{INV}}(\omega)$, the MSE at time n_0 is given from (17) by

$$E\left\{|\hat{x}[n_0]-x[n]|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\omega)|^{-2}S_w(\omega)d\omega \triangleq E_{\mathrm{INV}} \tag{26}$$

which is independent of n_0 and $x[n_0]$. On the other hand, if $G(\omega) = G_{\mathrm{MX}}(\omega)$ is the minimax MSE filter of (25), then the resulting MSE of (17) will depend on n_0 and $x[n_0]$. However, from the development of the previous section we have that for any $x[n] \in \mathcal{T}$

$$\begin{aligned}E_{\mathrm{MX}} &= E\left\{|\hat{x}[n_0]-x[n]|^2\right\} \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(|G_{\mathrm{MX}}(\omega)|^2 S_w(\omega) + L^2T^{-1}(\omega) \times |1-G_{\mathrm{MX}}(\omega)H(\omega)|^2\Big)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{L^2T^{-1}(\omega)S_w(\omega)}{S_w(\omega)+L^2T^{-1}(\omega)|H(\omega)|^2}d\omega. \end{aligned} \tag{27}$$

Since

$$\begin{aligned}\frac{L^2T^{-1}(\omega)S_w(\omega)}{S_w(\omega)+L^2T^{-1}(\omega)|H(\omega)|^2} &= \frac{S_w(\omega)}{|H(\omega)|^2}\frac{L^2T^{-1}(\omega)|H(\omega)|^2}{S_w(\omega)+L^2T^{-1}(\omega)|H(\omega)|^2} \\ &\le \frac{S_w(\omega)}{|H(\omega)|^2}\end{aligned} \tag{28}$$

with strict inequality if $S_w(\omega) > 0$, we conclude that $E_{\mathrm{MX}} < E_{\mathrm{INV}}$ for all $x[n] \in \mathcal{T}$, as long as $S_w(\omega) > 0$ over some frequency interval.

We summarize our results on minimax MSE filtering in the following theorem.

Theorem 1: Let $x[n]$ be the unknown deterministic signal in the model $y[n] = h[n]*x[n]+w[n]$ where $h[n]$ is a known filter and $w[n]$ is a zero-mean stationary noise process with power spectrum $S_w(\omega)$. Let $\hat{x}[n] = g[n]*y[n]$ denote an estimate of $x[n]$ where $g[n]$ is a discrete-time filter, and let $\mathcal{T} = \{x[n] : \int_{-\pi}^{\pi}T(\omega)|X(\omega)|^2 d\omega \le L^2\}$ for some $T(\omega) > 0$ and $L > 0$. Then the minimax filter that is the solution to the problem

$$\min_{g[n]}\max_{x[n]\in\mathcal{T}} E\left\{|\hat{x}[n_0]-x[n_0]|^2\right\}$$

$$\min_{g[n]}\max_{x[n]\in\mathcal{T}} E\left\{|\hat{x}[n_0]-x[n_0]|^2\right\} = \min_{G(\omega)}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(\omega)|^2 S_w(\omega)d\omega + \frac{1}{(2\pi)^2}\max_{X(\omega)\in\mathcal{T}}\left|\int_{-\pi}^{\pi}(1-G(\omega)H(\omega))X(\omega)e^{j\omega n_0}d\omega\right|^2\right\}. \tag{18}$$

is independent of n_0, and is given in the Fourier domain by

$$G_{\mathrm{MX}}(\omega)=\frac{L^2T^{-1}(\omega)H^*(\omega)}{S_w(\omega)+L^2T^{-1}(\omega)|H(\omega)|^2}.$$

In addition, we have the following.

1. The filter $G_{\mathrm{MX}}(\omega)$ can be interpreted as a Wiener filter matched to a signal power spectrum $S_x(\omega)=L^2T^{-1}(\omega)$.
2. At every time index n_0, the MSE $E\left\{|\hat{x}[n_0]-x[n_0]|^2\right\}$ using $G_{\mathrm{MX}}(\omega)$ is smaller than the MSE using the inverse filter $G_{\mathrm{INV}}(\omega)=H^{-1}(\omega)$ for all $x[n]\in\mathcal{T}$.

IV. Deconvolution of Stochastic Signals

A. Minimax MSE Wiener Filter

We now consider the case in which $x[n]$ and $w[n]$ are independent, zero-mean, stationary random processes, with power spectra specified by the uncertainty region defined in (11). Our problem is to design a robust filter with good performance for all $S_x(\omega)$ and $S_w(\omega)$ in the region of uncertainty.

A popular approach to robust filtering is to seek the solution that minimizes the worst case MSE over all possible values of $S_x(\omega)$ and $S_w(\omega)$ in the region of uncertainty. For the model (11), the minimax MSE filter reduces to a Wiener filter matched to $S_x(\omega)=u(\omega)$ and $S_w(\omega)=U(\omega)$. To see this, we note that for all ω and for all power spectra in the set $\mathcal{D}$ defined by

$$\mathcal{D}=\{S_x(\omega),S_w(\omega): l(\omega)\le S_x(\omega)\le u(\omega),\; L(\omega)\le S_w(\omega)\le U(\omega)\} \quad (29)$$

we have that

$$|1-G(\omega)H(\omega)|^2S_x(\omega)+|G(\omega)|^2S_w(\omega) \le |1-G(\omega)H(\omega)|^2u(\omega)+|G(\omega)|^2U(\omega). \quad (30)$$

Therefore, with $E(G,S_x,S_w)$ denoting the MSE $E\left\{|x[n]-\hat{x}[n]|^2\right\}$ defined by (8) using a filter with frequency response $G(\omega)$

$$\min_G \max_{S_x,S_w\in\mathcal{D}} E(G,S_x,S_w)=\min_G E(G,u,U), \quad (31)$$

and from (9) the optimal filter is

$$G(\omega)=\frac{H^*(\omega)u(\omega)}{u(\omega)|H(\omega)|^2+U(\omega)}. \quad (32)$$

We see that the minimax MSE filter is too conservative, since it minimizes the MSE for the worst possible choice of parameters. It also does not take the full uncertainty region into account, but rather considers only the upper bound of the uncertainty region. To try and compensate for the conservative nature of the minimax MSE approach, in Section IV-B we develop a minimax regret estimation filter, whose performance is as close as possible to that of the optimal Wiener filter for all values of $S_x(\omega)$ and $S_w(\omega)$ satisfying (11).

B. Minimax Regret Wiener Filter

If the power spectra $S_x(\omega)$ and $S_w(\omega)$ are known, then the filter $G(S_x,S_w)$ which depends on $S_x(\omega)$ and $S_w(\omega)$ minimizing the MSE is the Wiener filter of (9), and the smallest attainable MSE is given by (10). Note that the optimal MSE is a function of the unknown power spectra $S_x(\omega)$ and $S_w(\omega)$. When the power spectra are not known, the minimal MSE is not achievable. The regret $\mathcal{R}(S_x,S_w,G)$ is defined as the difference between the MSE using a filter $G(\omega)$ and the smallest possible MSE

$$\mathcal{R}(S_x,S_w,G)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|1-G(\omega)H(\omega)|^2S_x(\omega)+|G(\omega)|^2S_w(\omega)-\frac{S_w(\omega)S_x(\omega)}{S_x(\omega)|H(\omega)|^2+S_w(\omega)}\right)d\omega. \quad (33)$$

To try and uniformly approach the optimal MSE in the presence of power spectrum uncertainties, we seek an estimator that minimizes the worst case regret, i.e., it is the solution to the problem

$$\min_G \max_{S_x,S_w\in\mathcal{D}} \mathcal{R}(S_x,S_w,G) \quad (34)$$

where $\mathcal{D}$ is defined by (29). The form of the minimax regret filter is given in the following theorem.

Theorem 2: Let $x[n]$ be an unknown zero-mean, stationary random signal with power spectrum $S_x(\omega)$ in the model $y[n]=h[n]*x[n]+w[n]$, where $h[n]$ is a known filter and $w[n]$ is a zero-mean stationary noise process, independent of $x[n]$, with power spectrum $S_w(\omega)$. Let $\hat{x}[n]=g[n]*y[n]$ denote an estimate of $x[n]$ where $g[n]$ is a discrete-time filter, and let $\mathcal{D}$ denote the set of power spectra satisfying $u(\omega)\le S_x(\omega)\le l(\omega)$ and $U(\omega)\le S_w(\omega)\le L(\omega)$. Then the minimax regret filter $G_{\mathrm{REG}}(\omega)$ that is the solution to the problem

$$\min_{G(\omega)} \max_{S_x(\omega),S_w(\omega)\in\mathcal{D}} \mathcal{R}(S_x,S_w,G)$$

is given by

$$G_{\mathrm{REG}}(\omega)=\frac{H^*(\omega)}{\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}+\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}}\cdot\left(\frac{l(\omega)}{\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}}+\frac{u(\omega)}{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}}\right).$$

Before proving the theorem we note that if $L(\omega)=U(\omega)$ and $l(\omega)=u(\omega)$ so that there is no uncertainty in $S_x(\omega)$ and $S_w(\omega)$ then, as we expect $G_{\mathrm{REG}}(\omega)$ reduces to the Wiener filter of (9).

Proof: We develop the minimax regret filter by first expressing $G(\omega)$ as $|G(\omega)|e^{j\phi(\omega)}$, and noting that the regret $\mathcal{R}(S_x,S_w,G)$ depends on $\phi(\omega)$ only through the expression

$$|1-G(\omega)H(\omega)|^2 = 1+|G(\omega)H(\omega)|^2-2|G(\omega)H(\omega)|\cos(\phi(\omega)+\psi(\omega)) \quad (35)$$

where $H(\omega)=|H(\omega)|e^{j\psi(\omega)}$. Since $\cos(\phi(\omega)+\psi(\omega))\le 1$ with equality for $\phi(\omega)=-\psi(\omega)$, we have that

$$|1-G(\omega)H(\omega)|^2\ge 1+|G(\omega)H(\omega)|^2-2|G(\omega)H(\omega)| = (1-|G(\omega)H(\omega)|)^2. \quad (36)$$

Therefore, for any choice of $\phi(\omega)$

$$\max_{S_x,S_w\in\mathcal{D}}\mathcal{R}(S_x,S_w,G) \ge \max_{S_x,S_w\in\mathcal{D}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{M}(|G(\omega)|,S_x(\omega),S_w(\omega))d\omega \quad (37)$$

where

$$\mathcal{M}(|G(\omega)|,S_x(\omega),S_w(\omega)) = (1-|G(\omega)H(\omega)|)^2S_x(\omega)+|G(\omega)|^2S_w(\omega) - \frac{S_w(\omega)S_x(\omega)}{S_x(\omega)|H(\omega)|^2+S_w(\omega)} \quad (38)$$

with equality for

$$\phi(\omega)=-\psi(\omega). \quad (39)$$

It remains to determine the optimal value of $|G(\omega)|$, which is the solution to

$$\min_{|G|}\max_{S_x,S_w\in\mathcal{D}}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{M}(|G(\omega)|,S_x(\omega),S_w(\omega))d\omega. \quad (40)$$

Since the constraint set $\mathcal{D}$ is separable in ω

$$\min_{|G|}\max_{S_x,S_w\in\mathcal{D}}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{M}(|G(\omega)|,S_x(\omega),S_w(\omega))d\omega\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\min_{|G|}\max_{S_x,S_w\in\mathcal{D}}\{\mathcal{M}(|G(\omega)|,S_x(\omega),S_w(\omega))\}\right)d\omega. \quad (41)$$

For a fixed ω, let $g = |G(\omega)|$, $h = |H(\omega)|$, $\sigma_x = S_x(\omega)$, and $\sigma_w = S_w(\omega)$. Then, our problem becomes

$$\min_{g\geq 0} \max_{l\leq\sigma_x\leq u, L\leq\sigma_w\leq U} \left\{(1-gh)^2\sigma_x + g^2\sigma_w - \frac{\sigma_w\sigma_x}{\sigma_x h^2+\sigma_w}\right\} \quad (42)$$

where $h \geq 0$, $l = l(\omega)$, $u = u(\omega)$, $L = L(\omega)$, and $U = U(\omega)$. The solution to (42) follows from [18], and is given by the following lemma.

Lemma 2: The solution to the problem

$$\min_{g\geq 0} \max_{l\leq\sigma_x\leq u, L\leq\sigma_w\leq U} \left\{(1-gh)^2\sigma_x + g^2\sigma_w - \frac{\sigma_w\sigma_x}{\sigma_x h^2+\sigma_w}\right\} \quad (43)$$

is

$$g = \frac{h}{\sqrt{U+lh^2}+\sqrt{L+uh^2}}\left(\frac{l}{\sqrt{U+lh^2}} + \frac{u}{\sqrt{L+uh^2}}\right). \quad (44)$$

Proof: We first consider the case in which $h = 0$. In this case, the problem (43) becomes

$$\min_{g\geq 0} \max_{l\leq\sigma_x\leq u, L\leq\sigma_w\leq U} g^2\sigma_w \quad (45)$$

and clearly, the solution is $g = 0$. Next, suppose that $h > 0$. The proof of (44) in this case is based on a similar result developed in [18]. Specifically, in the proof of Theorem 4 in [18] it is shown that the value of d that is the solution to

$$\min_{d} \max_{l\leq\delta\leq u, L\leq\theta\leq U} \left\{(1-d)^2\delta + \frac{\theta d^2}{\sigma^2} - \frac{\delta\theta}{\delta\sigma^2+\theta}\right\} \quad (46)$$

where $\sigma^2 > 0$ is given by

$$d = \frac{\sigma^2}{\sqrt{U+l\sigma^2}+\sqrt{L+u\sigma^2}}\left(\frac{l}{\sqrt{U+l\sigma^2}} + \frac{u}{\sqrt{L+u\sigma^2}}\right). \quad (47)$$

It is immediate to see that our problem (43) is equivalent to that of (47) with $\sigma_x = \delta, \sigma_w = \theta, h = \sigma$ and $g = d/h$. Since the optimal solution d satisfies $d > 0$, the result follows.

The proof of the theorem then follows from combining Lemma 2 with (39).

C. Minimum MSE Interpretation of the Regret Filter

Substituting the power spectra

$$S_x(\omega) = \alpha(\omega)l(\omega) + (1-\alpha(\omega))u(\omega);$$
$$S_w(\omega) = \alpha(\omega)L(\omega) + (1-\alpha(\omega))U(\omega) \quad (48)$$

where

$$\alpha(\omega) = \frac{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}}{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}+\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}} \quad (49)$$

into the expression for the Wiener filter (9), results in the minimax regret filter of Theorem 2. Therefore, we can interpret the regret filter as a Wiener filter matched to the power spectra (48), which can be viewed as estimates of the true, unknown power spectra. Specifically, the signal spectrum $S_x(\omega)$ at a given frequency ω_0 is estimated as a weighted combination of the bounds $u(\omega_0)$ and $l(\omega_0)$, where the weights depend explicitly on the signal and noise uncertainty level at ω_0, and on the magnitude of the frequency response of the filter $|H(\omega_0)|$. The same holds true for the noise spectrum $S_w(\omega)$. Thus, in contrast with the minimax MSE filter, which is matched to power spectra that are equal to the upper bound, the minimax regret filter takes both the upper and lower bounds into account, as well as the frequency response $H(\omega)$. Since the minimax regret filter minimizes the regret for the power spectra given by (48), we may view these power spectra as the "least favorable" power spectra in the regret sense.

We therefore have the following corollary to Theorem 2.

Corollary 1: Consider the setup of Theorem 2. The minimax regret filter $G_{\mathrm{REG}}(\omega)$ can be viewed as a Wiener filter matched to the power spectra

$$S_x(\omega) = \alpha(\omega)l(\omega) + (1-\alpha(\omega))u(\omega)$$

and

$$S_w(\omega) = \alpha(\omega)L(\omega) + (1-\alpha(\omega))U(\omega)$$

where

$$\alpha(\omega) = \frac{l(\omega)\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}}{\sqrt{L(\omega)+u(\omega)|H(\omega)|^2}+\sqrt{U(\omega)+l(\omega)|H(\omega)|^2}}.$$

Some insight into the least favorable power spectra can be gained by considering the low and high signal-to-noise ratio (SNR) regions. If $l(\omega)|H(\omega)|^2 \gg U(\omega)$, then

$$\begin{aligned} S_x(\omega) &\approx \frac{l(\omega)\sqrt{u(\omega)|H(\omega)|^2}+u(\omega)\sqrt{l(\omega)|H(\omega)|^2}}{\sqrt{u(\omega)|H(\omega)|^2}+\sqrt{l(\omega)|H(\omega)|^2}} \\ &= \frac{\sqrt{u(\omega)l(\omega)}\left(\sqrt{u(\omega)|H(\omega)|^2}+\sqrt{l(\omega)|H(\omega)|^2}\right)}{\sqrt{u(\omega)|H(\omega)|^2}+\sqrt{l(\omega)|H(\omega)|^2}} \\ &= \sqrt{u(\omega)l(\omega)}. \end{aligned} \quad (50)$$

The least favorable signal spectrum is thus the geometric average of the lower and upper bounds. If, on the other hand, $u(\omega)|H(\omega)|^2 \ll L(\omega)$, then

$$S_x(\omega) \approx \frac{l(\omega)\sqrt{L(\omega)}+u(\omega)\sqrt{U(\omega)}}{\sqrt{L(\omega)}+\sqrt{U(\omega)}}. \quad (51)$$

Similarly

$$S_w(\omega) \approx \begin{cases} \frac{L(\omega)(\omega)\sqrt{l(\omega)}+U(\omega)\sqrt{u(\omega)}}{\sqrt{l(\omega)}+\sqrt{u(\omega)}}, & U(\omega) \ll l(\omega)|H(\omega)|^2 \\ \sqrt{U(\omega)L(\omega)}, & L(\omega) \gg u(\omega)|H(\omega)|^2. \end{cases} \quad (52)$$

D. Difference Regret Estimator for Known $S_x(\omega)$ or $S_w(\omega)$

We now consider two special cases of Theorem 2, in which either $S_x(\omega)$ or $S_w(\omega)$ are completely specified, so that the uncertainty is only in one of the power spectra.

Suppose first that there is no uncertainty in the noise power spectrum, so that $U(\omega) = L(\omega) = S_w(\omega)$. From Theorem 2, we get (53) at the bottom of the page, which can be interpreted as a Wiener filter matched to a signal power spectrum

$$S_x(\omega) = \frac{1}{|H(\omega)|^2}\left(\sqrt{S_w(\omega)+l(\omega)|H(\omega)|^2} \times\sqrt{S_w(\omega)+u(\omega)|H(\omega)|^2} - S_w(\omega)\right). \quad (54)$$

$$G_{\mathrm{REG}}(\omega) = \frac{1}{H(\omega)}\left(1 - \frac{S_w(\omega)}{\sqrt{S_w(\omega)+l(\omega)|H(\omega)|^2}\sqrt{S_w(\omega)+u(\omega)|H(\omega)|^2}}\right) \quad (53)$$

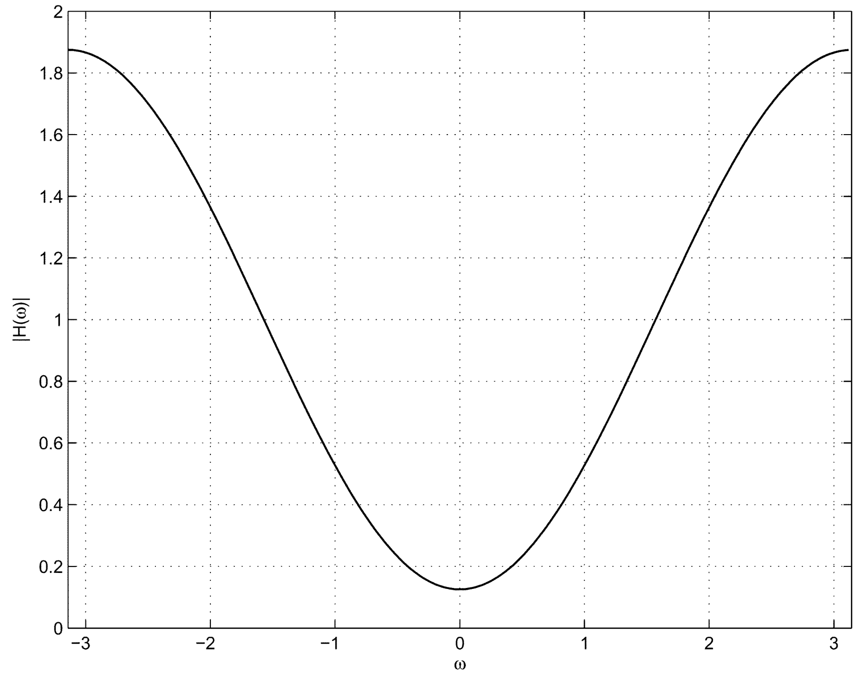

Fig. 3. Frequency response magnitude $|H(\omega)|$ of the filter given by (61).

From (50) and (51)

$$S_x(\omega) = \begin{cases} \frac{1}{2}(u(\omega)+l(\omega)), & u(\omega)|H(\omega)|^2 \ll S_w(\omega) \\ \sqrt{u(\omega)l(\omega)}, & l(\omega)|H(\omega)|^2 \gg S_w(\omega) \end{cases} \quad (55)$$

so that for low SNR, $S_x(\omega)$ is equal to the arithmetic average of the upper and lower bounds, and for high SNR, $S_x(\omega)$ is equal to the geometric average.

It is interesting to note that while the minimax MSE filter of (32) is matched to a power spectrum $u(\omega) \geq (u(\omega)+l(\omega))/2$, the minimax regret filter is matched to a power spectrum $S_x(\omega) \leq (u(\omega)+l(\omega))/2$. This follows from the inequality that $\sqrt{ab} \leq (a+b)/2$.

We next consider the case in which $u(\omega) = l(\omega) = S_x(\omega)$ so that the power spectrum of the signal is known, and the uncertainty is only in the noise power spectrum. In this case, the minimax regret filter is

$$G_{\mathrm{REG}}(\omega) = \frac{S_x(\omega)H^*(\omega)}{\sqrt{L(\omega)+S_x(\omega)|H(\omega)|^2}\sqrt{U(\omega)+S_x(\omega)|H(\omega)|^2}} \quad (56)$$

which can be viewed as a Wiener filter matched to a noise power spectrum

$$\begin{aligned} S_w(\omega) &= \sqrt{L(\omega)+S_x(\omega)|H(\omega)|^2}\sqrt{U(\omega)+S_x(\omega)|H(\omega)|^2} \\ &\quad - S_x(\omega)|H(\omega)|^2 \\ &\leq \frac{U(\omega)+L(\omega)}{2}. \end{aligned} \quad (57)$$

In analogy with (55)

$$S_w(\omega) = \begin{cases} \frac{1}{2}(U(\omega)+L(\omega)), & U(\omega) \ll S_x(\omega)|H(\omega)|^2 \\ \sqrt{L(\omega)U(\omega)}, & L(\omega) \gg S_x(\omega)|H(\omega)|^2. \end{cases} \quad (58)$$

E. Example

In this subsection, we illustrate the performance of the minimax MSE and the minimax regret filters. Clearly, the performance of these filters depends on the values of the unknown power spectra. If, for example, the true values of $S_x(\omega)$ and $S_w(\omega)$ are equal to $S_x(\omega) = u(\omega)$ and $S_w(\omega) = U(\omega)$, then the minimax MSE filter will provide the best performance, since it minimizes the MSE for this choice of power spectra. As suggested in [18], one possible way of assessing the performance of the filters, is to compute the MSE at the output of the each of the filters for the best possible choice of power spectra, the worst possible choice, and the nominal (average) choice. Obviously, the minimax MSE filter will optimize the performance for the worst choice of power spectra. However, as we will see in the example below, the minimax regret filter often performs only slightly worse than the minimax MSE filter in the worst case, but can provide a substantial performance improvement for the best choice of power spectra.

Consider the estimation problem in which

$$y[n] = h[n] * x[n] + w[n], \quad (59)$$

where $x[n]$ is a zero-mean stationary first-order autoregressive (AR) process with power spectrum

$$S_x^{\mathrm{O}}(\omega) = \frac{1}{|1-\rho e^{j\omega}|^2} \quad (60)$$

for some parameter ρ, and $w[n]$ is a zero-mean, uncorrelated random process with variance σ^2, where we assume for simplicity that σ^2 is known. The filter $h[n]$ is a finite impulse response (FIR) filter with taps

$$h[1] = 1, \quad h[\pm 1] = -7/16, \quad h[n] = 0, \quad |n| > 1. \quad (61)$$

The frequency response magnitude of the filter is depicted in Fig. 3. We assume that the signal spectrum $S_x(\omega)$ is not known exactly, however, we know that $l(\omega) \leq S_x(\omega) \leq u(\omega)$ with $l(\omega) = (1-\alpha)S_x(\omega)$ and $u(\omega) = (1+\alpha)S_x(\omega)$, where $0 < \alpha < 1$ is a parameter that defines the size of the uncertainty set.

For an arbitrary estimation filter $G(\omega)$ we can find the worst choice of $S_x(\omega)$, denoted $S_x^{\mathrm{WC}}(\omega)$, that maximizes the MSE, and the best choice of $S_x(\omega)$, denoted $S_x^{\mathrm{BC}}(\omega)$, that minimizes the MSE. Since for any $l(\omega) \leq S_x(\omega) \leq u(\omega)$ we have that

$$\begin{aligned} &|1-G(\omega)H(\omega)|^2 S_x(\omega) + |G(\omega)|^2 S_w(\omega) \\ &\qquad \leq |1-G(\omega)H(\omega)|^2 u(\omega) + |G(\omega)|^2 S_w(\omega) \end{aligned} \quad (62)$$

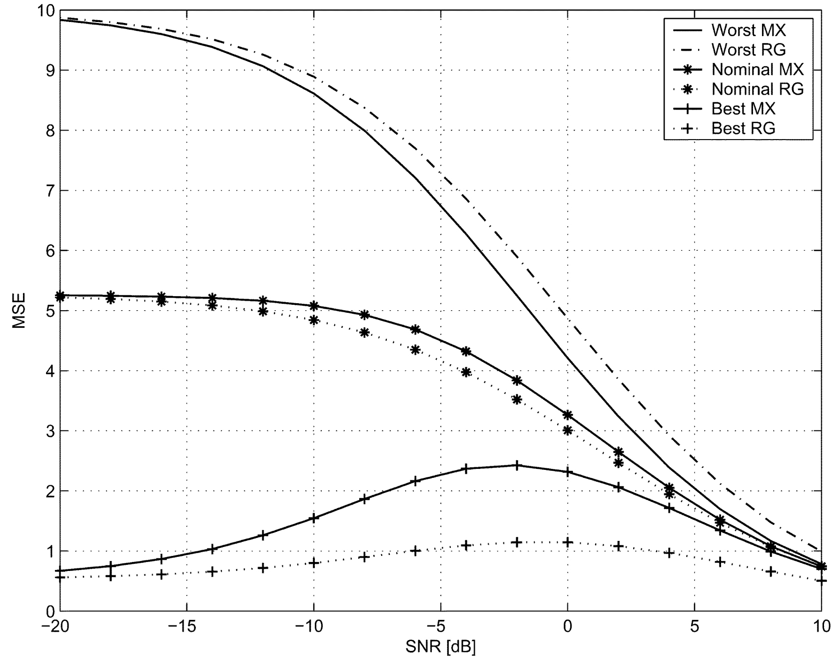

Fig. 4. MSE in estimating $x[n]$ as a function of SNR using the minimax regret filter and the minimax MSE filter, for $S_x(\omega)$ equal to $S_x^{\mathrm{WC}}(\omega)$, $S_x^{\mathrm{BC}}(\omega)$, and $S_x^{\mathrm{O}}(\omega)$.

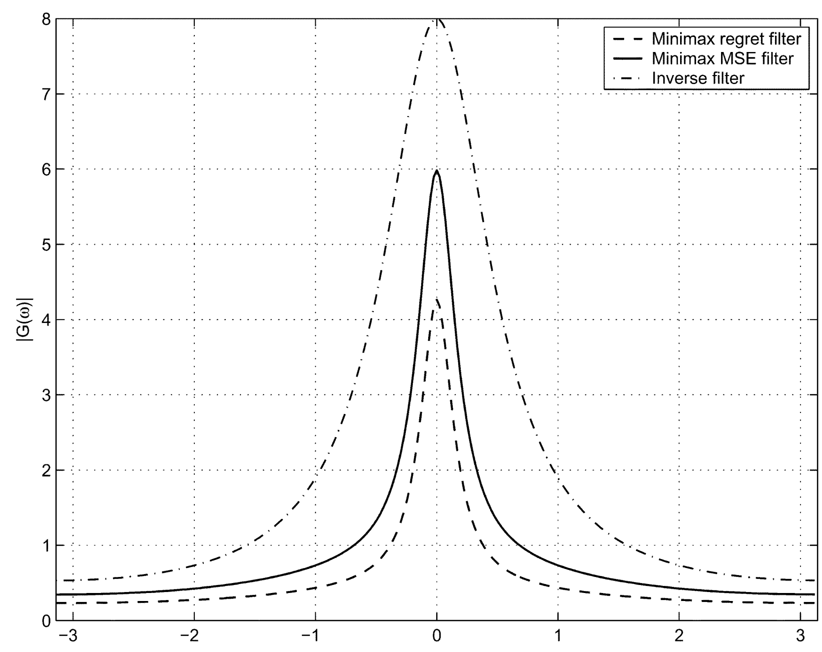

Fig. 5. Frequency response magnitude of the minimax regret, minimax MSE, and inverse filters.

and

$$|1-G(\omega)H(\omega)|^2 S_x(\omega)+|G(\omega)|^2 S_w(\omega) \geq |1-G(\omega)H(\omega)|^2 l(\omega)+|G(\omega)|^2 S_w(\omega) \quad (63)$$

the MSE defined by (8) is minimized when $S_x(\omega)=l(\omega)$ and is maximized when $S_x(\omega)=u(\omega)$, so that $S_x^{\mathrm{WC}}(\omega)=l(\omega)$ and $S_x^{\mathrm{BC}}(\omega)=u(\omega)$, regardless of the filter $G(\omega)$.

In Fig. 4, we plot the MSE of the minimax MSE filter (MX) of (32) and the minimax regret filter (RG) of Theorem 2 as a function of the SNR defined by $-10\log\sigma^2$ for $\rho = 0.9$, and $\alpha = 0.9$. The MSE of each of the filters is plotted for three different choices of $S_x(\omega)$: the worst case $S_x(\omega) = S_x^{\mathrm{WC}}(\omega)$, the best case $S_x(\omega) = S_x^{\mathrm{BC}}(\omega)$, and the nominal (true) value $S_x(\omega) = S_x^{\mathrm{O}}(\omega)$. As we expect, when $S_x(\omega)=S_x^{\mathrm{WC}}(\omega)$, the minimax MSE filter has the best performance. On the other hand, when $S_x(\omega) = S_x^{\mathrm{BC}}(\omega)$, the performance of the minimax MSE filter deteriorates considerably. In this example, we may prefer using the minimax regret filter over the minimax MSE filter since the loss in performance of the minimax MSE filter in the best case is much more significant then the loss in performance of the minimax regret filter in the worst case.

In Fig. 5, we plot the magnitude of the frequency responses of the minimax regret filter, the minimax MSE filter, and the inverse filter $G(\omega)=1/H(\omega)$ for an SNR of 0 dB.

V. Conclusion

We considered the problem of deconvolving a signal $x[n]$ from noisy, filtered observations $y[n] = x[n] * h[n] + w[n]$, both for the case in which $x[n]$ is a deterministic, bounded energy signal, and for the case in which $x[n]$ is a stationary random process independent of $w[n]$, with unknown power spectrum. In both settings, we designed an LTI estimation filter to minimize some measure of the worst case point-wise MSE.

In the case in which $x[n]$ is deterministic, we developed a minimax MSE filter that minimizes the worst case point-wise MSE over all bounded energy signals. We showed that the resulting filter has a similar form to the Wiener filter, and its MSE is always smaller than that of the least squares inverse (zero-forcing) filter, regardless of the value of the true signal $x[n]$.

We then treated the case in which $x[n]$ is a random process and the power spectra of $x[n]$ and $w[n]$ obey a band uncertainty model. We showed that for this uncertainty model, the minimax MSE approach is too pessimistic, and does not take the full uncertainty region into account. Thus, in contrast with the deterministic case in which the minimax MSE approach results in a filter that has some nice properties, such as resemblance to the Wiener filter and smaller MSE than the inverse filter, in the stochastic case, the minimax MSE approach is too conservative, and often does not lead to satisfactory performance. As an alternative to the minimax MSE approach, we considered a minimax regret approach in which we developed an estimation filter whose MSE is uniformly close to that of the Wiener filter that knows the power spectra. As we showed, the regret filter can also be viewed as a Wiener filter matched to a pair of least favorable power spectra, that explicitly take the uncertainty region as well as the frequency response of the filter $h[n]$ into account.

In future work, it would also be interesting to develop a minimax regret filter for the deterministic case, by first finding the optimal filter that minimizes the point-wise MSE, assuming that $X(\omega)$ is known, and then seeking the estimator that minimizes the worst case difference, over all bounded energy signals, between the MSE and the best possible MSE attainable when we allow the filter to depend on $X(\omega)$.

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