Dual Gabor Frames: Theory and Computational Aspects

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***Abstract*—We consider a general method for constructing dual Gabor elements different from the canonical dual. Our approach is based on combining two Gabor frames such that the generated frame-type operator $S_{g,\gamma}$ is nonsingular. We provide necessary and sufficient conditions on the Gabor window functions g and γ such that $S_{g,\gamma}$ is nonsingular for rational oversampling, considering both the continuous-time and the discrete-time settings. In contrast to the frame operator, the operator $S_{g,\gamma}$ is, in general, not positive. Therefore, all results in Gabor analysis that are based on the positivity of the frame operator cannot be applied directly. The advantage of the proposed characterization is that the algebraic system for computing the Gabor dual elements preserves the high structure of usual Gabor frames, leading to computationally efficient algorithms. In particular, we consider examples in which both the condition number and the computational complexity in computing the proposed dual Gabor elements decrease in comparison to the canonical dual Gabor elements.**

***Index Terms*—Frame theory, Gabor analysis, twisted convolution, window design.**

I. Introduction

GABOR analysis is a pervasive signal processing method for decomposing and reconstructing signals from their time-frequency projections. Gabor representation is used in many applications ranging from speech processing and texture segmentation to pattern and object recognition, among others [3], [4], [21], [28], [40].

One of the advantages of Gabor analysis is the highly structured system inherited from the uniform time-frequency lattice, which allows for efficient computational algorithms. A major part of Gabor analysis relies on frame theory, which deals with overcomplete sets [8]. When using an overcomplete Gabor system to decompose a signal, the reconstruction is no longer unique. The most popular choice of reconstruction is based on the canonical (minimal-norm) dual Gabor frame [33], [36]. Computing the canonical frame involves inverting the frame operator associated with the given Gabor frame. The frame operator has a lot of structure emerging from the commutivity with corresponding time-frequency shifts, which is a consequence of the uniformity of the underlying time-frequency lattice. This structure gives rise to many important and well-known results, such as the Wexler–Raz biorthogonality and the Janssen representation, that we briefly discuss in Section III. In the discrete time setting, the structure of the Gabor system is evident in the sparsity and the periodic distribution of the entries of the Gabor frame matrix that can be used to develop efficient numerical algorithms [31], [36]. In some applications, however, the frame operator may be poorly conditioned or require many operations to compute, cf. Section VI. In such cases, it may be possible to reduce the computational complexity and the condition number by considering dual elements different from the canonical dual.

A general characterization of dual Gabor elements was given in [24] based on the canonical dual. Each of the dual Gabor elements in this characterization depend explicitly on the canonical dual and therefore require computing the inverse of the frame operator. Different dual Gabor windows have also been studied for the continuous and the discrete case in [9] and [36], respectively. In both cases, the authors aimed at computing the dual window that has minimal (semi)-norm with respect to a special norm other than the L^2-norm. The resulting system, however, does not, in general, preserve the structure of the Gabor frame operator that is heavily exploited for deriving fast algorithms.

In this paper, we present a method for deriving dual Gabor elements that retain the high Gabor structure, without relying on the inversion of the frame operator. Specifically, we use a frame-type operator that enjoys the same commutivity properties as the standard Gabor frame operator in order to derive dual Gabor elements that generate an alternative dual Gabor frame. In the discrete time setting, this method can still access all fast inversion schemes that do not rely on positivity. The basic idea follows the general concept introduced in [13] of replacing the pseudo-inverse of a matrix with another left-inverse. In the context of general frames, this leads to dual frames different from the canonical choice. For Gabor frames, we are constrained to special left inverses in order to retain most of the structure that arises from uniform time-frequency lattices. In particular, we build a left inverse of the analysis operator of a given Gabor frame using another Gabor frame sequence such that the arising frame-type operator is nonsingular. We prove necessary and sufficient conditions for the underlying Gabor atoms to generate a left inverse for rational oversampling. An important case in applications is integer oversampling, for which the corresponding frame operator is sparse, leading to fast numerical algorithms [30], [31], [33], [41]. In the case of integer oversampling, our general condition reduces to an easily verifiable condition on the given windows. In our development, we consider both the continuous-time case and the discrete-time case.

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The paper is organized as follows. In Section II, we provide the basic definitions, formulate the problem in a general setting, and introduce Gabor frames. In Section III, we review fundamental results in Gabor analysis, outline the Janssen representation, and briefly state the Feichtinger algebra. These first three chapters are intended to provide a short overview of the facts needed in the present approach. Section IV contains the main results. Specifically, for rational lattice parameters we develop necessary and sufficient conditions on the Gabor windows such that the resulting frame-type operator is invertible on L^2. As we show, our general condition takes on a particularly simple form in the integer oversampling case. In Section V, we describe a finite dimensional model, and discuss the example of Gaussian windows. This context is important in applications, since in practice, we always have a finite set of data. Section VI provides numerical examples illustrating the advantages of the proposed method with respect to computational complexity and stability of the Gabor system compared with the standard frame operator described in [36].

The contribution of this paper is two-fold: On the theoretical side, we provide an alternative characterization of all dual Gabor elements that does not require the conventional frame operator. From a practical perspective, we develop concrete methods for Gabor expansions that can reduce the computational complexity and improve the stability with respect to existing approaches. The practical aspects are mainly in the discrete-setting, while the heart of the theory is in analyzing the continuous-time setting. Thus, while the results of Section IV are interesting on a theoretical level, the more practically oriented reader may focus on Sections V and VI.

II. Preliminaries and Problem Definition

A. Notation

We define the Fourier transform of an absolutely integrable function f by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx$$

where $\mathbb{R}$ denotes the reals. The Fourier transform extends to a unitary mapping on the Hilbert space L^2 of square integrable functions with inner product

$$\langle f, h \rangle = \int_{\mathbb{R}} f(x) \overline{h(x)} dx$$

where $\overline{h}$ denotes the complex conjugate of h. The Fourier series expansion of a α-periodic square-integrable function f is

$$f(x) = \alpha^{-1} \sum_{n\in\mathbb{Z}} a_n e^{\frac{2\pi i n x}{\alpha}} \tag{1}$$

where the coefficients are given by $a_n = \int_0^\alpha f(x) e^{-2\pi i n x/\alpha} dx$, and $\mathbb{Z}$ denotes the integers. If the coefficient sequence (a_n) is absolutely summable, then f is continuous.

For real numbers ν and y, we denote by

$$E_\nu f(x) = e^{2\pi i \nu x} f(x) \quad \text{and} \quad T_y f(x) = f(x-y)$$

the modulation (frequency translation) and translation operator, respectively. It is important to note that the two operators do not commute, but

$$E_\nu T_y = e^{2\pi i \nu y} T_y E_\nu. \tag{2}$$

In particular, we have

$$E_\mu T_y E_\nu T_z = e^{-2\pi i \nu y} E_{\mu+\nu} T_{y+z}. \tag{3}$$

This commutation rule plays an important role throughout the paper.

The space $\ell^p(\mathbb{Z}^d)$ for $1 \leq p < \infty$ and $d = 1, 2, \ldots$ contains all sequences $a = (a_n)_{n\in\mathbb{Z}^d}$ of complex numbers with

$$\|a\|_{\ell^p} = \left(\sum_{n\in\mathbb{Z}^d} |a_n|^p \right)^{\frac{1}{p}} < \infty.$$

Given two sequences $a, b \in \ell^1(\mathbb{Z}^2)$, we define the twisted convolution (with parameter θ) by

$$(a \natural b)_{m,n} = \sum_{k,l\in\mathbb{Z}} a_{k,l} b_{m-k,n-l} e^{-2\pi i \theta (m-k) l}. \tag{4}$$

For $\theta \in \mathbb{Z}$, twisted convolution reduces to the standard convolution:

$$a * b = \sum_{k,l\in\mathbb{Z}} a_{k,l} b_{m-k,n-l}. \tag{5}$$

We will later see that twisted convolution arises naturally from the commutation rule (3).

B. Frames

We introduce two basic concepts in Hilbert space theory that generalize the very useful tool of orthonormal bases. In this section, $\mathcal{H}$ denotes a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Definition 2.1: A sequence $\{g_k\}_{k\in\mathbb{Z}}$ of a Hilbert space $\mathcal{H}$ is a Riesz basis of its closed linear span if there exist bounds A, $B > 0$ such that

$$A\|c\|_{\ell^2}^2 \leq \left\| \sum_{k\in\mathbb{Z}} c_k g_k \right\|^2 \leq B\|c\|_{\ell^2}^2, \quad c \in \ell^2.$$

Riesz bases preserve many properties of orthonormal sets [5]. An extension to overcomplete (linearly dependent) sets is the concept of frames.

Definition 2.2: A sequence $\{g_k\}_{k\in\mathbb{Z}}$ of a Hilbert space $\mathcal{H}$ is a frame of $\mathcal{H}$ if there exist bounds $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, g_k \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{6}$$

A sequence $\{g_k\}_{k\in\mathbb{Z}}$ satisfying the right inequality is called a Bessel sequence.

The synthesis map $G : \ell^2 \to \mathcal{H}$ of a frame $\{g_k\}$ is defined by

$$G : (c_k) \to \sum_k c_k g_k.$$

Its adjoint G^* is the analysis operator $G^*f = (\langle f, g_k\rangle)$. Due to (6), G^*, and thus, G are bounded. The *frame operator* S is defined by

$$Sf = GG^*f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle g_k, \quad f \in \mathcal{H}.$$

By (6), the frame operator satisfies

$$A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle, \quad f \in \mathcal{H}$$

and is, therefore, positive and invertible.

Given a frame, there exists a *dual frame* $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$, which is a frame itself, such that

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{g}_k\rangle g_k = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle \tilde{g}_k, \quad f \in \mathcal{H}. \tag{7}$$

The dual frame is unique if and only if $\{g_k\}$ is a Riesz basis. For a frame that is not a Riesz basis, the canonical choice of its dual frame is the sequence $\{S^{-1}g_k\}$. It provides the minimal ℓ^2-norm coefficients in the expansion (7) [12]. Many results about frames can be found in [5]. For example, a Bessel sequence $\{\gamma_k\}$ is a dual frame of $\{g_k\}$ if and only if

$$f = \sum_{k} \langle f, \gamma_k\rangle g_k = \sum_{k} \langle f, g_k\rangle \gamma_k, \quad f \in \mathcal{H}. \tag{8}$$

In this paper, we focus on dual frames other than the canonical dual. The reason for choosing alternative dual frames is that for particular applications one might be interested in different features than minimal ℓ^2-norm coefficients. For example, we might be interested in a dual frame whose elements are of a special shape or are easy to compute.

For alternative dual frames, there exist constructive approaches that rely on the canonical dual. In [6], [7], and [25], it is shown that any dual frame of $\{g_k\}$ can be written as

$$S^{-1}g_k + h_k - \sum_{j\in\mathbb{Z}} \langle S^{-1}g_k, g_j\rangle h_j \tag{9}$$

where $\{h_k\}$ is a Bessel sequence. Note, however, that the representation (9) involves inverting the frame operator. An alternative approach based on the combination of two different frames that does not rely on the canonical dual was suggested in [13], and we outline it here.

Given two frames $\{g_k\}$ and $\{h_k\}$ for a Hilbert space $\mathcal{H}$ with synthesis operators G and H, respectively, we observe that if HG^* is invertible, then

$$(HG^*)^{-1}H$$

is a left-inverse of G^*, and the vectors $\gamma_k = (HG^*)^{-1}h_k$ and $k \in \mathbb{Z}$ behave similarly to a dual frame in the sense that

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle \gamma_k, \quad f \in \mathcal{H}.$$

Pursuing this idea leads to dual frames as follows. If HG^* is invertible, then $\{\gamma_k\}$ defined by

$$\gamma_k = (HG^*)^{-1}h_k \tag{10}$$

is a dual frame for $\{g_k\}$. This follows from the fact that $\{\gamma_k\}$ is a Bessel sequence, and

$$\begin{aligned} f &= (HG^*)^{-1}HG^*f = \sum_k \langle f, g_k\rangle \gamma_k \\ &= GH^*(GH^*)^{-1}f = \sum_k \langle f, \gamma_k\rangle g_k \end{aligned}$$

which verifies (8). If we assume that $\{h_k\}$ is already a dual frame for $\{g_k\}$, then HG^* is obviously the identity, and therefore, (10) trivially leads to the dual frame $\{h_k\}$ itself. We conclude that instead of the characterization of (9), we can characterize all duals by (10), where H is chosen such that HG^* is invertible.

At this point, we may raise the question under which conditions HG^* is invertible. The following characterization, given in a slightly more general context in [13], is rather abstract.

Proposition 2.3: Let $\{g_k\}$ and $\{h_k\}$ be frames for a Hilbert space $\mathcal{H}$ with synthesis operator $G, H : \ell^2 \to \mathcal{H}$, respectively. Then, the operator HG^* is invertible on $\mathcal{H}$ if and only if every sequence $c \in \ell^2$ has a unique representation of the form $c = G^*h + d$ for some $h \in \mathcal{H}$ and $d \in \ell^2$ with $Hd = 0$.

Proposition 2.3 provides a very general condition that is difficult to verify. One purpose of this paper is to derive explicit conditions for invertibility of HG^* in the case in which G and H represent Gabor frames.

C. Gabor Frames

A function $g \in L^2$ generates a *Gabor frame* with the (lattice) parameters $\alpha > 0$ (time parameter) and $\beta > 0$ (frequency parameter) if the sequence of uniform time-frequency translates $\{E_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ is a frame for L^2. To denote Gabor systems, we use the notation

$$\mathcal{G}(g, \alpha, \beta) = \{E_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}.$$

The prototype of a function generating Gabor frames is the Gaussian

$$\psi(x) = e^{-\pi x^2}. \tag{11}$$

The Gaussian generates a frame if and only if $\alpha\beta < 1$ [27], [35]. It can be easily derived that the dilated Gaussian $e^{-\pi x^2/\sigma^2}$, $\sigma \in \mathbb{R}$ also generates a Gabor frame if and only if $\alpha\beta < 1$.

In general, it is complicated to find all lattice parameters for which a function constitutes a Gabor frame. For instance, the B-splines of any order do not generate a Gabor frame whenever $\beta = 2, 3, \ldots$ [19].

It is well-known that the canonical dual of a Gabor frame is itself a Gabor frame given by $\mathcal{G}(S^{-1}g, \alpha, \beta)$, with the generating function $S^{-1}g$. Functions that generate a dual Gabor frame are called *dual Gabor elements*. Besides the canonical dual, there exists other dual Gabor frames, as shown in [24] by means of (9), which involves the canonical dual. Our aim is to study approaches for deriving dual Gabor elements without relying on the canonical dual that requires the inversion of the frame operator.

Transferring the idea of different left inverses described in Section II-B to a Gabor setting leads to dual Gabor elements that can be efficiently computed without using the traditional frame operator. In this way, we gain some freedom in optimizing Gabor algorithms with respect to numerical issues such as stability and computational complexity.

For fixed parameters α, β, we denote by T_g the synthesis operator of the Gabor system $\mathcal{G}(g,\alpha,\beta)$ given by

$$T_g c = \sum_{m,n\in\mathbb{Z}} c_{m,n} E_{m\beta} T_{n\alpha} g, \quad c \in \ell^2(\mathbb{Z}^2).$$

If $\mathcal{G}(g,\alpha,\beta)$ is a frame, then the adjoint operator T_g^* is bounded, and so is T_g. The frame operator is simply given by $S_g = T_g T_g^*$.

According to the general concept (10), we consider Gabor frames $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\gamma,\alpha,\beta)$ with synthesis operators T_g and T_γ, respectively. Our main purpose is to find conditions on the functions g and γ such that the (bounded) frame-type operator

$$S_{g,\gamma} = T_\gamma T_g^*$$

is invertible on L^2. Theorem 4.1 characterizes the invertibility of $S_{g,\gamma}$ for the continuous time case in terms of the coefficient sequence $(\langle \gamma, E_{m/\alpha} T_{n/\beta} g\rangle)_{m,n\in\mathbb{Z}}$. This characterization boils down to a rather simple formula for integer oversampling, i.e., the case $(\alpha\beta)^{-1} \in \mathbb{N}$; see Corollary 4.3. An analog result for the discrete time case is given in Corollary 5.2.

If the frame-type operator is invertible, then the Gabor system $\mathcal{G}(S_{g,\gamma}^{-1}\gamma,\alpha,\beta)$ is a dual Gabor frame for $\mathcal{G}(g,\alpha,\beta)$. This follows from the facts that $\mathcal{G}(S_{g,\gamma}^{-1}\gamma,\alpha,\beta)$ is a Bessel sequence and that (8) is easily verified since $S_{g,\gamma}$ and, therefore, $S_{g,\gamma}^{-1}$ commute with all time-frequency shifts $E_{m\beta}T_{n\alpha}$ [9], [23], i.e.,

$$E_{m\beta}T_{n\alpha}S_{g,\gamma} = S_{g,\gamma}E_{m\beta}T_{n\alpha}, \quad m,n \in \mathbb{Z}. \tag{12}$$

We emphasize that this commutation rule is the core property for many results in Gabor analysis.

An advantage of the present approach is that due to (12), $S_{g,\gamma}$ preserves the commutivity properties of the standard frame operator and, therefore, fast methods that have been developed for inverting the frame operator that do not rely on the positivity can also be used to invert $S_{g,\gamma}$, as discussed in Section V. Furthermore, by choosing γ appropriately, we may be able to further decrease the computational complexity of establishing $S_{g,\gamma}$ and increase the stability. An example in the discrete case is the combination of two Gaussian functions of different spread for which we explicitly show in Proposition 5.3 that $S_{g,\gamma}$ is invertible. We also provide numerical simulations, in Section VI, that confirm the potential improvement of stability.

Before proceeding to the detailed development, in the next section, we summarize the main results on Gabor analysis that will be used in our derivations in Sections IV and V.

III. Basic Results in Gabor Analysis

A. Characterizations

In the following, we describe fundamental results in Gabor analysis that are important to understand the alternative approach that we present for computing dual Gabor elements. For basic and more advanced properties of Gabor frames, we refer to [5], [15], and [18].

An important result in Gabor analysis is the Wexler–Raz biorthogonality relation that characterizes all dual Gabor elements for a given Gabor frame [5], [18], [39].

Theorem 3.1: Assume that $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(\gamma,\alpha,\beta)$ are Gabor frames for L^2. Then, they are dual frames if and only if

$$\left\langle \gamma, E_{\frac{m}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle = \alpha\beta\delta_{m,n}, \quad m,n \in \mathbb{Z}. \tag{13}$$

It is interesting to see that the condition for dual Gabor elements uses the so-called *dual lattice* $\mathbb{Z}/\beta \times \mathbb{Z}/\alpha$. In fact, the dual lattice plays a central role in many parts of Gabor theory. The first important relationship between the lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$ and its dual lattice goes back to the Ron–Shen duality [34].

Theorem 3.2: Let $g \in L^2$ and $\alpha, \beta > 0$ be given. Then, the Gabor system $\mathcal{G}(g,\alpha,\beta)$ is a frame for L^2 with bounds A, B if and only if $\mathcal{G}(g,1/\beta,1/\alpha)$ is a Riesz basis for its closed linear span with bounds $\alpha\beta A$, $\alpha\beta B$.

These two theorems are the basis of important results of conditions on the parameters α and β, where $\mathcal{G}(g,\alpha,\beta)$ is a frame. We only mention a central result and refer to [18] and [26] for detailed discussions.

Theorem 3.3: Assume that $\mathcal{G}(g,\alpha,\beta)$ is a frame. Then, $\alpha\beta = 1$. Moreover, $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis for L^2 if and only if $\alpha\beta = 1$.

In this paper, we are only interested in the case $\alpha\beta < 1$ (oversampling) since we want to make sure that more than one dual Gabor element exists in contrast to Riesz bases where the dual or biorthogonal basis is always unique.

Assume that for some α, β with $\alpha\beta < 1$, $\mathcal{G}(g,\alpha,\beta)$ is a frame of L^2 with the canonical dual Gabor element g_0. It follows that the closed linear span of $\mathcal{G}(g,1/\beta,1/\alpha)$, say $\mathcal{H}$, is a proper subspace of L^2 and $g_0 \in \mathcal{H}$ [18]. Note that the dual Gabor element of minimal L^2-norm is precisely g_0, which is therefore also called the minimal dual [18].

B. Janssen Representation and the Feichtinger Algebra

Every operator S that commutes with all time-frequency shifts $E_{m\beta}T_{n\alpha}$ has a representation of the form

$$S = \sum_{m,n} b_{m,n} E_{\frac{m}{\alpha}} T_{\frac{n}{\beta}} \tag{14}$$

for a unique sequence $b = (b_{m,n}) \in \ell^2(\mathbb{Z}^2)$ [9], [23]. This is often referred to as the *Janssen representation*. If b is an absolutely converging sequence, i.e., $b \in \ell^1(\mathbb{Z}^2)$, then the Janssen representation is convergent in L^2. Obviously, the sequence of the Janssen representation for the identity operator is the Dirac sequence δ_{00}. The fact that our construction of duals involves the operator $S_{g,\gamma}$ that commutes with all time-frequency shifts will allow us to exploit the Janssen representation. In particular, if two operators S and Q commute with all $E_{m\beta}T_{n\alpha}$, then the product SQ also commutes with $E_{m\beta}T_{n\alpha}$ and assuming that both operators S, and Q have a Janssen representation of the form (14) with $a, b \in \ell^1(\mathbb{Z}^2)$, respectively, we obtain

$$SQ = \sum_{m,n\in\mathbb{Z}} (a\natural b)_{m,n} E_{\frac{m}{\alpha}} T_{\frac{n}{\beta}} \tag{15}$$

by means of the commutation rule (3), as derived, for instance, in [14] and [18]. Here, it becomes evident how the twisted convolution comes into the game. This representation will be used in deriving our results in the next section.

For the special case of $S_{g,\gamma}$, the coefficient sequence of the Janssen representation, which we denote by $a = (a_{m,n})$, is shown in [9] and [23] to be given by

$$a_{m,n} = \left\langle \gamma, E_{\frac{m}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle, \quad m,n \in \mathbb{Z}. \tag{16}$$

Throughout the paper, a always denotes the sequence of (16), unless otherwise stated.

To avoid technical details that do not contribute to a better insight of the problem, we only consider functions g, γ such that $a \in \ell^1$. A class of functions with this property is the *Feichtinger algebra* S_0 defined by

$$S_0 = \left\{ f \in L^2 | \|f\|_{S_0} := \int\limits_{\mathbb{R}^2} |V_\psi f(x,\omega)| \, dx d\omega < \infty \right\}$$

where $V_\psi f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{\psi(t-x)} e^{-2\pi i \omega t} dt$ denotes the short-time Fourier transform with respect to the Gaussian window ψ defined in (11). Examples of functions in S_0 are the Gaussian and continuous B-splines of any order.

The Feichtinger algebra is an extremely useful space of test functions and of "good" window functions in the sense of time-frequency localization. Rigorous descriptions of S_0 can be found in [16] and references therein.

IV. Invertibility of the Frame-Type Operator on L^2

We now derive necessary and sufficient conditions on the functions g and γ such that the frame-type operator $S_{g,\gamma} = T_\gamma T_g^*$ is invertible on L^2. As we have seen, if $S_{g,\gamma}$ is invertible, then the Gabor system $\mathcal{G}(S_{g,\gamma}^{-1}\gamma, \alpha, \beta)$ is a dual Gabor frame for $\mathcal{G}(g, \alpha, \beta)$. We will see in Section V that the discrete finite representation leads to simpler results than the standard L^2-setting.

A. Rational Case

As in many parts of Gabor analysis [18], [42], we only consider the rational case $\alpha\beta = p/q$ with p, q relatively prime. In contrast to irrational relations of the parameters, the rational case preserves important periodicity conditions such as (18) that are exploited in the proofs. Tackling the problem for all α, β requires completely different methods [20] from those used here and are of little practical significance.

In our approach, we use standard time-frequency methods. Many definitions and derivations are described in [18, Ch. 13].

Let $g, \gamma \in S_0$, and define the entries of the bi-infinite matrix valued function $G(x)$ by the correlation functions

$$G_{j,l}(x) = \sum_{k\in\mathbb{Z}} \overline{g\left(x - \frac{l}{\beta} - \alpha k\right)} \gamma\left(x - \frac{j}{\beta} - \alpha k\right), \quad j,l \in \mathbb{Z}. \tag{17}$$

Because of $g, \gamma \in S_0$, $T_{l/\beta}\overline{g} \cdot T_{j/\beta}\gamma$ is in S_0 since S_0 is closed under translation and under pointwise multiplication. Moreover, the periodization $G_{j,l}(x)$ of period a is continuous. As shown in [18, Ch. 6] by virtue of Schur's test,[1] the bi-infinite matrix $G(x)$ is a bounded operator on $\ell^2(Z)$ for all x.

We state two important properties of $G(x)$:

$$G_{j,l}(x) = G_{j+pk,l+pk}(x), \tag{18}$$

$$G_{j,l}(x) = G_{0,l-j}\left(x - \frac{j}{\beta}\right). \tag{19}$$

The first property is based on the fact that $q\alpha = p/\beta$. The second relation shows that $G_{j,l}(x)$ can be derived from $G_{0,l}(x)$. Indeed, we can also define $G(x)$ by the Fourier series

$$G_{0,n}(x) = \alpha^{-1} \sum_{m\in\mathbb{Z}} a_{m,n} e^{\frac{2\pi i m x}{\alpha}} \tag{20}$$

where $a_{m,n} = \langle \gamma, E_{m/\alpha} T_{n/\beta} g \rangle$, and use (19) to extend $G_{0,n}(x)$ to $G_{j,n}(x)$. Equation (20) follows from the fact that

$$\begin{aligned} &\left\langle \gamma, E_{\frac{m}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle \\ &= \int\limits_{\mathbb{R}} \gamma(x) e^{-\frac{2\pi i m x}{\alpha}} \overline{g\left(x - \frac{n}{\beta}\right)} dx \\ &= \int\limits_0^\alpha \sum_{k\in\mathbb{Z}} \gamma(x-\alpha k) \overline{g\left(x - \frac{n}{\beta} - \alpha k\right)} e^{-\frac{2\pi i m x}{\alpha}} dx \\ &= \int\limits_0^\alpha G_{0,n}(x) e^{-\frac{2\pi i m x}{\alpha}} dx \end{aligned}$$

represent the Fourier coefficients, as introduced in (1).

The correlation functions $G_{0,n}(x)$ provide an important representation of the frame-type operator $S_{g,\gamma}$: the so-called *Walnut representation*

$$S_{g,\gamma} f = \alpha \sum_{n\in\mathbb{Z}} G_{0,n} T_{\frac{n}{\beta}} f, \quad f \in L^2 \tag{21}$$

that follows directly from the Janssen representation (14) of $S_{g,\gamma}$. Since g, $\gamma \in S_0$, the series of the Walnut representation converges unconditionally[2] in L^2 [18], [38]. We will also make use of

$$S_{g,\gamma}^* f = S_{\gamma,g} f = \alpha \sum_{n\in\mathbb{Z}} \overline{G}_{n,0} T_{\frac{n}{\beta}} f. \tag{22}$$

Next, we define the entries of the $(p \times p)$-matrix valued function $\hat{G}(\omega, x)$ by

$$\hat{G}_{\mu,\nu}(\omega,x) = \sum_{k\in\mathbb{Z}} G_{\mu+pk,\nu}(x) e^{-\frac{2\pi i k \omega}{\beta}}, \quad \mu,\nu = 0,1,\ldots,p-1. \tag{23}$$

As shown in [18, Ch. 13] (replace one g by γ), $\hat{G}(\omega, x)$ has an absolutely converging Fourier series expansion and is therefore continuous and of period (β, α). Moreover, $G(x)$ is invertible on $\ell^2(\mathbb{Z})$ if and only if the $(p \times p)$-matrix $\hat{G}(\omega, x)$ is invertible for every (ω, x).

The last statement is useful in order to transfer the question of the invertibility of $S_{g,\gamma}$ to $\hat{G}(\omega, x)$ and leads to the core result of the continuous case.

[1] The ℓ^1-norm of the rows and of the columns are uniformly bounded.

[2] Any sequence of increasing finite sums converges to the same limit.

Theorem 4.1: For $g, \gamma \in S_0$, and $\alpha\beta = p/q$, set $a_{m,n} = \langle \gamma, E_{m/\alpha}T_{n/\beta}g\rangle$, and define the $(p \times p)$ matrix-valued function $\Phi(\omega, x) = (\phi_{\mu,\nu}(\omega, x))_{\mu,\nu=0}^{p-1}$ with

$$\phi_{\mu,\nu}(\omega, x) = \alpha^{-1} \sum_{m,n} \alpha_{m,\nu-\mu+pn} e^{-\frac{2\pi i m\mu}{\alpha\beta}} e^{2\pi i\left(\frac{mx}{\alpha}+\frac{n\omega}{\beta}\right)}. \tag{24}$$

Then, $S_{g,\gamma} = \sum_{m,n} a_{m,n} E_{m/\alpha}T_{n/\beta}$ is invertible if and only if $\det(\Phi(\omega, x)) \neq 0$ for all (ω, x).

Proof: First, we show that $\Phi(\omega, x)$ coincides with the matrix-valued function $\hat{G}(\omega, x)$ defined in (23). Using (18)–(20), we compute

$$\begin{aligned}
\hat{G}_{\mu,\nu}(\omega, x) &= \sum_{n\in\mathbb{Z}} G_{\mu+pn,\nu}(x) e^{-\frac{2\pi i n\omega}{\beta}} \\
&= \sum_{n\in\mathbb{Z}} G_{0,\nu-\mu-pn}\left(x - \frac{\mu}{\beta}\right) e^{-\frac{2\pi i n\omega}{\beta}} \\
&= \alpha^{-1} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} a_{m,\nu-\mu-pn} e^{\frac{2\pi i m\left(x-\frac{\mu}{\beta}\right)}{\alpha}} e^{-\frac{2\pi i n\omega}{\beta}} \\
&= \alpha^{-1} \sum_{m,n} a_{m,\nu-\mu+pn} e^{-\frac{2\pi i m\mu}{\alpha\beta}} e^{2\pi i\left(\frac{mx}{\alpha}+\frac{n\omega}{\beta}\right)} \\
&= \phi_{\mu,\nu}(\omega, x).
\end{aligned}$$

Now, we transfer the statement of the theorem to $\hat{G}(\omega, x)$.

Assume first that $S_{g,\gamma}$ is invertible. By virtue of Wiener's Lemma, cf. [20], there exists a sequence $b \in \ell^1(\mathbb{Z}^2)$ such that $S_{g,\gamma}^{-1} = \sum_{m,n} b_{m,n} E_{m/\alpha}T_{n/\beta}$. Furthermore, we derive from (15)

$$a \natural b = b \natural a = \delta_{0,0}. \tag{25}$$

We use the sequence b for defining

$$H_{0,n}(x) = \alpha \sum_{m\in\mathbb{Z}} b_{m,n} e^{\frac{2\pi i m x}{\alpha}} \text{ and } H_{j,l}(x) = H_{0,l-j}\left(x - \frac{j}{\beta}\right). \tag{26}$$

In the Appendix, in Lemma 7.1, we prove that the matrix-valued function $H(x)$ is the inverse of $G(x)$. This shows that $G(x)$, and therefore, $\hat{G}(\omega, x)$ is invertible for all (ω, x).

Conversely, assume that $\hat{G}(\omega, x)$ is invertible for all (ω, x). We define

$$\hat{H}(\omega, x) = \hat{G}^{-1}(\omega, x).$$

By using a generalized version of Wiener's lemma, it has been shown in [18] that all entries $\hat{H}_{\mu,\nu}(\omega, x)$ have an absolutely convergent Fourier series. In particular, the entries of $\hat{H}(\omega, x)$ are continuous and of period (β, α). We further define a matrix-valued function $H(x)$ by

$$H_{\mu,\nu+pk}(x) = \beta^{-1} \int_0^\beta \hat{H}_{\mu,\nu}(\omega, x) e^{-\frac{2\pi i k\omega}{\beta}} d\omega, \qquad \mu, \nu = 0, 1, \ldots, p-1, k \in \mathbb{Z} \tag{27}$$

and $H_{\mu+pn,\nu+pk}(x) = H_{\mu,\nu+p(k-n)}(x)$ for $n \in \mathbb{Z}$. As it can easily be seen, there exists a constant $C > 0$ such that

$$|H_{j,l}(x)| \leq C, \quad j, k \in \mathbb{Z}. \tag{28}$$

Recall that by definition (23)

$$G_{\mu+pk,\nu}(x) = \beta^{-1} \int_0^\beta \hat{G}_{\mu,\nu}(\omega, x) e^{\frac{2\pi i k\omega}{\beta}} d\omega, \qquad \mu, \nu = 0, 1, \ldots, p-1.$$

Lemma 7.2 shows explicitly that $H(x)$ is the inverse of $G(x)$.

In order to obtain the inverse of $S_{g,\gamma}$ from the matrix-valued function $H(x)$, we define the sesqui-linear form

$$\beta \int_0^{\frac{1}{\beta}} \sum_{j,l} H_{jl}(x) T_{\frac{l}{\beta}} f(x) T_{\frac{j}{\beta}} \overline{h(x)} dx \tag{29}$$

for $f, h \in L^2$. By means of (28), we have

$$\begin{aligned}
&\sum_{j,l} \int_0^{\frac{1}{\beta}} \left| H_{jl}(x) T_{\frac{l}{\beta}} f(x) T_{\frac{j}{\beta}} \overline{h(x)} \right| dx \\
&\leq C \sum_{j,l} \int_0^{\frac{1}{\beta}} \left| T_{\frac{l}{\beta}} f(x) T_{\frac{j}{\beta}} \overline{h(x)} \right| dx \\
&\leq C \sum_{j,l} \int_0^{\frac{1}{\beta}} \left| T_{\frac{l}{\beta}} f(x) \right|^2 dx \int_0^{\frac{1}{\beta}} \left| T_{\frac{j}{\beta}} h(x) \right|^2 dx \\
&= C\|f\|^2\|h\|^2
\end{aligned}$$

and, therefore, (29) is bounded.

By means of (29), we define the operator S on L^2 implicitly through

$$\langle Sf, h\rangle = \beta \int_0^{\frac{1}{\beta}} \sum_{j,l} H_{jl}(x) T_{\frac{l}{\beta}} f(x) T_{\frac{j}{\beta}} \overline{h(x)} dx. \tag{30}$$

Finally, Lemma 7.3 shows that S is the inverse of $S_{g,\gamma}$. In particular, $S_{g,\gamma}$ is invertible on $L^2(\mathbb{R})$.

Altogether, we showed that $S_{g,\gamma}$ is invertible if and only if $\Phi(\omega, x)$ is invertible for all (ω, x), which is equivalent to the claim of the theorem. □

The following statement results from the fact that if $g, \gamma \in S_0$, then $S_{g,\gamma}$ maps S_0 into itself [16]. It basically says that if g and γ are well-localized in time and frequency, so is the dual window $S_{g,\gamma}^{-1}\gamma$.

Proposition 4.2: If $g, \gamma \in S_0$, and $S_{g,\gamma}$ is invertible for some α, β, then $S_{g,\gamma}^{-1}\gamma$ is also in S_0.

Remark 1: Theorem 4.1 can also be stated only in terms of the coefficient sequence a without referring to special constructs of Gabor analysis. Indeed, we can use (20) in order to define (17). Then, we obtain (19) by definition and derive (18) easily from $\alpha\beta = p/q$. These are the only properties that we used in our computations. In this context, Theorem 4.1 says that the Janssen representation of an ℓ^1 sequence a is invertible if and only if $\det(\Phi(\omega, x))$ never vanishes.

Remark 2: In the case $g = \gamma$, Theorem 4.1 states a characterization for g generating a Gabor frame, which coincides with the derivations in [18, p. 286].

Remark 3: The fact that $\mathcal{G}(\gamma, \alpha, \beta)$ is a Gabor frame must be fulfilled for $\mathcal{G}(S_{g,\gamma}^{-1}\gamma, \alpha, \beta)$ being a frame, although it is not explicit in the above derivation. This follows easily from the frame inequality (6) and the fact that $S_{g,\gamma}^{-1}$ commutes with all time-frequency shifts of the form $E_{m\beta}T_{n\alpha}$ such that

$$\langle f, E_{m\beta}T_{n\alpha}S_{g,\gamma}^{-1}\gamma\rangle = \langle (S_{g,\gamma}^{-1})^* f, E_{m\beta}T_{n\alpha}\gamma\rangle \qquad m,n \in \mathbb{Z}, f \in L^2.$$

A different but less transparent characterization of the invertibility of $S_{g,\gamma}$ is given by the so-called Zibulski–Zeevi representation [42]. This representation uses the *Zak transform*

$$Z_\alpha f(x,\omega) = \sum_{k\in\mathbb{Z}} f(x-\alpha k)e^{2\pi i\alpha k\omega}, \quad f \in L^2(\mathbb{R}) \tag{31}$$

that is widely known in signal processing [22].

The Zibulski–Zeevi representation is as follows. For $\alpha\beta = p/q$, define the matrix-valued function $A(x,\omega)$ by

$$A_{r,s}(x,\omega) = \alpha\sum_{j=0}^{q-1}\overline{Z_\alpha g\left(x+\frac{\alpha s}{p}, \omega-\beta j\right)}Z_\alpha\gamma \times\left(x+\frac{\alpha r}{p}, \omega-\beta j\right)e^{\frac{2\pi i j(r-s)}{q}}$$

for r, $s = 0,1,\ldots,p-1$, and the vector valued Zak transform by

$$(\mathcal{Z}_\alpha f(x,\omega))_j = Z_\alpha f\left(x+\frac{\alpha j}{p},\omega\right), \quad j=0,1,\ldots,p-1.$$

Then

$$S_{g,\gamma} = \mathcal{Z}_\alpha^{-1}A\mathcal{Z}_\alpha. \tag{32}$$

Hence, $S_{g,\gamma}$ is invertible if and only if the matrix-valued function $A(x,\omega)$ is invertible for almost all x, ω [18], [42].

The Zibulski–Zeevi representation is a powerful tool in Gabor analysis and provides an alternative characterization of the invertibility of the frame-type operator $S_{g,\gamma}$. While the Zibulski–Zeevi representation uses the Zak transform, our approach is based on the series expansion of frame-type operators (Janssen representation) transferring the analysis over to the corresponding coefficients. These coefficients, which are hidden in the Zibulski–Zeevi representation, reveal many properties of frame-type operators and are well suited for further studies. For instance, preconditioning concepts of the Gabor frame operator are developed on the basis of manipulating coefficients in the Janssen representation [1]. Therefore, analysis based on these coefficients can be important in our aspects of Gabor theory as well.

As a final remark of this section, we point out that both the Zibulski–Zeevi representation and Theorem 4.1 are inappropriate for studying the numerical behavior of the Gabor frame-type operator in the presence of small changes of the lattice parameters α, β since p and q may get arbitrary large, leading to computational drawbacks. This would suggest to use the old standard von Neumann-type inversion, as described in early papers about frames [12]. In most practical applications, however, one tries to avoid large p and usually sticks to the integer oversampling case.

B. Integer Oversampling

The case $\alpha\beta < 1$ with $(\alpha\beta)^{-1} = 2,3,\ldots$ is called integer oversampling and corresponds to the rational case $p = 1$. This case has been studied extensively in the literature since it leads to efficient computational algorithms [30], [31], [41]. As Corollary 4.3 below shows, in this case, we can obtain an easily verifiable condition on the windows g and γ such that $S_{g,\gamma}$ is invertible. This is also a consequence of the fact that in the integer oversampling case, the operators $T_{\alpha n}$ and $E_{\beta m}$ commute.

Corollary 4.3: For g, $\gamma \in S_0$, and $(\alpha\beta)^{-1} = 2,3,\ldots$, set $a_{m,n} = \langle\gamma, E_{m/\alpha}T_{n/\beta}g\rangle$. Then, $S_{g,\gamma} = \sum_{m,n} a_{m,n}E_{m/\alpha}T_{n/\beta}$ is invertible if and only if the function

$$\phi(\omega,x) = \alpha^{-1}\sum_{m,n} a_{m,n}e^{2\pi i\left(\frac{mx}{\alpha}+\frac{n\omega}{\beta}\right)} \tag{33}$$

never vanishes.

Proof: Since $p = 1$ leaves μ, $\nu = 0$, expression (24) of Theorem 4.1 simply reduces to (33). □

Example: We now consider a general class of functions that satisfy the conditions of Corollary 4.3. Suppose that for some constant C, we have the following:

1) $\alpha \leq C$, and $\beta \leq 1/C$.
2) supp(g), and supp(γ) $\subset [-C/2, C/2]$.

Then, it follows that $\langle\gamma, E_{m/\alpha}T_{n/\beta}g\rangle = 0$ for all $n \neq 0$, which shows that $\phi(\omega,x)$ is constant in the first variable since the condition in Corollary 4.3 reduces to

$$\phi(\omega,x) = \alpha^{-1}\sum_m a_{m,0}e^{\frac{2\pi i m x}{\alpha}}. \tag{34}$$

Using $a_{m,0} = \langle\gamma, E_{m/\alpha}g\rangle = \hat{f}(m/\alpha)$ with $f = \gamma\bar{g}$, we can apply the Poisson summation formula to (34) and obtain

$$\phi(\omega,x)=\alpha^{-1}\sum_m \hat{f}\left(\frac{m}{\alpha}\right)e^{\frac{2\pi i m x}{\alpha}}=\sum_m \gamma(x+\alpha m)\overline{g(x+\alpha m)}.$$

If g and γ have overlapping supports and are real-valued and non-negative, for instance, B-splines, then there always exists α small enough such that ϕ is strictly positive, and hence, Corollary 4.3 applies.

V. Discrete Finite Representation

In the previous section, we considered the case of continuous-time Gabor representations. In this section, we consider the finite discrete-time case. In applications, we can only process a finite number of data. Therefore, we introduce a finite discrete model of signals defined on the cyclic (index) group $\{0,1,\ldots,L-1\}$ of dimension L according to the excellent treatise on finite-dimensional Gabor systems in [36].

One approach to deriving the results in the discrete setting is to "sample" the continuous-time results. However, it is important to note that the Gabor frame-type matrix has an additional sparsity structure that is not present in the continuous model [30], [36]. This observation allows for easier characterizations by treating the discrete model directly without relying on the continuous-time results. Nonetheless, we will show in the case of integer oversampling that the condition for invertibility of the frame-type operator is analogous to the continuous-time condition of Corollary 4.3 and how this result can be applied to the example of Gaussian windows.

What we will show by examples is that the presented alternative approach for dual Gabor windows leaves some freedom that can be used to reduce computational complexity and to improve stability in the sense of controlling the condition number of the Gabor frame-type matrix.

The notation of the finite discrete setting mainly follows [33] and [36]. A signal of length L is a vector of complex numbers $f[k]$, $k=0,1,\dots,L-1$. The vector is extended periodically on $\mathbb{Z}$, that is

$$f[k+qL]=f[k], \quad q\in\mathbb{Z}.$$

The inner product of two signals $f,g\in\mathbb{C}^L$ is

$$\langle f,g\rangle=\sum_{k=0}^{L-1} f[k]\overline{g[k]}.$$

Given two integer parameters α,β with $\alpha\beta<L$ that are divisors of L, we define $\tilde{\alpha}=L/\alpha$ and $\tilde{\beta}=L/\beta$. (Note that $\tilde{\alpha}$ and $\tilde{\beta}$ are integers.) If the set of vectors

$$g_{m,n}[k]=g[k-\alpha n]e^{\frac{2\pi i\beta mk}{L}},$$

$m=0,1,\dots,\tilde{\beta}-1$, $n=0,1,\dots,\tilde{\alpha}-1$ spans $\mathbb{C}^L$, then it is called a Gabor frame. The case that $\alpha\beta$ divides L corresponds to integer oversampling. We note that in the discrete case, we always have that $\alpha\beta$ is an integer and that $\alpha\beta$ divides L^2.

For two Gabor frames with generating vectors g and γ, the associated frame-type operator $S_{g,\gamma}$ is an $(L\times L)$-matrix with jlth entry

$$(S_{g,\gamma})_{jl} = \begin{cases}\tilde{\beta}\sum_{n=0}^{\tilde{\alpha}-1}\gamma[j-\alpha n]\overline{g[l-\alpha n]}, & \text{if } |j-l| \text{ is divided by } \tilde{\beta}\\ 0, & \text{otherwise}\end{cases} \tag{35}$$

which is called the Walnut representation of $S_{g,\gamma}$ for the discrete case [33]. The discrete Walnut representation implies the following properties of $S_{g,\gamma}$:

1) Only every $\tilde{\beta}$th subdiagonal of $S_{g,\gamma}$ is nonzero.
2) Entries along a subdiagonal are α-periodic.
3) $S_{g,\gamma}$ is a block circulant matrix of the form

$$S_{g,\gamma}=\begin{bmatrix} A_0 & A_1 & \dots & A_{\tilde{\alpha}-1}\\ A_{\tilde{\alpha}-1} & A_0 & \dots & A_{\tilde{\alpha}-2}\\ \vdots & \vdots & \ddots & \vdots\\ A_1 & A_2 & \dots & A_0\end{bmatrix}$$

where A_s are noncirculant $\alpha\times\alpha$ matrices, with

$$(A_s)_{j,l}=(S_{g,\gamma})_{j+s\alpha,l+s\alpha} \tag{36}$$

for $s=0,1,\dots,\tilde{\alpha}-1$ and $j,l=0,1,\dots,a-1$, [36].

In the following, we exploit the high structure of the matrix $S_{g,\gamma}$ in order to derive statements about the invertibility of $S_{g,\gamma}$ similar to the main results in the continuous case where this sparsity structure is not present.

For a Block circulant matrix A of square matrices A_s, $s=0,1,\dots,n-1$, we write $A=\mathcal{C}(A_0,\dots,A_{n-1})$. We define the Fourier transform of a block circulant matrix A by $\mathcal{F}(A)=\mathcal{C}(\hat{A}_0,\dots,\hat{A}_{n-1})$ with

$$\hat{A}_r=\sum_{s=0}^{n-1} e^{-\frac{2\pi irs}{n}}A_s, \quad r=0,\dots,n-1.$$

The invertibility of a circulant matrix can be characterized by means of its Fourier transform [2], [11], [37].

Lemma 5.1: The block circulant matrix $A=\mathcal{C}(A_0,\dots,A_{n-1})$ is invertible if and only if $\hat{A}_r$ is invertible for all $r=0,\dots,n-1$. In this case, $A^{-1}=\mathcal{C}(B_0,\dots,B_{n-1})$ with

$$B_s=\frac{1}{n}\sum_{r=0}^{n-1} e^{\frac{2\pi irs}{n}}\hat{A}_r^{-1}, \quad s=0,\dots,n-1.$$

In this context, the main Theorem 4.1 of the continuous case reduces to the verification of the invertibility of the $(\alpha\times\alpha)$-matrices A_s, $s=0,\dots,\tilde{\alpha}-1$ in the discrete case as a consequence of the additional structure of the Gabor matrix.

Exploiting the structure of circulant matrices leads to very efficient inversion methods. Indeed, as shown in [36] for the Gabor matrix, the inversion of $S_{g,\gamma}$ can be carried out in $\mathcal{O}(\alpha\beta^2\log\beta+\alpha\beta)$ operations, provided that $S_{g,\gamma}$ is invertible.

Applying Lemma 5.1 to the frame-type operator for integer oversampling leads to a simple explicit invertibility characterization that is slightly different to what is stated in [32] and [33].

Corollary 5.2: Assume that $L/(\alpha\beta)$ is an integer. Then, the frame-type operator

$$S_{g,\gamma}=\mathcal{C}(A_0,\dots,A_{\alpha-1})$$

is invertible if and only if the diagonal entries of $\hat{A}_s$ are not zero for all indices $s=0,\dots,\tilde{\alpha}-1$.

Proof: Since $\tilde{\beta}$ is assumed to be divisible by α, it follows from the properties of $S_{g,\gamma}$ that

$$A_0, A_{\frac{\tilde{\beta}}{\alpha}}, A_{\frac{2\tilde{\beta}}{\alpha}},\dots,A_{\frac{(\beta-1)\tilde{\beta}}{\alpha}}$$

are the only nonzero diagonal matrices. Therefore

$$\hat{A}_r=A_0+e^{-\frac{2\pi ir}{\beta}}A_{\frac{\tilde{\beta}}{\alpha}}+\dots+e^{-\frac{2\pi ir(\beta-1)}{\beta}}A_{\frac{(\beta-1)\tilde{\beta}}{\alpha}} \tag{37}$$

are also diagonal matrices for $r=0,\dots,\tilde{\alpha}-1$ and invertible if and only if each diagonal entry is not zero. Applying Lemma 5.1 concludes the proof. □

The difference of Corollary 5.2 from Corollary 4.3 emerges only from the additional structure of the Gabor frame-type matrix, whereas the statement is indeed the equivalent formulation of Corollary 4.3 for the discrete setting.

Example: A popular choice of a Gabor window is the Gaussian window $g[k]=e^{-k^2/\sigma^2}$. It is well known that the Gaussian window is optimal in the sense that it is most localized in time and frequency [10], [18], [29].

Consider choosing g and γ as Gaussian windows of different spread, that is

$$g[k]=e^{-\frac{k^2}{\sigma_1^2}} \quad\text{and}\quad \gamma[k]=e^{-\frac{k^2}{\sigma_2^2}} \tag{38}$$

for some $\sigma_1, \sigma_2 > 0$. We explicitly show that in the case of integer oversampling, the corresponding frame-type matrix $S_{g,\gamma}$ is invertible for all choices of σ_1 and σ_2. Thus, in applications where σ_1 is fixed, we may choose $\sigma_2 \neq \sigma_1$ to obtain a frame-type operator that is better conditioned or computed with fewer operations.

Proposition 5.3: If $L/(\alpha\beta)$ is an integer, then the frame-type operator $S_{g,\gamma}$ of g, γ defined in (38) is invertible.

Proof: In accordance with Corollary 5.2, we have to show that the diagonal entries of the matrices $\hat{A}_r, r = 0, 1, \ldots, \tilde{\alpha}-1$, given by (37), are nonzero. By combining (35)–(37), we derive

$$\begin{aligned}(\hat{A}_r)_{k,k} &= \sum_{q=0}^{\beta-1} e^{-\frac{2\pi i r q}{\beta}} \left(A_{\frac{q\tilde{\beta}}{\alpha}}\right)_{k,k} \\ &= \tilde{\beta} \sum_{q=0}^{\beta-1} e^{-\frac{2\pi i r q}{\beta}} \sum_{n=0}^{\tilde{\alpha}-1} e^{-\frac{(k-\alpha n+q\tilde{\beta})^2}{\sigma_2^2}} e^{-\frac{(k-\alpha n+q\tilde{\beta})^2}{\sigma_1^2}} \\ &= \tilde{\beta} \sum_{q=0}^{\beta-1} e^{-\frac{2\pi i r q}{\beta}} \sum_{n=0}^{\tilde{\alpha}-1} e^{-\frac{(k-\alpha n+q\tilde{\beta})^2\left(\sigma_1^2+\sigma_2^2\right)}{\sigma_1^2\sigma_2^2}}.\end{aligned}$$

What we obtain is a sum of the discrete Fourier transform of shifted Gaussians, which never vanishes. □

VI. Computational Aspects

In the discussion about computational aspects, we consider the discrete finite setting described in the previous section. In particular, we use the example of Gaussian windows given in (38) and provide arguments for computational savings when calculating dual elements. Numerical examples finally show that it is possible to improve the condition number of the Gabor frame-type matrix.

In some applications, the spread of one Gaussian window is fixed, and we can choose the second Gaussian window with a much smaller spread. In this way, the sum (35) for the entries of the frame-type matrix reduces to a few terms, and the computational complexity for establishing the matrix can be reduced significantly. For example, if we choose the effective spread of the second Gaussian to be within the interval $[-\alpha, \alpha]$, i.e., all values outside this interval are negligibly small, then the sum (35) becomes a single term, and the number of operations for establishing the frame-type matrix $S_{g,\gamma}$ is reduced from $(\alpha+1)\alpha\beta$ when all terms of (35) are considered to $2\alpha\beta$, leading to computational savings.

Figs. 1–3 show examples of different dual elements, including the canonical dual for the Gaussian window defined in (38) with $\sigma_1 = 32$, $L = 256$, $\alpha = 8$, and $\beta = 8$ (integer oversampling).

What we observe is that changing the spread parameter of the Gaussian window γ leads to different but similarly shaped dual elements.

In Fig. 2, we compare the condition number of the frame-type matrices when varying the spread σ. The parameters are as above, except $\sigma_1 = 128$.

The example in Fig. 2 reveals that it is possible to improve the stability of Gabor systems when replacing the canonical dual by different dual elements. In the given example, we reduce the condition number by a factor of 100 when choosing $\sigma_2 = 8$.

The final figure shows the improvement of the condition number by a factor of 100 for the Gaussian example for noninteger oversampling. Here, we chose $L = 240$, $\alpha = 8$, $\beta = 12$, and $\sigma_1 = 32$.

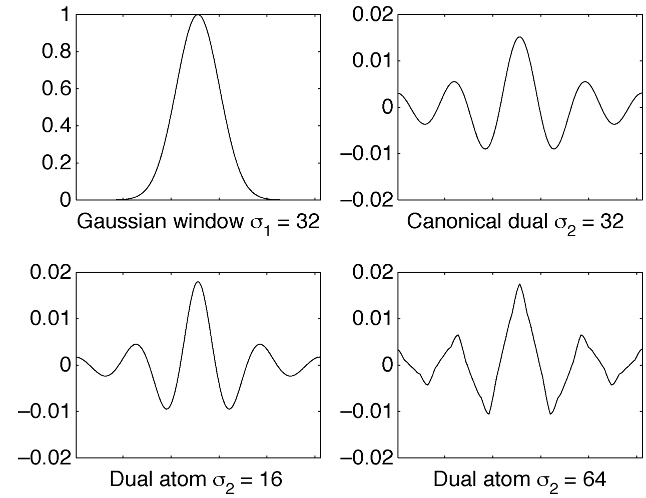

Fig. 1. Gaussian windows.

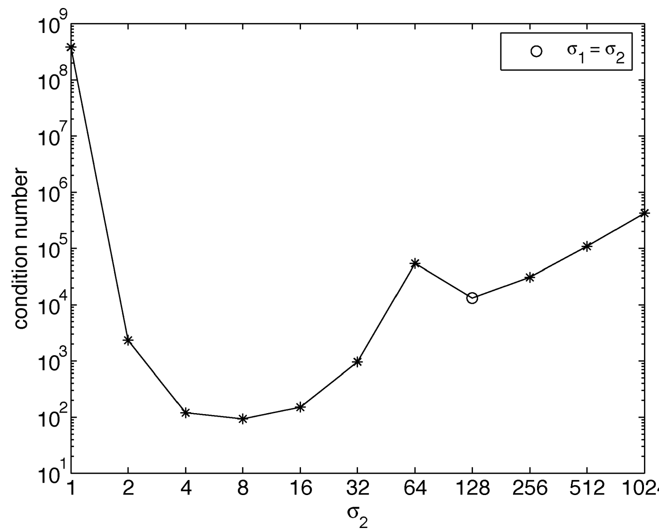

Fig. 2. Condition number versus spread σ_2.

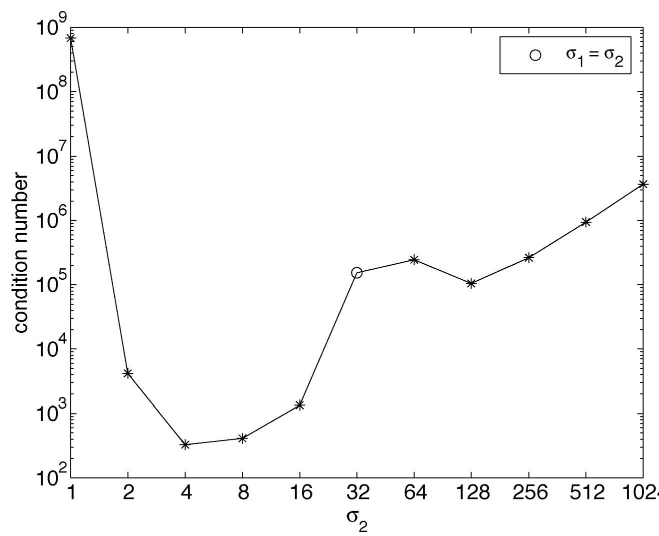

Fig. 3. Condition number versus spread σ_2.

VII. Conclusion

In this paper, we have presented a method for computing alternative dual Gabor elements without relying on the canonical dual. The approach is based on the general idea that replacing the pseudo-inverse of the analysis operator of a frame by another left inverse leads to alternative dual frames.

In the theoretical part, we provide necessary and sufficient conditions on the window functions for the frame-type operator to be invertible in the rational case and derive a simple formulation for integer oversampling. In the discrete finite setting, we exploit the Gabor structure in order to derive invertibility statements directly from the frame-type matrix. Our approach has the potential for computational savings and stability improvement when calculating the dual element, as substantially shown in examples.

Appendix

Lemma 7.1: The matrix-valued function $H(x)$ defined in (26) is the inverse of the matrix-valued function $G(x)$ defined in (20).

Proof: We fix x, and by means of (25), we compute

$$
\begin{aligned}
&(H(x)G(x))_{j,k}\\
&\quad = \sum_l H_{j,l}(x)G_{l,k}(x)\\
&\quad = \sum_l H_{0,l-j}\left(x-\frac{j}{\beta}\right)G_{0,k-l}\left(x-\frac{l}{\beta}\right)\\
&\quad = \sum_l \sum_m b_{m,l-j}e^{\frac{2\pi i m\left(x-\frac{j}{\beta}\right)}{\alpha}} \sum_n a_{n,k-l}e^{\frac{2\pi i n\left(x-\frac{l}{\beta}\right)}{\alpha}}\\
&\quad = \sum_l \sum_m \sum_n b_{m,l-j}a_{n,k-l}e^{\frac{2\pi i\left((m+n)x-\frac{(mj+nl)}{\beta}\right)}{\alpha}}\\
&\quad\quad l\rightarrow l+j,\quad n\rightarrow j-m-n\\
&\quad = \sum_n \sum_{l,m} b_{m,l}a_{(j-n)-m,(k-j)-l}e^{-\frac{2\pi i(j-n-m)l}{\alpha\beta}}\\
&\quad\quad \times e^{2\pi i\left(\frac{(j-n)x}{\alpha}-\frac{j(j-n)}{\beta}\right)}\\
&\quad = \sum_n b\natural a_{j-n,k-j}e^{2\pi i\left(\frac{(j-n)x}{\alpha}-\frac{j(j-n)}{\beta}\right)}\\
&\quad = \sum_n \delta_{j-n,k-j}e^{2\pi i\left(\frac{(j-n)x}{\alpha}-\frac{j(j-n)}{\beta}\right)}\\
&\quad = \delta_{j,k}\sum_n \delta_{j,n}e^{2\pi i\left(\frac{(j-n)x}{\alpha}-\frac{j(j-n)}{\beta}\right)} = \delta_{j,k}.
\end{aligned}
$$

The same holds true for $(G(x)H(x))_{j,k}$. Note that since $a, b \in \ell^1$, the interchanges of the sums are justified. $\square$

The above computation gives an insight into the connection of the definitions of the matrix-valued functions $G(x)$ and the twisted convolution.

Lemma 7.2: The matrix-valued function $H(x)$ defined in (27) is the inverse of the matrix-valued function $G(x)$ defined in (17).

Proof: Let $r=\mu+pm$, $s=\nu+pn$ be arbitrary integers. Applying the Poisson summation formula [17], we compute

$$
\begin{aligned}
&(H(x)G(x))_{r,s}\\
&\quad = \sum_{l\in\mathbb{Z}} H_{r,l}(x)G_{l,s}(x)\\
&\quad = \sum_{\eta=0}^{p-1}\sum_{k\in\mathbb{Z}} H_{\mu+pm,\eta+pk}(x)G_{\eta+pk,\nu+pn}(x)\\
&\quad = \sum_{\eta=0}^{p-1}\sum_{k\in\mathbb{Z}} H_{\mu,\eta+p(k-m)}(x)G_{\eta+p(k-n),\nu}(x)\\
&\quad = \beta^{-2}\sum_{\eta=0}^{p-1}\sum_{k\in\mathbb{Z}}\int_0^\beta \hat{H}_{\mu,\eta}(\omega,x)e^{-\frac{2\pi i(k-m)\omega}{\beta}}d\omega\\
&\quad\quad \times\int_0^\beta \hat{G}_{\eta,\nu}(\xi,x)e^{\frac{2\pi i(k-n)\xi}{\beta}}d\xi\\
&\quad = \beta^{-2}\sum_{\eta=0}^{p-1}\int_0^\beta\int_0^\beta \hat{H}_{\mu,\eta}(\omega,x)\hat{G}_{\eta,\nu}(\xi,x)e^{\frac{2\pi i(m\omega-n\xi)}{\beta}}\\
&\quad\quad \times\sum_{k\in\mathbb{Z}} e^{\frac{2\pi i k(\xi-\omega)}{\beta}}d\xi d\omega\\
&\quad = \beta^{-1}\sum_{\eta=0}^{p-1}\int_0^\beta\int_0^\beta \hat{H}_{\mu,\eta}(\omega,x)\hat{G}_{\eta,\nu}(\xi,x)e^{\frac{2\pi i(m\omega-n\xi)}{\beta}}\\
&\quad\quad \times\delta(\xi-\omega)d\,xid\omega\\
&\quad = \beta^{-1}\int_0^\beta e^{\frac{2\pi i(m-n)\omega}{\beta}}\sum_{\eta=0}^{p-1}\hat{H}_{\mu,\eta}(\omega,x)\hat{G}_{\eta,\nu}(\omega,x)d\omega\\
&\quad = \delta_{\mu,\nu}\beta^{-1}\int_0^\beta e^{\frac{2\pi i(m-n)\omega}{\beta}}d\omega = \delta_{\mu,\nu}\delta_{m,n} = \delta_{r,s}.
\end{aligned}
$$

The same computation goes through for $(G(x)H(x))_{r,s}$. $\square$

Lemma 7.3: The operator S defined in (30) is the inverse of $S_{g,\gamma}$.

Proof: Using the Walnut representation (21) of $S_{g,\gamma}$, we compute

$$
\begin{aligned}
&\langle SS_{g,\gamma}f,h\rangle\\
&\quad = \beta\int_0^{\frac{1}{\beta}}\sum_{j,l}H_{jl}(x)T_{\frac{l}{\beta}}\left(S_{g,\gamma}f(x)\right)T_{\frac{j}{\beta}}\overline{h(x)}dx\\
&\quad = \int_0^{\frac{1}{\beta}}\sum_{j,l}H_{jl}(x)T_{\frac{l}{\beta}}\left(\sum_k G_{0,k}(x)T_{\frac{k}{\beta}}f(x)\right)T_{\frac{j}{\beta}}\overline{h(x)}dx\\
&\quad = \int_0^{\frac{1}{\beta}}\sum_k\sum_{j,l}H_{jl}(x)G_{l,k+l}(x)T_{k+\frac{l}{\beta}}f(x)T_{\frac{j}{\beta}}\overline{h(x)}dx\\
&\quad\quad k\rightarrow k-l\\
&\quad = \int_0^{\frac{1}{\beta}}\sum_{j,k}\left(\sum_l H_{jl}(x)G_{l,k}(x)\right)T_{\frac{k}{\beta}}f(x)T_{\frac{j}{\beta}}\overline{h(x)}dx\\
&\quad = \int_0^{\frac{1}{\beta}}\sum_j T_{\frac{j}{\beta}}f(x)T_{\frac{j}{\beta}}\overline{h(x)}dx = \langle f,h\rangle.
\end{aligned}
$$

In the same way, by using (22), we have $\langle S_{g\gamma}Sf, h\rangle = \langle Sf, S^*_{g,\gamma}h\rangle = \langle f, h\rangle$. Since this holds true for all $h \in L^2$, we obtain $SS_{g,\gamma} = S_{g,\gamma}S = Id$. □

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REFERENCES

[1] P. Balazs, H. G. Feichtinger, M. Hampejs, and G. Kracher, "Double preconditioning of the Gabor frame operator," *IEEE Trans. Signal Process.*, submitted for publication.

[2] A. Beck, Y. C. Eldar, and A. Ben-Tal, "Minimax mean-squared error estimation of multichannel signals," IEEE Trans. Inf. Theory, submitted for publication.

[3] J. ben Arie and Z. Wang, "Gabor kernels for affine-invariant object recognition," in *Gabor Analysis and Algorithms: Theory and Applications*, H. G. Feichtinger and T. Strohmer, Eds. Boston, MA: Birkhäuser, 1998, pp. 409–426.

[4] J. ben Arie, Z. Wang, and K. R. Rao, "Affine invariant shape representation and recognition using Gaussian kernels and multi-dimensional indexing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 1996, pp. 3470–3473.

[5] O. Christensen, *An Introduction to Frames and Riesz Bases*. Boston, MA: Birkhäser, 2003.

[6] O. Christensen and Y. C. Eldar, "Generalized shift-invariant frames and duals for subspaces," *J. Fourier Anal. Appl.*, vol. 11, pp. 299–313, 2005.

[7] ——, "Oblique dual frames and shift-invariant spaces," *Appl. Comput. Harm. Anal.*, vol. 17, no. 1, pp. 48–68, Jul. 2004.

[8] I. Daubechies, A. Grossmann, and Y. Meyer, "Painless nonorthogonal expansions," *J. Math. Phys.*, vol. 27, no. 5, pp. 1271–1283, 1986.

[9] I. Daubechies, H. J. Landau, and Z. Landau, "Gabor time-frequency lattices and the Wexler–Raz Identity," *J. Four. Anal. Appl.*, vol. 1, no. 4, pp. 437–478, 1995.

[10] J. G. Daugman, "Complete discrete 2D Gabor transforms by neural networks for image analysis and compression," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 36, no. 7, pp. 1169–1179, Jul. 1988.

[11] P. J. Davis, *Circulant Matrices*, Second ed. New York: AMS Chelsea, 1994.

[12] R. Duffin and A. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, vol. 72, pp. 341–366, 1952.

[13] Y. C. Eldar and O. Christensen, "Characterization of oblique dual frame pairs," *J. Appl. Signal Process.*, to be published.

[14] H. G. Feichtinger and N. Kaiblinger, "Varying the time-frequency lattice of Gabor frames," *Trans. Amer. Math. Soc.*, vol. 356, no. 5, pp. 2001–2023, 2003.

[15] H. G. Feichtinger and T. Strohmer, *Gabor Analysis and Algorithms: Theory and Applications*. Boston, MA: Birkhäuser, 1998.

[16] H. G. Feichtinger and G. Zimmermann, "A Banach space of test functions for Gabor analysis," in *Gabor Analysis and Algorithms: Theory and Applications*, H. G. Feichtinger and T. Strohmer, Eds. Boston, MA: Birkhäuser, 1998, pp. 123–170.

[17] G. B. Folland, *Fourier Analysis and Its Applications*. Florence, KY: Wadsworth, 1992.

[18] K. Gröchenig, *Foundations of Time-Frequency Analysis*. Boston, MA: Birkhäuser, 2001.

[19] K. Gröchenig, A. J. E. M. Janssen, N. Kaiblinger, and G. E. Pfander, "Note on B-splines, wavelet scaling functions, and Gabor frames," *IEEE Trans. Inf. Theory*, vol. 49, no. 12, pp. 3318–3320, Dec. 2003.

[20] K. Gröchenig and M. Leinert, "Wiener's Lemma for twisted convolution and Gabor frames," *J. Amer. Math. Soc.*, vol. 17, no. 1, pp. 1–18, 2003.

[21] A. K. Jain and F. Farrokhnia, "Unsupervised texture segmentation using Gabor Filters," *Pattern Recogn.*, vol. 24, no. 12, pp. 1167–1186, 1991.

[22] A. J. E. M. Janssen, "The Zak transform: A signal transform for sampled time-continuous signals," *Philips J. Res.*, vol. 43, no. 1, pp. 23–69, 1988.

[23] ——, "Duality and biorthogonality for Weyl-Heisenberg frames," *J. Four. Anal. Appl.*, vol. 1, no. 4, pp. 403–436, 1995.

[24] S. Li, "On general frame decompositions," *Numer. Funct. Anal. Optimiz.*, vol. 16, no. 9, pp. 1181–1191, 1995.

[25] S. Li and H. Ogawa, "Pseudoframes for subspaces with applications," *J. Four. Anal. Appl.*, vol. 10, no. 4, pp. 409–431, 2004.

[26] F. Luef, The Density Theorem of Gabor Analysis, submitted for publication.

[27] Y. I. Lyubarskii, "Frames in the Bargmann space of entire functions," *Adv. Sov. Math.*, vol. 11, pp. 167–180, 1992.

[28] B. S. Manjunath and W. Y. Ma, "Texture features for browsing and retrieval of image data," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 18, no. 8, pp. 837–842, Aug. 1996.

[29] S. Marcelja, "Mathematical description of the responses of simple cortical cells," *J. Opt. Soc. Amer.*, vol. 70, no. 11, pp. 1297–1300, 1980.

[30] P. Prinz, "Theory and Algorithms for Discrete 1-Dimensional Gabor Frames," Master's thesis, Univ. Vienna, Wien, Austria, 1996.

[31] S. Qiu, "Block-circulant Gabor-matrix structure and discrete Gabor transforms," *Opt. Eng.*, vol. 34, no. 10, pp. 2872–2878, 1995.

[32] ——, "Discrete Gabor transforms: the Gabor-Gram matrix approach," *J. Four. Anal. Appl.*, vol. 4, no. 1, pp. 1–17, 1998.

[33] S. Qiu and H. G. Feichtinger, "Discrete Gabor structures and optimal representation," *IEEE Trans. Signal Process.*, vol. 43, no. 10, pp. 2258–2268, Oct. 1995.

[34] A. Ron and Z. Shen, "Weyl–Heisenberg frames and Riesz basis on $L^2(\mathbb{R}^d)$," *Duke Math. J.*, vol. 89, no. 2, pp. 273–282, 1997.

[35] K. Seip and R. Wallsten, "Sampling and interpolation in the Bargmann-Fock space II," *J. Reine Angew. Math.*, vol. 429, pp. 107–113, 1992.

[36] T. Strohmer, "Numerical algorithms for discrete Gabor expansions," in *Gabor Analysis and Algorithms: Theory and Applications*, H. G. Feichtinger and T. Strohmer, Eds. Boston, MA: Birkhäuser, 1998, pp. 267–294.

[37] R. Vescovo, "Inversion of block circulant matrices and the circular array approach," *IEEE Trans. Antennas Propag.*, vol. 45, no. 10, pp. 1565–1567, Oct. 1997.

[38] D. F. Walnut, "Continuity properties of the Gabor frame operator," *J. Math. Anal. Appl.*, vol. 165, no. 2, pp. 479–504, 1992.

[39] J. Wexler and S. Raz, "Discrete Gabor expansions," *Signal Process.*, vol. 21, no. 3, pp. 207–220, 1990.

[40] P. J. Wolfe, M. Dörfler, and S. J. Godsill, "Multi-Gabor dictionaries for audio time-frequency analysis," in *Proc. IEEE Workshop Signal Process. Audio Acoust.*, Mohonk, NY, Oct. 2001, pp. 43–46.

[41] M. Zibulski and Y. Y. Zeevi, "Oversampling in the Gabor scheme," *IEEE Trans. Signal Process.*, vol. 41, no. 8, pp. 2679–2687, Aug. 1993.

[42] ——, "Analysis of multiwindow Gabor-type schemes by frame methods," *Appl. Comp. Harm. Anal.*, vol. 4, no. 2, pp. 188–221, 1997.

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