Wavelet Based Image Restoration Using Cross-Band Operators

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Layout

- **Introduction** - Wavelet-Based Image Restoration
- LTI Systems Representation in the WPT Domain
- Multiplicative Operators in the WPT Domain
- Image Restoration with Cross-band Filters
- Conclusion
Image Restoration

True Image

Observed Image

Estimated Image

\[ x(m, n) \rightarrow a(m, n) \rightarrow y(m, n) \rightarrow \hat{x}(m, n) \]

\[ e \sim N\left(0, \sigma_e^2\right) \]

\[ y = Ax + e \]
Wavelet Based Image Restoration Using Cross-Band Operators

**Space Domain Image Restoration**

**Quadratic Regularization**

- Convex optimization, closed form solution
- Linear space-invariant restoration
  - Singular regions like edges, contain most of the perceptual information
  - In low SNR, strong regularization smears these local features

\[
J(x) = \|y - Ax\|^2 + \lambda \|Qx\|^2
\]

\[
\hat{x} = (A^T A + \lambda Q^T Q)^{-1} A^T y
\]

**Nonquadratic Regularization**

- Total variation, Gibbs distribution, etc.
- Compound GMRF

\[
h(x) = \sum_{m=1}^{M} \omega_m \sum_{i_1, i_2} \phi\left(\left[Q^{(m)} x\right]_{i_1, i_2}\right)
\]

\[
\phi(t) - edge\ preserving\ penalty\ function
\]

Nonlinear solution, sometimes even nonconvex target function.
Wavelet Domain Image Restoration

Related Work

- **Expectation-Maximization (EM) Algorithm**

- **Gaussian Scale Mixtures (GSM)**

- **ForWaRD**
Statistical Model of the Wavelet Coefficients

Individual Wavelet Coefficient
- Heavy-tailed distributions describe the **sparsity** of wavelet representation.
- Few large coefficients dominate the signal’s energy.
- Most of the coefficients are nearly zero.

Dependence Between Wavelet Coefficients
- Wavelet coefficients are roughly decorrelated.
- Although spatial and cross-scale dependence exists, statistical **independence** is often assumed.

\[
p(\tilde{x}) = \prod_{k=1}^{\ell} \prod_{p_1, p_2} p(x_{p_1, p_2, k})
\]

- Dependence between scales can be described with hidden Markov trees [Crouse et al. 1998].
Wavelet-Based Image Restoration

Bayesian Restoration

- MAP estimation

\[ \hat{x}_{\text{MAP}} = \arg \min_{\tilde{x}} \left\{ \| \tilde{y} - \mathbf{W}^T \mathbf{A} \mathbf{W} \tilde{x} \|_2^2 - 2\sigma_e^2 \log p_\theta (\tilde{x}) \right\} \]

\[
\begin{align*}
\text{2D-DFT domain} & \quad \text{Wavelet domain} \\
\mathbf{d}_D &= \mathbf{D}^H \mathbf{A} \mathbf{D} \mathbf{x}_D \\
\tilde{\mathbf{d}} &= \mathbf{W}^T \mathbf{A} \mathbf{W} \tilde{x}
\end{align*}
\]

- Unlike \( \mathbf{A} \) alone, \( \tilde{\mathbf{A}} = \mathbf{W}^T \mathbf{A} \mathbf{W} \) is not block-circulant, and cannot be diagonalized by the 2D-DFT matrix.
- The convolution matrix doesn’t have a sparse structure.
- For images of size \( P \times P \), the convolution matrix size is \( P^2 \times P^2 \).
EM Algorithm for Wavelet-Based IR

**MPLE/MAP - Maximum Penalized Likelihood Estimator**

\[
\hat{x} = \arg \max_{\hat{x}} \left\{ \log p(y | \hat{x}) + p_\theta (\hat{x}) \right\}
\]

\[
= \arg \max_{\hat{x}} \left\{ -\frac{\|y - AW\hat{x}\|^2}{2\sigma_e^2} + p_\theta (\hat{x}) \right\}
\]

- Unlike \( A \) alone, \( AW \) is not block-circulant.

- An equivalent decoupled scheme, defining:

\[
e = \alpha A e_1 + e_2 \Rightarrow \begin{cases} 
e_1 \sim N(0, I) \\
e_2 \sim N\left(0, \sigma_e^2 I - \alpha^2 AA^T\right) \quad \alpha^2 \leq \frac{\sigma_e^2}{\lambda_{\text{max}}(AA^T)}
\end{cases}
\]

- State model:

\[
y = Ax + e \Rightarrow \begin{cases} 
z = W\tilde{x} + \alpha e_1 \\
y = Az + e_2
\end{cases}
\]
EM Algorithm for Wavelet-Based IR

- **Direct maximization**:

\[
\hat{x} = \arg \max_x \left\{ -\frac{\|y - AW\hat{x}\|^2}{2\sigma_e^2} + p_\theta(\hat{x}) \right\}
\]

- **Expectation-Maximization**:

\[
\begin{align*}
Q(\tilde{x}, \hat{x}^{(t)}) &= E_z \left[ \log p(y, z | \tilde{x}) | y, \hat{x}^{(t)} \right] \\
\hat{x}^{(t+1)} &= \arg \max_\theta \left\{ Q(\tilde{x}, \hat{x}^{(t)}) + p_\theta(\tilde{x}) \right\}
\end{align*}
\]

\[
\begin{align*}
\text{E-step:} & \quad \hat{x}^{(t)} = W\tilde{x}^{(t)} \Rightarrow z^{(t)} = \hat{x}^{(t)} + \frac{\alpha^2}{\sigma_e^2} A^T (y - A\hat{x}^{(t)}) \\
\text{M-step:} & \quad \tilde{z}^{(t)} = W^T z^{(t)} \Rightarrow \hat{x}^{(t+1)} = \arg \max_{\tilde{x}} \left\{ \|\tilde{x} - \tilde{z}^{(t)}\|^2 - 2\alpha^2 p_\theta(\tilde{x}) \right\}
\end{align*}
\]
Wavelet Based Image Restoration Using Cross-Band Operators

Gaussian Scale Mixtures for Wavelet-Based IR

- **Bayesian Restoration**

\[
\tilde{x} = \arg \max_x \left\{ \frac{\|y - AW\tilde{x}\|^2}{2\sigma_e^2} + p(\tilde{x}) \right\}
\]

\[
p(\tilde{x}) = \prod_{k=1}^{t} \prod_{P_1,P_2} p\left(x_{P_1,P_2,k}\right)
\]

- **Gaussian Scale Mixtures (GSM) prior**

\[
p(x) = \int_0^\infty p_x(z) p_z(z) dz
\]

\[
x|z \sim N(0, z)
\]

- **GEM Algorithm**

E-step: \[d^{(t)}(x^{(t)}_{P_1,P_2,k}) = E\left(z^{-1}\left|x^{(t)}_{P_1,P_2,k}\right|\right) \rightarrow D^{(t)}\]

M-step: \[\tilde{x}^{(t+1)} = \left(\sigma_e^2 D^{(t)} + W^T A^T A W\right)^{-1} W^T A^T y\]

In order to implement the M-step, second-order stationary iterative method is applied.
Wavelet Based Image Restoration Using Cross-Band Operators

**Fourier-Wavelet Regularized Deconvolution (ForWaRD)**

**Deconvolution by transform-domain shrinkage**

\[ \hat{x}_\lambda = \sum_{k=0}^{n-1} \left( \langle x, b_k \rangle + \langle A^{-1}e, b_k \rangle \right) \lambda_k b_k \]

- Lower bound for MSE:

\[ \text{MSE} \leq \hat{x}_\lambda = \sum_{k=0}^{n-1} \min \left( \left| \langle x, b_k \rangle \right|^2 + \sigma_k^2 \right) \]

- Small MSE only when most of the signal energy and colored noise energy are captured by just a few transform-domain coefficients.
  - Fourier basis economically represents the colored stationary noise.
  - Wavelet basis economically represents classes of signals that contain singularities.

“..deconvolution techniques employing shrinkage in a single transform domain cannot yield adequate estimates in many deconvolution problems of interest.”
Wavelet Based Image Restoration Using Cross-Band Operators

Fourier-Wavelet Regularized Deconvolution (ForWaRD)

ForWaRD

Employing scalar shrinkage both in the Fourier domain and in the wavelet domain:
- First, ForWaRD employs a small amount of Fourier shrinkage to significantly attenuate the amplified noise components with a minimal loss of signal components.
- Subsequent wavelet shrinkage in Step 2 effectively estimates the retained signal from the low-variance leaked noise.

ForWaRD’s hybrid approach yields robust solutions to a wide variety of deconvolution problems

Signals with more economical wavelet representations should require less Fourier shrinkage.
Wavelet-Based Image Restoration

Bayesian Restoration

- MAP estimation

\[ \hat{x}_{\text{MAP}} = \arg \min_{\tilde{x}} \left\{ \left\| \tilde{y} - \underbrace{W^T A W \tilde{x}}_\Lambda \right\|^2 - 2\sigma_e^2 \log p_\theta (\tilde{x}) \right\} \]

- Unlike \( A \) alone, \( \tilde{A} = W^T A W \) is not block-circulant, and cannot be diagonalized by the 2D-DFT matrix

- The convolution matrix doesn’t have a sparse structure

- For images of size \( p \times p \), the convolution matrix size is \( p^2 \times p^2 \)

**2D-DFT domain**

\[ d_D = D^H A D x_D \]

\[ \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \]

**Wavelet domain**

\[ \tilde{d} = \underbrace{W^T A W \tilde{x}}_\Lambda \]

\[ \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \]
Wavelet-Based Image Restoration

The Goal

To find a sparse representation of LTI systems in the wavelet packet domain:

\[ d(n) = \sum_{i=L_{se}}^{L_{se}-1} a(i) x(n-i) \quad \longleftrightarrow \quad \tilde{d} = \tilde{A} \tilde{x} \]

Previous work

- Uniform decomposition [Banham et al. 1993]
- DTWT (Pyramid) decomposition [Zervakis et al. 1995]

Our work

- Applicable for any admissible wavelet-packet decomposition
- Relies on cross-band filtering notation
- Can be easily extended to two-dimensions using separable bases
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**Wavelet Packet Transform (WPT)**

**Equivalent Nonuniform Filter-Bank**

\[
x(n) = \sum_{k=1}^{\ell} \sum_{p \in \mathbb{Z}} x_{p,k} f^k \left( n - 2^k p \right)
\]

\[
\sum_{k=1}^{\ell} 2^{-j^k} = 1
\]

\[
f^k(n) = h^k(-n)
\]

\[
|F^k(\theta)| = |H^k(\theta)|
\]
Representation of LTI Systems in the WPT Domain

**Time Domain**

\[ d(n) = \sum_{i=-L_{\text{nc}}}^{L_{\text{nc}}-1} a(i) x(n - i) \]

**WPT Domain**

\[ d_{p,k} = \varphi(x_{p,1}, \ldots, x_{p,\ell}) \quad p' \in \square, 1 \leq k \leq \ell \]
Cross-band filters are used on system identification in the STFT domain for acoustic echo cancellation [Avargel and Cohen 2007]

In the wavelet domain, the cross-band filters are time varying
Representation of LTI Systems in the WPT Domain

Cross-band filtering notation

\[ d_{p,k,k'} = \sum_{p'} x_{p',k'} \cdot a_{n,k,k'} \bigg|_{n=2^{j_k}p-2^{j_{k'}}p'} \]

- Divide into 3 cases:
  - \( j^k > j^{k'} \)
  - \( j^k = j^{k'} \)
  - \( j^k < j^{k'} \)

Any cross-term between two distinct subbands is a multirate filter, which depends on the subbands scales ratio.

Band-to-band terms between any subband to itself are always time-invariant filters.
Representation of LTI Systems in the WPT Domain

Cross-band filtering notation

\[ \bar{a}_{p,k,k'} = a(n) * h^k(n) * f^{k'}(n) \bigg|_{p=2^{\min(j,k')}} \]

Band-to-band filter
Representation of LTI Systems in the WPT Domain

Cross-band filters energy

\[ a_{n,k,k'} = a(n) * h^k(n) * f^{k'}(n) \]

\[ e_{k,k'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(\theta)H^k(\theta)F^{k'}(\theta)|^2 d\theta \]

\[ N = 4 \]

\[ N = 32 \]
Wavelet Based Image Restoration Using Cross-Band Operators

Representation of LTI Systems in the WPT Domain

Matrix-vector notation

\[ \tilde{d} = \tilde{A} \tilde{x} \]

For \( j^k > j^{k'} \)

\[ \tilde{d}_{k,k'} = \begin{pmatrix} \tilde{A}_{k,k'}^{(0)} & \tilde{A}_{k,k'}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{x}_{k}^{(0)} \\ \tilde{x}_{k}^{(1)} \end{pmatrix} \]

For \( j^k = j^{k'} \)

\[ \tilde{d}_{k,k'} = \tilde{A}_{k,k'} \tilde{x}_{k} \]

For \( j^k < j^{k'} \)

\[ \begin{pmatrix} \tilde{d}_{k,k'}^{(0)} \\ \tilde{d}_{k,k'}^{(1)} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{k,k'}^{(0)} \\ \tilde{A}_{k,k'}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{x}_{k}^{(0)} \\ \tilde{x}_{k}^{(1)} \end{pmatrix} \]
Representation of LTI Systems in the WPT Domain

Matrix-vector notation

\[
\begin{pmatrix}
\tilde{d}_1 \\
\vdots \\
\tilde{d}_\ell
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{A}_{1,1} & \cdots & \tilde{A}_{1,\ell} \\
\vdots & \ddots & \vdots \\
\tilde{A}_{\ell,1} & \cdots & \tilde{A}_{\ell,\ell}
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_1 \\
\vdots \\
\tilde{x}_\ell
\end{pmatrix}
\]

\[\tilde{A}\]  Block-semi-circulant - \(\text{BSC}\left(2^j \times 2^j, 2^{-j} P\right)\)

Matrix of size \(P \times P\) having arbitrary block structure composed of \(2^j \times 2^j\) circulant matrices of size \(2^{-j} P \times 2^{-j} P\) each

Multichannel Diagonalization

\[
\tilde{D}\tilde{d} = \tilde{D}^H \tilde{A} \tilde{D} \tilde{D}^H \tilde{x}
\]

\[\tilde{D}\]  Diagonal-block - \(\text{DB}\left(2^j \times 2^j, 2^{-j} P\right)\)

\(2^j P\) non-zero elements out of \(P^2\)

\[\left(2^j \otimes P\right)\]

\[
\begin{align*}
J &= 0 \\
J &= 1 \\
J &= 2
\end{align*}
\]
Extension to 2D-LSI systems

\[ d(n_1, n_2) = \sum_{i_1 = -L_{ne}}^{L_{ne}-1} \sum_{i_2 = -L_{ne}}^{L_{ne}-1} a(i_1, i_2) x(n_1 - i_1, n_2 - i_2) \]

Cross-band filters notation

\[ d_{p,k} = \sum_{k=1}^{\ell} \sum_{p_1, p_2} x_{p_1, p_2, k} \cdot a_{n_1, n_2, k, k'} \bigg|_{n=2^j \cdot p, -2^j \cdot p'} \]

\[ a_{n_1, n_2, k, k'} = a(n_1, n_2) * h^k(n_1, n_2) * f^{k'}(n_1, n_2) \]

Matrix-vector notation

\[ \tilde{d}_D = \tilde{A} \tilde{x}_D \]

diagonal-block - DB\(4^j \times 4^j, 4^j \times P^2\)

\(4^j P^2\) non-zero elements out of \(P^4\) \(2^j \times P\)

\[ P = 256, \ J=3 \rightarrow P^4 \approx 4 \cdot 10^9 \]
\[ \rightarrow 4^j P^2 \approx 4 \cdot 10^6 \]
Multiplicative Operators in the WPT Domain

Cross-band filters notation

\[
d_{p,k} = \sum_{k' = 1}^{\ell} \sum_{p'} x_{p',k'} \cdot a_{p,k,k'} \cdot x_{p-2^p_p-2^k_p}
\]

- \( \ell \times \ell \) cross-band filters
- Multirate operations between subbands
- Sparsity depends on the number of decomposition levels

Multiplicative transfer function (MTF) operator

\[
d = \tilde{C}(x) \rightarrow d_{p,k} = c_k x_{p,k}
\]

- Cross-terms are neglected
- Band-to-band filters are replaced with scalar multiplication
Wavelet Based Image Restoration Using Cross-Band Operators

Multiplicative Operators in the WPT Domain

Multiplicative transfer function (MTF) operator

\[ d = \tilde{C}(x) \rightarrow d_{p,k} = c_k x_{p,k} \]

- Incapable of attaining an accurate representation of LTI systems
- Time-varying operator, due to the downsampling on each subband
Multiplicative Operators in the WPT Domain

Averaged MTF operator

\[ C_R \triangleq \frac{1}{R} \sum_{r=0}^{R-1} \tau_{-r} \circ \tilde{C} \circ \tau_r \]

The averaged MTF operator over \( 2^j \) translations is a zero-phase LTI system with frequency-response:

\[ C_{2^j}(\theta) \triangleq \sum_{k=1}^{t} c_k \frac{|H^k(\theta)|^2}{2^j} \]

Shift operator
Multiplicative Operators in the WPT Domain

LTI systems approximation using averaged MTF operator

\[ d(n) = \sum_{i=-L_{nc}}^{L_{nc}-1} a(i)x(n-i) \]

The parameters set \{c_i\}_{i=1}^t that minimizing the least-squares frequency response error:

\[ \varepsilon = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ A(\theta) - C_{2^j}(\theta) \right]^2 d\theta} \]

is given by the solution of a linear equation system:

\[ \Phi \mathbf{c} = \mathbf{b} \]

where:

\[ \Phi_{k,m} = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| H^m(\theta) \right|^2 \left| H^k(\theta) \right|^2}{2^{j^2}} d\theta} \]

\[ \mathbf{b}_k = \overline{a}_{p,k,k} \bigg|_{p=0} \]

\[ a_{p,k} = \frac{1}{2^{j^2}} \int_{-\pi}^{\pi} \frac{\left| H^m(\theta) \right|^2 \left| H^k(\theta) \right|^2}{2^{j^2}} d\theta \]
Multiplicative Operators in the WPT Domain

LTI systems approximation using averaged MTF operator

\[ N = 2 \]

\[ N = 32 \]
The averaged CMTF operator over $2^j$ translations is an LTI system with frequency-response:

$$C_{2^j}(\theta) = \sum_{k,k'=1}^c c_{k,k'} \frac{H^k(\theta)F^{k'}(\theta)}{2^{\max\{j^k,j^{k'}\}}}$$
Multiplicative Operators in the WPT Domain

LTI systems approximation using averaged CMTF operator

The parameters set \( \{c_{k,k'}\}_{k,k'=1}^{\ell} \) that minimizing the least-squares frequency response error:

\[
\varepsilon = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ A(\theta) - C_{2'}(\theta) \right]^2 d\theta
\]

is given by the solution of a linear equation system:

\[
\begin{pmatrix}
\Phi_{1,1} & \cdots & \Phi_{1,\ell} \\
\vdots & \ddots & \vdots \\
\Phi_{\ell,1} & \cdots & \Phi_{\ell,\ell}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_\ell
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
\vdots \\
b_\ell
\end{pmatrix}
\]

where:

\[
\left[ \Phi_{k,m} \right]_{k'=m'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{H^m(\theta) F_{j'}^m(\theta) H^k(\theta) F_{j'}^k(\theta)}{2^\max\{j',j\}} d\theta
\]

\[
\left[ b_k \right]_{k'} = \left. \alpha_{p,k,k'} \right|_{p=0}
\]
Multiplicative Operators in the WPT Domain

LTI systems approximation using averaged CMTF operator

$N = 4$

$N = 12$
Introduction - Wavelet-Based Image Restoration

LTI Systems Representation in the WPT Domain

Multiplicative Operators in the WPT Domain

Image Restoration with Cross-band Filters

Conclusion
Wavelet Based Image Restoration Using Cross-Band Operators

**Image Restoration with Cross-band Filters**

- Observation model - Additive white Gaussian noise

\[ p(\tilde{y}|\tilde{x}) \propto \exp\left\{-\frac{\|\tilde{y} - \tilde{A}\tilde{x}\|^2}{2\sigma^2}\right\} \]

- Prior model - Laplace iid on each subband

\[
p_\rho(\tilde{x}) = \prod_{k=1}^{l} \prod_{p_1, p_2} p_\rho(x_{p_1, p_2, k})
\]

\[
p_\rho(x_{p_1, p_2, k}) = \frac{1}{2\theta_x} \exp\left\{-\frac{|x_{p_1, p_2, k}|}{\theta_x}\right\}
\]

- Optimization problem

\[
\hat{x}_{MAP} = \arg \min_x \left\{ \|\tilde{y} - \tilde{A}\hat{x}\|^2 + 2\sigma^2 \sum_{k=1}^{l} \sum_{p_1, p_2} \frac{|x_{p_1, p_2, k}|}{\theta_x} \right\}
\]
Image Restoration with Cross-band Filters

\[ J_\theta (\tilde{x}) = \left\| \tilde{y} - \tilde{A} \tilde{x} \right\|^2 + 2s^2 \sum_{k=1}^L \frac{1}{\theta_k} \sum_{p_1, p_2} \left| x_{p_1, p_2, k} \right| \]

- Conjugate gradient iterations

\[ \tilde{g}^{(t)} = \tilde{A}^T \left( \tilde{A} \tilde{x}^{(t)} - \tilde{y} \right) + \tilde{F}_\theta \text{sign} \left( \tilde{x}^{(t)} \right) \]
\[ \tilde{d}^{(t)} = -\tilde{g}^{(t)} + \beta^{(t)} \tilde{d}^{(t-1)} \]
\[ \tilde{x}^{(t+1)} = \tilde{x}^{(t)} + \eta^{(t)} \tilde{d}^{(t)} \]

where:

\[ \tilde{F}_\theta = \text{diag} \left\{ \frac{s^2}{\theta_k} 2^{c_k} I_{2^{c_k}} \right\} \]
\[ \beta^{(t)} = \frac{\tilde{g}^{(t)} \left( \tilde{g}^{(t)} - \tilde{g}^{(t-1)} \right)}{\left\| \tilde{g}^{(t-1)} \right\|^2} \]
\[ \eta^{(t)} = \arg \min_{\eta > 0} J_\theta \left( \tilde{x}^{(t)} + \eta \tilde{d}^{(t)} \right) \]

- Efficient implementation in the multichannel-DFT domain

\[ \tilde{g}_D^{(t)} = \tilde{A}_D^H \tilde{A}_D \tilde{x}^{(t)} - \tilde{A}_D^H \tilde{y}_D + \tilde{F}_\theta \text{sign} \left( \tilde{D} \tilde{x}^{(t)} \right) \]
Image Restoration with Cross-band Filters

\[
\hat{g}_D^{(t)} = \tilde{A}_x^t \tilde{A}_D \tilde{x}_D^{(t)} - \tilde{A}_y^t \tilde{y}_D + \tilde{F}_\varphi \text{sign} \left\{ \tilde{D} \tilde{x}_D^{(t)} \right\}
\]

\[
\tilde{F}_\varphi = \text{diag} \left\{ \frac{\sigma^2}{\theta_k} I_{2^r,2^k} \right\}^T
\]

- Hyperparamaters set selection:

\[
\hat{\theta}_k = \sum_{P_1,P_2} \left| x_{P_1,P_2,k} \right|
\]

- The true image, and therefore \( x_{P_1,P_2,k} \) is not available

- Using the MTF approximation:

\[
y_{P_1,P_2,k} \approx c_k x_{P_1,P_2,k}
\]

\[
\hat{\theta}_k = \frac{1}{|c_k| + \lambda} \sum_{P_1,P_2} \left| y_{P_1,P_2,k} \right|
\]

- MTF approximation needs to be justified
Experimental Results - Setup 1

Horizontal Motion Blur (1×7)
BSNR - 30 dB
$N = 12$

True Image

Observed Image
Experimental Results - Setup 1

Horizontal Motion Blur (1×7)
BSNR - 30 dB
N = 12

Iterative Wiener Filter
ISNR - 6.06 dB

WaveCBF
ISNR - 7.55 dB
Experimental Results - Setup 2

Uniform Blur (9×9)
BSNR - 30 dB
N = 8

True Image

Observed Image
Experimental Results - Setup 2

Uniform Blur \((9 \times 9)\)
BSNR - 30 dB
\(N = 8\)

Iterative Wiener Filter
ISNR - 4.18 dB

WaveCBF
ISNR - 5.29 dB
Wavelet Based Image Restoration Using Cross-Band Operators

Layout

- **Introduction** - Wavelet-Based Image Restoration
- **LTI Systems Representation in the WPT Domain**
- **Multiplicative Operators in the WPT Domain**
- **Image Restoration with Cross-band Filters**
- **Conclusion**
Summary

- **LTI systems representation using cross-band filters**
  - Perfect representation
  - Each cross-term is a multirate filter, depends on the scales ration between the frequency subbands
  - Efficient implementation in the multichannel DFT domain
  - Applicable for any admissible wavelet packet decomposition

- **Multiplicative operators in the WPT domain**
  - Cross-band filtering is replaced with scalar multiplication
  - Very low computational complexity
  - LTI systems approximation using averaged MTF/CMTF operator

- **Image restoration with cross-band filters**
  - Cross-band filters notation enables fast implementation
  - MTF approximation is used for hyperparameters set selection
Future Research

- Integration of cross-band filters usage with other state-of-the-art image priors
  - Better performance at the cost of computational efficiency

- Image restoration in wavelet packet bases
  - A considerable degree of freedom
  - Finding a “best basis” for restoration is nontrivial

- Extension to time-varying systems