EFFICIENT SAMPLING SCHEMES FOR PULSE STREAMS WITH APPLICATION TO ULTRASOUND IMAGING

Kfir Gedalyahu, Ronen Tur and Yonina C. Eldar

Department of Electrical Engineering
Technion—Israel Institute of Technology, Haifa 32000, Israel
Emails: {kfirge@techunix,ronentur@techunix,yonina@ee}.technion.ac.il.

ABSTRACT

We consider the problem of sampling signals which are comprised of pulse streams, a model which belongs to the recently introduced framework of signals with finite rate of innovation (FRI). Although sampling of pulse streams was treated in various works, either the rate of innovation (ROI) was not achieved, or the pulse shape was limited to diracs. In this work we present two sampling architectures for pulse streams with arbitrary shape. Our first scheme is a single channel method, operates at rate of only 2.5 times the ROI. The second technique, which is based on multichannel sampling, achieves the ROI exactly. Both configurations are flexible, and exhibit better noise robustness than previous approaches.

***Keywords*—** Finite rate of innovation, sub-Nyquist sampling, time delay estimation.

1. INTRODUCTION

Signals comprised of a stream of short pulses appear in many applications including bio-imaging, radar, and ultrawideband communication. A recently proposed framework has suggested to address such signals as parametric signals, with a finite number degrees of freedom per unit time, referred to as the rate of innovation (ROI). The ROI provides a lower bound on the sampling rate, required in order to perfectly recover such pulses from its samples. This rate can be significantly lower than the traditional Nyquist rate.

Current approaches in the literature do not provide a complete solution to the problem, allowing sampling of pulse streams at the ROI. The work in [10] proposed an efficient scheme, based on ideal low-pass filter, operating at the rate of innovation. However, this scheme supports only periodic pulse streams. An alternative approach [1] based on polynomial or exponentials reproducing kernels, allows sampling of finite and infinite streams of pulses. This method is limited by the fact that its sampling rate is significantly higher than the ROI for infinite streams. Other multichannel based schemes [3, 6, 7], achieve the ROI for the infinite setting, but support only diracs as pulse shapes. Moreover, as we demonstrate in this work, several of the existing techniques [1, 3, 6] suffer from poor noise robustness, especially for high model orders.

In this paper we propose two schemes for low rate sampling of pulse streams. The first method is based on a new class of compactly supported filters, referred to as Sum of Sincs (SoS) kernels [8]. This single channel configuration, operates at a rate close to the ROI and supports pulses with arbitrary shape. Our second system is based on multichannel sampling and can operate exactly at the ROI, while also supporting general pulse shapes [2]. This approach is based on modulating of the input signal with a set of properly chosen waveforms, followed by integration. These ideas are motivated by the recent Xampling framework for low rate sampling of multiband signals [4, 5]. We demonstrate the superior noise robustness of the proposed schemes, over the previous approaches using simulations. In addition, we apply our results to ultrasound imaging data, and show that our techniques result in substantial rate reduction with respect to traditional ultrasound sampling schemes.

The remainder of this paper is organized as follows. In Section 2 we present a single channel method for pulse streams sampling. In Section 3 we derive a multichannel scheme for the problem. Finally, in Section 4, we demonstrate our method on real ultrasound data, and explore the performance of our approach in the presence of noise.

2. SINGLE-CHANNEL SAMPLING SCHEME

We denote matrices and vectors by bold font, with lowercase letters corresponding to vectors and uppercase letters to matrices. The nth element of a vector $\mathbf{a}$ is written as $\mathbf{a}_n$, and $\mathbf{A}_{ij}$ denotes the ijth element of a matrix $\mathbf{A}$. Superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ represent complex conjugation, transposition and conjugate transposition, respectively. The Moore-Penrose pseudo-inverse of a matrix $\mathbf{A}$ is written as $\mathbf{A}^\dagger$. The continuous-time Fourier transform (CTFT) of a continuous-time signal $x(t) \in L_2$ is defined by $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{d}t$.

2.1. Efficient Sampling of FRI Signals

Consider a T-periodic stream of pulses, defined as

$$x(t) = \sum_{m \in \mathbb{Z}} \sum_{l=1}^{L} a_l h(t - t_l - mT), \tag{1}$$

where $h(t)$ and T are the known pulse shape and period, respectively, and $\{t_l, a_l\}_{l=1}^{L}$, $t_l \in [0, T)$, $a_l \in \mathbb{C}$, $l = 1 \ldots L$ are the unknown delays and amplitudes. Our goal is to sample and reconstruct $x(t)$ efficiently.

Expanding $x(t)$ to its Fourier series we have

$$x(t) = \sum_{k \in \mathbb{Z}} X[k] e^{j2\pi kt/T}, \tag{2}$$

where we denoted

$$X[k] = \frac{1}{T} H\left(\frac{2\pi k}{T}\right) \sum_{l=1}^{L} a_l e^{-j2\pi kt_l/T}, \tag{3}$$

and $H(\cdot)$ denotes the CTFT of $h(t)$. The right hand side of (3) is a sum of exponentials with frequencies $\{t_l\}$, which can be found using standard spectral analysis methods as long as we have a set of Fourier coefficients $\{X[k]\}$, with cardinality greater than $2L$ [10].

2.2. Sampling Schemes and SoS Filters

In order to obtain a set $\{X[k]\}$, we propose uniformly sampling $x(t)$ with a any sampling kernel satisfying:

$$S(\omega) = \begin{cases} 0 & \omega = 2\pi k/T, k \notin \mathcal{K} \\ \text{nonzero} & \omega = 2\pi k/T, k \in \mathcal{K} \\ \text{arbitrary} & \text{otherwise}, \end{cases} \tag{4}$$

For which it can be shown that the resulting samples are

$$c[n] = \sum_{k \in \mathcal{K}} X[k] e^{j2\pi knT_s/T} S^*(2\pi k/T), \tag{5}$$

where $\mathcal{K}$ is an index set of the Fourier coefficients for which $H\left(\frac{2\pi k}{T}\right) \neq 0$, $\forall k \in \mathcal{K}$, and T_s is the sampling period. The set of equations in (5) can be solved for $\{X[k]\}$ as long as N, the number of samples, is larger than $|\mathcal{K}|$.

Following the general condition presented in (4), we propose a compactly supported filter which consists of a sum of sincs in the frequency domain:

$$G(\omega) = \frac{T}{\sqrt{2\pi}} \sum_{k \in \mathcal{K}} b_k \operatorname{sinc}\left(\frac{\omega}{2\pi/T} - k\right), \tag{6}$$

where $b_k \neq 0$, $k \in \mathcal{K}$, are parameters of the filter. Switching to the time domain the compact support is made evident

$$g(t) = \operatorname{rect}\left(\frac{t}{T}\right) \sum_{k \in \mathcal{K}} b_k e^{j2\pi kt/T}. \tag{7}$$

The compact support of the SoS filter class enables extension of our noise robust solution to the infinite setting, discussed in the next section.

2.3. Infinite Pulse Streams

We now consider the case of an infinite stream of pulses

$$z(t) = \sum_{l \in \mathbb{Z}} a_l h(t - t_l), \quad t_l, a_l \in \mathbb{R}. \tag{8}$$

We assume that the infinite signal has a bursty character, i.e., the signal has two distinct phases: a) bursts of maximal duration T containing at most L pulses, and b) quiet phases between bursts. For the sake of clarity we begin with the case $h(t) = \delta(t)$.

Consider uniformly sampling $z(t)$ with a filter comprising three periods of $g(t)$ as follows

$$g_{3p}(t) \triangleq g(t - T) + g(t) + g(t + T). \tag{9}$$

If the minimal spacing between any two consecutive bursts is $3T/2$, then we are guaranteed that each sample taken during the burst is influenced by one burst only, as depicted in Fig. 1. Here we exploited the compact support of the SoS filter.

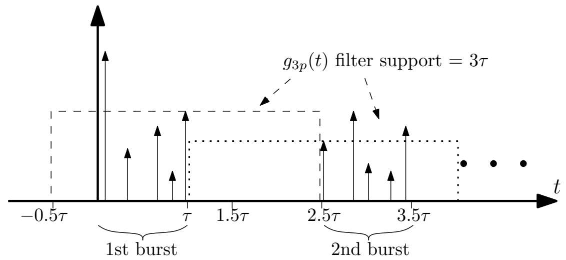


Fig. 1. Bursty signal $z(t)$. Spacing of $3T/2$ between bursts ensures that the influence of the current burst ends before taking the samples of the next burst. This is due to the finite support, $3T$ of the sampling kernel $g_{3p}^*(-t)$.

Once we reduced the infinite problem to a sequence of finite pulse streams, it can be shown that the samples obtained by the sampling kernel (9) form a set of equations which allows to obtain the set $\{X[k]\}$ exactly as in the periodic case, for each burst independently. Therefore the unknown delays and amplitudes can be determined throughout the infinite signal $z(t)$. We summarize our results in the following theorem.

Theorem 1 *Consider a signal $z(t)$ which is a stream of bursts consisting of delayed and weighted diracs. The maximal burst duration is T, and the maximal number of pulses within each burst is L. Then, the samples given by*

$$c[n] = \langle g_{3p}(t - nT_s), z(t) \rangle, \quad n \in \mathbb{Z}$$

where $g_{3p}(t)$ is defined by (9)*, are a sufficient characterization of $z(t)$ as long as the spacing between two adjacent bursts is greater than $3T/2$, and the burst locations are known.*

The scheme presented here achieves a rate which is close to the ROI of the input signals. The ROI of pulse streams addressed by Theorem 1 is equal to $2L/2.5T$. Our method operates at a rate of $1/T$, which is only 2.5 times larger than this minimal rate.

The extension to arbitrary $h(t)$ is possible as long as the pulse $h(t)$ has finite support R, which is a rather weak condition, since our primary interest is in very short pulses which have wide, or even infinite, frequency support. In this case we filter $z(t)$ with a filter

$$g_r(t) = \sum_{m=-r}^{r} g(t + mT), \tag{10}$$

where r is defined by

$$r = \left\lceil \frac{R/T + 3}{2} \right\rceil - 1. \tag{11}$$

3. MULTICHANNEL SAMPLING SCHEME

In the previous section we derived a new sampling method for pulse streams based on the SoS filter. We have shown that this technique can reduce the sampling rate down to 2.5 of the ROI. In this section we present a multichannel configuration which can operate exactly at the ROI.

3.1. Signal Model

We consider an infinite stream of pulses defined by

$$x(t) = \sum_{l \in \mathbb{Z}} a_l h(t - t_l), \quad t_l \in \mathbb{R}, \, a_l \in \mathbb{C}. \tag{12}$$

We assume that there are no more than L pulses in any interval $I_m \subset [(m-1)T, mT]$, $m \in \mathbb{Z}$ and that within each interval the delays satisfy the following condition:

$$h(t - t_l) = 0, \quad \forall t \notin I_m \quad l = 1 \dots L, \tag{13}$$

i.e., the pulses in each period are confined to the time-window I_m. Since in each interval of length T the signal is defined by $2L$ parameters, the ROI of a signal of the form (12) equal to $2L/T$.

3.2. Proposed Scheme

Our aim now is to design a sampling and reconstruction method which perfectly reconstructs the signal (12) when operating at the ROI. We note that (13) suggests that each interval I_m of the signal is independent of adjacent periods. We therefore address the infinite stream as a concatenation of finite streams of L pulses. In each period, the signal is processed and reconstructed separately.

In a similar way to the single-channel architecture of Section 2, we present a scheme which obtains the signal's Fourier coefficients $X[k]$. In the system depicted in Fig. 2, in each channel the signal is modulated using a waveform $s_i(t) = \sum_{k \in \mathcal{K}} s_{ik} e^{-j\frac{2\pi}{T}kt}$ followed by integration over the interval I_m. Various signals can be used as modulation waveforms. Examples include cosine and sine functions (tones), filtered rectangular pulses modulated by ± 1 [4] and more.

It can be easily shown that the proposed scheme produce at its outputs a mixture of the signal's Fourier coefficient. By proper selection of s_{ik} a unique recovery of the Fourier coefficients from the samples can be ensured, where $p \geq K$ is a necessary condition.

We summarize our results in the following theorem.

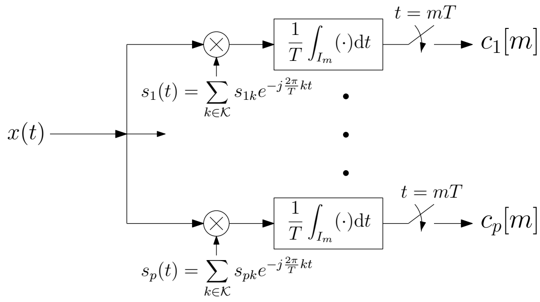
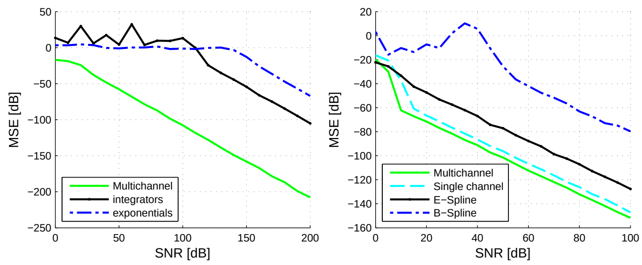


Fig. 2. Multichannel scheme for pulse streams.



(a) (b)

Fig. 3. Stream of $L = 4$ pulses. (a) Recovery using $p = 9$ samples. (b) Recovery using $p = 64$ samples.

Theorem 2 *Consider an infinite stream of pulses given by* (12). *Assume that there are no more than L pulses within any interval* $I_m \triangleq [(m-1)T, mT]$, $m \in \mathbb{Z}$, *and that condition* (13) *holds for all intervals. Consider the multichannel sampling scheme depicted in Fig. 2. Then, the signal* $x(t)$ *can be perfectly reconstructed from the samples* $\{c_i[m]\}_{i=1}^{p}$, $m \in \mathbb{Z}$ *as long as* $p \geq 2L$, *and the matrix* $\mathbf{S}$ *in constructed from the coefficients* s_{ik} *is left invertible.*

Theorem 2 presents the first sampling scheme for general pulse shapes, operating at the ROI.

4. SIMULATIONS

We demonstrate the performance of our approach in the presence of noise in comparison to various FRI methods, for a signal which consists of $L = 4$ Diracs. Figure 3(a) compares our multichannel scheme to the integrators [3] and exponential filters [6] methods, when working at the ROI. The figure shows that our configuration outperforms these two techniques. In Figure 3(b) we compare our schemes to the B-splines and E-splines [9] sampling kernels of [1], when working at a rate of $64/T$. Here again both our methods exhibit better noise robustness than the ones of [1].

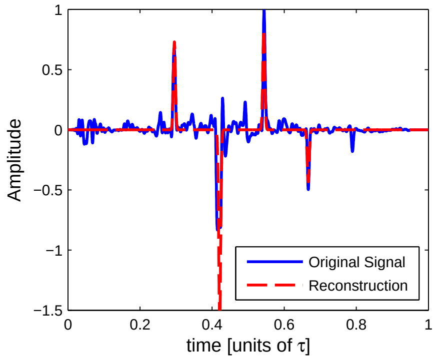


Fig. 4. Applying our single-channel method on real ultrasound imaging data. Results are shown vs. original demodulated signal. Reconstructed signal $N = 17$ samples only as opposed to 4160 samples used in current ultrasound systems.

An interesting application of our scheme is ultrasound imaging, in which the signal received from the tissue under test comprises a stream of short Gaussian pulses. Applying our scheme on data recorded with GE Healthcare's Vivid-i system, we reconstructed the original signal as depicted in Figure 4. The reconstruction is based on $N = 17$ only, whereas current ultrasonic imaging systems use for the same scenario 4000 samples, emphasizing the potential of our scheme in reducing sampling rate in such systems.

5. CONCLUSION

In this work we proposed new sampling schemes for pulse streams, allowing the system designer to trade-off between sampling rate and hardware complexity. Our multichannel architecture operates at the ROI, while our single channel scheme works at a rate 2.5 higher, but requires less sampling channels. As we demonstrate by simulations, our methods exhibits better noise robustness than the ones presented in [1, 3, 6]. We also demonstrate our method on real ultrasound imaging data.

6. REFERENCES

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