XAMPLING OF UNKNOWN PULSES

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ABSTRACT

We develop sub-Nyquist sampling systems for signals comprised of several, possibly overlapping, finite duration pulses with unknown shapes and time positions. To the best of our knowledge, stable and low-rate sampling strategies for a superposition of unknown pulses without knowledge of the pulse locations have not been derived. We propose a multichannel scheme based on Gabor frames that exploits the sparsity of signals in time and enables sampling at sub-Nyquist rates. Moreover, if the signal is additionally essentially multiband, then the sampling scheme can be adapted to lower the sampling rate without knowing the band positions. Our approach is based on modulating the input signal in each channel with a properly chosen waveform, followed by integration.

***Index Terms*—** sub-Nyquist sampling, compressed sensing, time-frequency analysis, Gabor frames

1. INTRODUCTION

One of the common assumptions in sampling theory suggests that in order to perfectly reconstruct a bandlimited analog signal from its samples, it must be sampled at the Nyquist rate, that is twice its highest frequency. In practice, however, all real life signals are necessarily of finite duration, and consequently cannot be perfectly bandlimited, due to the uncertainty principle. The Nyquist rate is therefore dictated by the essential bandwidth of the signal, that is by the desired accuracy of the approximation: the higher the rate, meaning the more samples are taken, the better the reconstruction.

In this paper we are interested in sampling a special class of time limited signals: signals consisting of a stream of short pulses, referred to as multipulse signals. Since the pulses occupy only a small portion of the signal support, intuitively less samples, then those dictated by the essential bandwidth, should suffice to reconstruct the signal. Our main goal is to design a minimal rate sampling and reconstruction scheme for multipulse signals that exploits the inherent structure of these signals, without knowing the pulse shapes and their locations. We show that when signals additionally exhibit certain sparsity in the frequency domain, in particular radar signals, then the sampling rate can be further reduced.

A special case was considered in [1] in which the signal is composed of shifts of a single known pulse shape. Such signals are defined by a finite number of parameters, and fall under the class of finite rate of innovation (FRI) signals introduced in [2]. The sampling scheme proposed in [1] operates at the minimal sampling rate required for such signals, which equals the number of unknown time delays and amplitudes. In this case without noise, perfect recovery is possible due to the finite dimensionality of the problem.

A more general class of multipulse signals results when the pulse shapes are not known. In this scenario perfect reconstruction from a finite number of samples is impossible, as there are generally infinitely many parameters defining the signal. Nonetheless, the reconstruction error can be made sufficiently small with just a finite number of samples. Two known sampling methods when the pulse locations are known are pointwise samples, or Fourier series. However, neither approaches can be used to reduce the sampling rate when the pulse positions are unknown.

Our contribution is an efficient sampling architecture for multipulse essentially multiband signals, that does not require the knowledge of the pulse shapes, their locations nor the locations of the essential bands. The only knowledge we assume is that our signal is comprised of N pulses, each of maximal width W in the interval $[-\beta/2, \beta/2]$, and that it is essentially concentrated on S frequency bands of width no more than Ω_W within its essential bandwidth $[-\Omega/2, \Omega/2]$. Despite the complete lack of knowledge on the signal shape, we achieve low sampling rate, proportional to $WN \times \Omega_W S$, that is, the actual time-frequency occupancy, whereas Nyquist sampling suggests taking approximately $\beta\Omega$ samples. We achieve this rate reduction by combining the well established theory of Gabor frames [3] with the recently proposed Xampling paradigm [4] which is a framework for sub-Nyquist sampling of analog signals.

The paper is organized as follows. In Section 2 we introduce notation and define our problem. In Section 3 we derive the multichannel scheme for multipulse signals, and provide error bounds on the reconstruction. Section 5 relates our sampling scheme to other sampling schemes that fall into the Xampling paradigm. Finally, in Section 6, we present simulation results.

2. PROBLEM FORMULATION

2.1. Notation and Definitions

We denote by $L_2(\mathbb{R})$ the Hilbert space of complex square integrable functions, with norm $\|f\|_2^2 = \langle f, f\rangle$, where $\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt$, and $\overline{g(t)}$ denotes the complex conjugate of $g(t)$. The Fourier transform of $f(t)$ is defined as $\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\, dt$. Two important operators that play a central role in Gabor theory, are the translation and modulation operators defined for $x, \omega \in \mathbb{R}$ as $T_x f(t) := f(t-x)$ and $M_\omega f(t) := e^{2\pi i\omega t} f(t)$, respectively. A function $g \in L_2(\mathbb{R})$, together with parameters $a, b > 0$ forms a Gabor frame if there exist constants $0 < A_1 \le A_2 < \infty$ such that

$$A_1\|f\|_2^2 \le \sum_{k,l\in\mathbb{Z}} |\langle f, M_{bl}T_{ak}g\rangle|^2 \le A_2\|f\|_2^2$$

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for all $f \in L_2(\mathbb{R})$, [3].

We denote matrices by boldface capital letters, for example $\mathbf{C}$, $\mathbf{D}$, and vectors by boldface low case letters, such as $\mathbf{X}$, $\mathbf{Z}$.

2.2. Problem Formulation

We consider signals that are a sum of short, finite duration pulses:

$$f(t) = \sum_{n=1}^{N} h_n(t)\,, \quad \text{where} \quad \max_{n} |\text{supp}\, h_n| \le W\,. \tag{1}$$

The number of pulses N and their maximal width W are assumed known, and the pulses may overlap in time. We assume that $f(t)$ is supported on an interval $[-\beta/2, \beta/2]$ with $NW \ll \beta$. Our goal is to recover $f(t)$ from the minimal number of samples possible.

Due to the uncertainty principle, finite duration functions cannot be perfectly bandlimited. However, in practice the main frequency content is typically confined to some finite interval $[-\Omega/2, \Omega/2]$, with the energy of $\widehat{f}(\omega)$ outside that interval being less than $\epsilon_\Omega \|f\|_2$ for some $\epsilon_\Omega < 1$. A special subclass of such signals are those whose frequency content is concentrated on only a few bands within $[-\Omega/2, \Omega/2]$. We refer to such signals as essentially multiband and denote them by $\mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$ if there are at most S bands, each of width no more then Ω_W.

We aim at designing a sampling system for signals from $\mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$ that satisfies the following properties: (i) the system has no prior knowledge on the locations or shapes of the pulses nor the bands; (ii) the number of samples should be as low as possible; (iii) the reconstruction from the samples should be simple; (iv) the reconstructed signal should be close to the original signal.

To achieve these goals we combine the well established theory of Gabor frames [3] with the recently proposed Xampling paradigm [4] which is a framework for sub-Nyquist sampling of analog signals. By choosing a proper frame, a multipulse essentially multiband signal admits a sparse representation in that frame. Moreover, we show that sampling and reconstruction can be made simple [3].

2.3. Gabor Frames

We consider Gabor frames whose windows are elements of the space S_0. A window $g \in L_2(\mathbb{R})$ belongs to S_0 if $\|g\|_{S_0} := \int_{-\infty}^{\infty} |\langle f, M_\omega T_x \psi\rangle|\, dx\, d\omega < \infty$, where $\psi(t)$ is a Gaussian.

Let $\mathcal{G}(g, a, b) = \{M_{bl}\, T_{ak}\, g\}$ be a Gabor frame [3] with $g \in S_0$ compactly supported on an interval $[-W/2, W/2]$, $a = \mu W$ and $b = 1/W$ for some $\mu \in (0, 1)$. Procedures for constructing such frames are presented in [5]. Given such a frame, every function $f(t)$ time limited to $[-\beta/2, \beta/2]$ admits a decomposition [3]

$$f = \sum_{k=-K_0}^{K_0} \sum_{l \in \mathbb{Z}} z_{k,l} M_{bl}\, T_{ak}\, \gamma\,, \tag{2}$$

where $\gamma(t)$ is a dual window and K_0 denotes the smallest integer such that the sum in (2) contains all possible non-zero coefficients $z_{k,l} = \langle f, M_{bl}\, T_{ak} g\rangle$. Moreover, if $f(t)$ is essentially bandlimited and the window $g(t)$ possesses good decay in the frequency domain, then the signal $f(t)$ can be well approximated by a truncated Gabor series [6]. The number of frequency samples $|\ell| \le L_0$ is dictated by the desired accuracy of the approximation and depends on the essential bandwidths of the signal and the Gabor window. Therefore, a total of KL, with $K = 2K_0 + 1$ and $L = 2L_0 + 1$, Gabor samples suffices to approximate $[-\beta/2, \beta/2]$ time limited and $[-\Omega/2, \Omega/2]$ essentially bandlimited signals.

For signals from $\mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$ many Gabor coefficients in the series (2) are zero. Indeed, if the support of $g(t - ak)$ does not overlap any active region of $f(t)$, then $z_{k,l} = 0$ for all $l \in \mathbb{Z}$. Since at most $\lceil 2\mu^{-1} \rceil$ shifts of $g(t)$ by $ak = \mu W k$ overlap one pulse of $f(t)$, and there are altogether N pulses present, the number of nonzero coefficients with respect to the index k is at most $\lceil 2\mu^{-1} \rceil N$. Hence, a finite number, $\lceil 2\mu^{-1} \rceil NL$ with $L = 2L_0 + 1$, of Gabor coefficients suffices to approximate any $f \in \mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$ with desired accuracy. If $f(t)$ is additionally essentially multiband, then this number can be further reduced. Indeed, if the essential support of $\widehat{g}(\omega - bl)$ does not overlap any essential band of $f(t)$, then $|z_{k,l}| = |V_g f(ak, bl)| = |V_{\widehat{g}} \widehat{f}(bl, -ak)|$ is small for all $-K_0 \le k \le K_0$. At most $\lceil (\Omega_W + B)W \rceil$ shifts of essential bandwidth $[-B/2, B/2]$ of $\widehat{g}(\omega)$ by $bl = l/W$ overlap one essential band of $f(t)$, and there are altogether S bands present. Therefore the number of dominant coefficients with respect to the index ℓ is at most $\lceil (\Omega_W + B)W \rceil S$, and the number of samples necessary for a good reconstruction becomes $\lceil 2\mu^{-1} \rceil \times \lceil (\Omega_W + B)W \rceil S$.

In the next section we introduce a sampling scheme that samples multipulse signals at the minimal rate which still allows to obtain the necessary dominant Gabor coefficients.

3. SAMPLING OF MULTIPULSE SIGNALS

3.1. Sampling System

Our system, shown in Fig. 1, exploits the sparsity of multipulse signals, $\mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$, in time and frequency. The signal $f(t)$ enters JM channels simultaneously. In the (j, m)th channel, $f(t)$ is multiplied by a mixing function $q_{j,m}(t)$, followed by an integrator. The role of the mixing functions is to gather together all the information in $f(t)$ over the entire interval $[-\beta/2, \beta/2]$. Namely, $f(t)$ is windowed with shifts of some compactly supported function, and all the windowed versions are summed together with different weights. Simultaneously, the frequency content of $f(t)$ is shifted and filtered with some essentially low-pass filter, and all the filtered shifts are summed together with different weights.

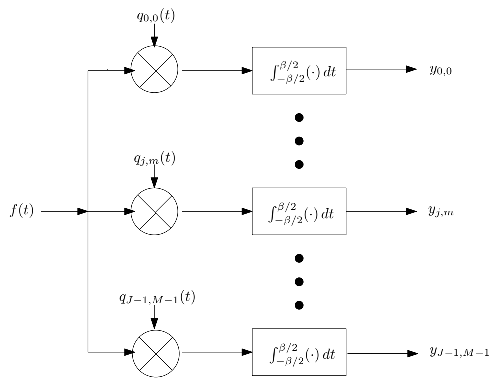

Fig. 1: An efficient sampling system for multipulse signals.

The functions $q_{j,m}(t)$ are constructed from the Gabor frame. Let $\mathcal{G}(g, a, b)$ be a Gabor frame with window $g(t)$ supported on the interval $[-W/2, W/2]$, essentially bandlimited to $[-B/2, B/2]$, and with sampling parameters $a = \mu W$ and $b = 1/W$ for some

$0 < \mu < 1$. Then $q_{j,m}(t) = w_j(t)s_m(t)$ where

$$w_j(t) = \sum_{l=-L_0}^{L_0} d_{j,l} e^{-2\pi i b l t}, \quad s_m(t) = \sum_{k=-K_0}^{K_0} c_{m,k} \overline{g(t-ak)}$$

with $j = 0,\ldots,J-1$, $m = 0,\ldots,M-1$, $K_0 = \lceil(\beta + W)/(2W\mu)\rceil - 1$ and $L_0 = \lceil((\Omega + B)W)/2\rceil - 1$. To specify $q_{j,m}(t)$ completely, it remains to choose the coefficients $d_{j,l}$ and $c_{m,k}$. To do so, we first analyze the effect of the sampler on the unknown signal and derive the relation between the samples $y_{j,m}$ and the signal $f(t)$.

Consider the (j,m)th channel:

$$y_{j,m} = \int_{-\beta/2}^{\beta/2} f(t) q_{j,m}(t)\, dt = \sum_{l=-L_0}^{L_0} d_{j,l} \sum_{k=-K_0}^{K_0} c_{m,k} z_{k,l}\,, \quad (3)$$

where $z_{k,l} = \langle f, M_{bl}T_{ak}g\rangle$. The above relation ties the known $y_{j,m}$ to the unknown Gabor coefficients of $f(t)$ with respect to $\mathcal{G}(g,a,b)$. This relation is key to the recovery of $f(t)$. As can be seen, the goal of the modulator $q_{j,m}(t)$ is to create mixtures of the unknown Gabor coefficients $z_{k,l}$. These mixtures, when chosen appropriately, will allow to recover $z_{k,l}$ from a small number JM measurements by exploiting the sparsity of these coefficients in time and/or in frequency, and relying on ideas of compressed sensing. If we were to collect pure Gabor coefficients then we would have to take KL measurements, more then necessary as many of them are zero or negligible for a good reconstruction.

3.2. Signal Recovery

It is convenient to write (3) in matrix form as

$$\mathbf{Y} = \mathbf{D}(\mathbf{C}\mathbf{Z})^T\,, \quad (4)$$

where $\mathbf{Y}$ is a matrix of size $M \times J$ with mjth element equal $y_{j,m}$, for $j = 0,\ldots,J-1$ and $m = 0,\ldots,M-1$. The unknown Gabor coefficients are gathered in the $K \times L$ matrix $\mathbf{Z}$ with columns $\mathbf{Z}[l] = [z_{-K_0,l},\ldots,z_{K_0,l}]^T$, $l = -L_0,\ldots,L_0$. The $J \times L$ matrix $\mathbf{D}$ contains the coefficients $\mathbf{D}_{j,l} = d_{j,l-L_0}$, $j = 0,\ldots,J-1$, $l = 0,\ldots,2L_0$, and the $M \times K$ matrix $\mathbf{C}$ contains the coefficients $\mathbf{C}_{m,k} = c_{m,k-K_0}$, $m = 0,\ldots,M-1$, $k = 0,\ldots,2K_0$.

The choice of a frame $\mathcal{G}(g,a,b)$ guarantees that for every ℓ, the column vectors $\mathbf{Z}[l]$ have only $\lceil 2\mu^{-1}\rceil N$ out of K nonzero entries, and the nonzero entries correspond to the locations of the pulses, as discussed in Subsection 2.3. We conclude that each $\mathbf{Z}[l]$ is $\lceil 2\mu^{-1}\rceil N-$sparse and all $\mathbf{Z}[l]$ have nonzero entries on the same rows due to the structure of $f(t)$. Moreover, if the signal is known to be essentially multiband, then the rows vectors $\mathbf{Z}[k] = [z_{k,-L_0},\ldots,z_{k,L_0}]$ have only $\lceil(\Omega_W + B)W\rceil S$ out of L dominant entries, and the dominant entries correspond to the locations of the essential bands of $f(t)$. Therefore, each $\mathbf{Z}[k]$ is $\lceil(\Omega_W + B)W\rceil S-$dominant and all $\mathbf{Z}[k]$ have dominant entries on the same columns due to the structure of $f(t)$ in the frequency domain. The $K \times L$ matrix $\mathbf{Z}^C$ with all but $\lceil(\Omega_W + B)W\rceil S$ nonzero columns corresponding to the $\lceil(\Omega_W + B)W\rceil S$ greatest columns, with respect to ℓ^2-norm, is referred to as the best $\lceil(\Omega_W + B)W\rceil S$ column approximation of $\mathbf{Z}$.

The following theorem states the conditions under which one can reconstruct the Gabor coefficients $z_{k,l}$, $|k| \le K_0$ and $|l| \le L_0$, from the outputs $y_{j,m}$. Let us fix $\mu \in (0,1)$ and a Gabor frame $\mathcal{G}(g,a,b)$, with $a = W\mu$, $b = 1/W$, and $g \in S_0$ compactly supported on $[-W/2, W/2]$ and ϵ_B-bandlimited to $[-B/2, B/2]$ in the S_0 norm.

Theorem 3.1 ([7]). *Let $f \in \mathcal{MP}(N, W, \beta; S, \Omega_W, \Omega)$ be sampled using the sampling scheme of Fig. 1 with the following parameters:*

1) $K_0 = \lceil(\beta + W)/(2W\mu)\rceil - 1$;
2) $L_0 = \lceil(\Omega + B)W/2\rceil - 1$;
3) $q_{j,m}(t) = w_j(t)s_m(t)$ *where* $m = 0,\ldots,M-1$ *and* $j = 0,\ldots,J-1$;
4) $M \ge \lceil 2\mu^{-1}\rceil N$ *for non-blind reconstruction or* $M \ge 2\lceil 2\mu^{-1}\rceil N$ *for blind;*
5) $J \ge \lceil(\Omega_W + B)W\rceil S$ *for non-blind reconstruction or* $J \ge 2\lceil(\Omega_W + B)W\rceil S$ *for blind.*

If the matrix $\mathbf{D}$ has RIP[1] constant δ of order $2\lceil(\Omega_W + B)W\rceil S$ such that $\delta \le \sqrt{2} - 1$ and every set of $2\lceil 2\mu^{-1}\rceil N$ columns of $\mathbf{C}$ are linearly independent, then (4) has a unique sparse solution $\widetilde{\mathbf{Z}}$. Moreover, the function $\widetilde{f} = \sum_{k=-K_0}^{K_0}\sum_{l=-L_0}^{L_0} \widetilde{z}_{k,l} M_{bl}\, T_{ak}\gamma$, with $\gamma \in S_0$ denoting the dual atom of $g(t)$, reconstructed from the obtained coefficients satisfies

$$\|f - \widetilde{f}\|_2 \le \widetilde{C}_0(\epsilon_\Omega + \epsilon_B)\|f\|_2 + \widetilde{C}_1 \sum_{l=-L_0}^{L_0} \left\|\mathbf{Z}[l] - \mathbf{Z}^C[l]\right\|_2, \quad (5)$$

where $\widetilde{C}_0$ and $\widetilde{C}_1$ are constants depending on the chosen Gabor frame, and $\widetilde{C}_1$ additionally on the RIP constant δ. The matrix $\mathbf{Z}^C$ is the best $\lceil(\Omega_W + B)W\rceil S-$term approximation of $\mathbf{Z}$.

In the case of known positions of the pulses and bands, referred to as non-blind, the minimal sampling rate for the desired accuracy of the approximation and a given frame is when $M = \lceil 2\mu^{-1}\rceil N$ and $J = \lceil(\Omega_W + B)W\rceil S$. In the blind setting, when the locations of the pulses and the bands are unknown, the sampling rate increases by a factor of four (a factor of two in each domain). Note, that the scheme is efficient only when the pulses occupy less then half of the overall support of the signal in time and it is concentrated on less then half of its bandwidth.

As we have seen, the matrix $\mathbf{Z}$ that we would like to recover from the measurements $\mathbf{Y}$ is both row sparse and column compressible. The recovery of $\mathbf{Z}$ is performed in two stages. First, we recover $\lceil 2\mu^{-1}\rceil N-$row sparse matrix $\mathbf{U}$ from the relation $\mathbf{Y}^T = \mathbf{C}\mathbf{U}$. This problem is referred to as a multiple measurement vector (MMV) problem. Several algorithms have been developed that exploit this structure to recover $\mathbf{U}$ efficiently from $\mathbf{Y}^T$ in polynomial time when M is increased beyond $M = 2\lceil 2\mu^{-1}\rceil N$ [8], [9]. Therefore, $\mathbf{U}$ can be recovered using any one of these known methods and, since the solution is unique, it equals $\mathbf{Z}\mathbf{D}^T$. Knowing $\mathbf{U}^T = \mathbf{D}\mathbf{Z}^T$, we solve another MMV problem for a unique $\lceil(\Omega_W + B)W\rceil S-$column sparse matrix $\widetilde{\mathbf{Z}}$. This time we obtain an approximation of $\mathbf{Z}$, as $\mathbf{Z}$ itself is not strictly column sparse. This approximation is proportional to the best $\lceil(\Omega_W + B)W\rceil S-$column approximation of $\mathbf{Z}$ [9].

When no information about the frequency sparsity of the signal is known, we take the matrix $\mathbf{D}$ to be left invertible, a necessary condition being $J \ge L$. The relation (4) then reduces to $\mathbf{D}^\dagger\mathbf{Y} = (\mathbf{C}\mathbf{Z})^T$ and, with the assumptions on the matrix $\mathbf{C}$, we are able to retrieve $\mathbf{Z}$.

[1] $\widetilde{\mathbf{D}}$ has a RIP constant of order S if $(1-\delta)\|x\|_2^2 \le \|\widetilde{\mathbf{D}}x\|_2^2 \le (1+\delta)\|x\|_2^2$ for all $S-$sparse vecors x.

4. WAVEFORM DESIGN

The functions $s_m(t)$ are pulse sequence modulations, and therefore simple to implement. The sequences are chosen such that they form a valid CS matrix. An example of a valid CS matrix is a matrix whose entries are ± 1 drawn independently and with equal probability. The waveforms $w_j(t)$ are created from $1/b-$periodic waveforms by low-pass filtering. More precisely, let

$$\widetilde{w}_j(t) = \alpha_j[i], \quad \frac{i}{bI} \le t \le \frac{i+1}{bI} \tag{6}$$

where $\alpha_j[i] = \pm 1$ and $i = 0, \ldots, I-1$. The filter $u(t)$ is designed so that $|\widehat{u}(\omega)| = 1$ for $\omega = bl$ and $|l| \le L_0$, $|\widehat{u}(\omega)| = 0$ for $\omega = bl$ and $|l| \ge L_0$ and taking arbitrary values otherwise. Depending on the application, we choose appropriate starting waveforms $w_j(t)$. For the matrix $\mathbf{D}$ to be left invertible a necessary condition is that $J \ge I \ge L$ and the sequences $\alpha_j[i]$ are chosen such that the matrix $\mathbf{A}$, whose jith element is $\alpha_j[i]$, has full column rank [1]. For example, if $J = I = L$, then the rows of $\mathbf{A}$ can be created from cyclic shifts of one basic sequence. On the other hand, for a matrix $\mathbf{D}$ to be a valid CS matrix, meaning to have RIP property with high probability, the values $\alpha_j[i] = \pm 1$ are chosen independently with equal probability and $I \ge L \ge J$ [10].

5. RELATED WORK

Our sampling scheme follows the philosophy in much of the recent work in analog compressed sensing, termed Xampling, which provides a framework for incorporating structure in analog signals to reduce the sampling rates, without the need for discretization [4].

A pioneer sub-Nyquist system of this type is the modulated wideband converter (MWC) introduced in [10]. This scheme targets low rate sampling of multiband signals. The MWC enables perfect recovery of any multiband function from its samples at rates far below Nyquist, without knowledge of the band locations. Sub-Nyquist sampling is achieved by applying modulation waveforms to the analog input prior to uniformly sampling at the low rate. Real life signals are by necessity of finite duration, hence are not strictly bandlimited, and the MWC does not provide perfect recovery in this case. Using our approach, through Gabor frames, we are able to compute reconstruction errors and generalize the MWC to other then ideal low pass filters [7].

Another application of MWC, with slightly modified waveforms, was used recently in [1] to treat multipulse signals with a known pulse shape. The purpose of the waveforms in that scheme is to simplify the hardware implementation and improve robustness, while in the MWC the waveforms are used to reduce the sampling rate relative to the Nyquist rate. In [7] we show that with proper choice of the $q_{j,m}(t)$ waveforms our scheme reduces to that of [1].

In our system, the the MWC like waveforms are the waveforms $w_j(t)$. Their purpose is twofold: reduce the sampling rate (when the signal in known to be sparse in frequency) and simplify the hardware implementation (when no specific knowledge on the frequency content is available).

6. SIMULATIONS

We examine the performance of our sampling scheme on a signal comprising three pulses, that is additionally essentially multiband with two bands. We use a tight Gabor frame with cosine window and $\mu = 1/2$. When sparsity in time and in frequency is not taken into account we need $KL = 5084$, with $K = 124$ and $L = 41$, Gabor coefficients to obtain the relative error $\|f - \widetilde{f}\|/\|f\|_2$ of 0.006. The same error can be achieved if we reduce $K = 124$ to $M = 30$, achieving fivefold improvement. On the other hand, when multibandedness of the signal is additionally taken into account, then $L = 41$ can be reduced to $J = 19$ achieving overall tenfold improvement with respect to KL. The relative error in this case increases, to 0.06, as Theorem 3.1 suggests.

Fig. 2 compares the performance of our sampling system for different number of channels with respect to sparsity in time (M) and sparsity in frequency (J). The results are averaged over 100 trials.

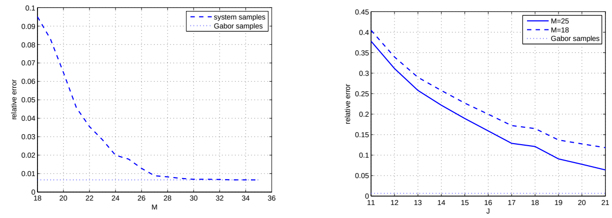

Fig. 2: Comparison of the relative error with respect to the number of channels. Dotted line represents the error when all Kl Gabor coefficients are used in the reconstruction.

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