On the Uniqueness of FROG Methods

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***Abstract*—The problem of recovering a signal from its power spectrum, called *phase retrieval*, arises in many scientific fields. One of many examples is ultra-short laser pulse characterization in which the electromagnetic field is oscillating with $\sim 10^{15}$ Hz and phase information cannot be measured directly due to limitations of the electronic sensors. Phase retrieval is ill-posed in most cases as there are many different signals with the same Fourier transform magnitude. To overcome this fundamental ill-posedness, several measurement techniques are used in practice. One of the most popular methods for complete characterization of ultra-short laser pulses is the Frequency-Resolved Optical Gating (FROG). In FROG, the acquired data is the power spectrum of the product of the unknown pulse with its delayed replica. Therefore the measured signal is a quartic function of the unknown pulse. A generalized version of FROG, where the delayed replica is replaced by a second unknown pulse, is called blind FROG. In this case, the measured signal is quadratic with respect to both pulses. In this letter we introduce and formulate FROG-type techniques. We then show that almost all band-limited signals are determined uniquely, up to trivial ambiguities, by blind FROG measurements (and thus also by FROG), if in addition we have access to the signals power spectrum.**

***Index Terms*—phase retrieval, quartic system of equations, ultra-short laser pulse measurements, FROG**

I. Introduction

In many measurement systems in physics and engineering one can only acquire the power spectrum of the underlying signal, namely, its Fourier transform magnitude. The problem of recovering a signal from its power spectrum is called *phase retrieval* and it arises in many scientific fields, such as optics, X-ray crystallography, speech recognition, blind channel estimation and astronomy (see for instance, [1], [2], [3], [4], [5], [6] and references therein). Phase retrieval for one-dimensional (1D) signals is ill-posed for almost all signals. Two exceptions are minimum phase signals [7] and sparse signals with structured support [8], [9]. Additional information on the sought signal can be used to guarantee uniqueness. For instance, the knowledge of one signal entry or the magnitude of one entry in the Fourier domain, in addition to the the power spectrum, determines almost all signals [10], [11].

For general signals, many algorithms and measurement techniques were suggested to make the problem well-posed. These methods can be classified into two categories. The first utilizes some prior knowledge (if it exists) on the underlying structure of the signal, such as sparsity (e.g. [8], [12], [13]) or knowledge on a portion of the signal (e.g. [2], [10], [11]). The second uses techniques that generate redundancy in the acquired data by taking additional measurements. These measurements can be obtained for instance using radom masks [14], [15] or by multiplying the underlying signal with shifted versions of a known reference signal, leading to short-time Fourier measurements [16], [17], [18].

An important application for phase retrieval is ultra-short laser pulse characterization. Since the electromagnetic field is oscillating at $\sim 10^{15}$ Hz, phase information cannot be measured directly due to limitations of the electronic sensors. To overcome the fundamental ill-posedness of the phase retrieval problem, a popular approach is to use Frequency-Resolved Optical Gating (FROG). This technique measures the power-spectrum of the product of the signal with a shifted version of itself or of another unknown signal. The inverse problem of recovering a signal from its FROG measurements can be thought of as *high-order phase retrieval problem*. The first goal of this letter is to introduce and formulate such FROG-type methods.

Our second contribution is to derive a uniqueness result for FROG-type models. Namely, conditions such that the underlying signal is uniquely determined from the acquired data. A common statement in the optics community, supported by two decades of experimental measurements, is that a laser pulse can be determined uniquely from FROG measurements if the power spectrum of the unknown signal is also measured. To the best of our knowledge, the uniqueness of FROG methods was analyzed only in [19] under the assumption that we have access to the full continuous spectrum. In this letter we analyze the discrete setup as it typically appears in applications.

The letter is organized as follows. Section II introduces the FROG problem and formulates it mathematically. Section III presents our uniqueness result, which is proved in Section IV. Section V concludes the letter.

II. Model and Background

We consider two laser pulse characterization techniques, called FROG and its generalized version blind FROG. These methods are used to generate redundancy in ultra-short laser pulse measurements. FROG is probably the most commonly used approach for full characterization of ultra-short optical pulses due to its simplicity and good experimental per-

This work was funded by the European Union's Horizon 2020 research and innovation program under grant agreement No. 646804-ERCCOG-BNYQ, by the Israel Science Foundation under Grant no. 335/14 and by a research grant from the Sylvia Schaefer Alzheimer's Research Fund. T.B. was partially funded by the Andrew and Erna Finci Viterbi Fellowship.

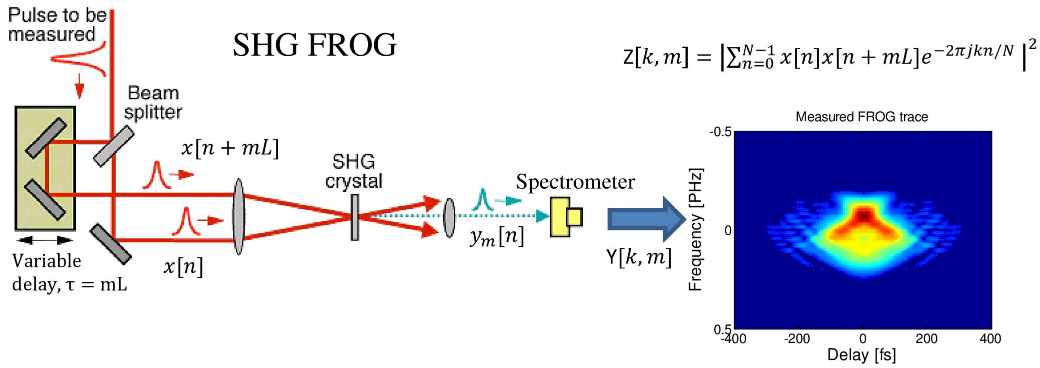

Figure II.1: Illustration of the SHG FROG technique.

formance [20], [21]. A FROG apparatus produces a two-dimensional (2D) intensity diagram of an input pulse by interacting the pulse with delayed versions of itself in a nonlinear-optical medium, usually using a second harmonic generation (SHG) crystal [22]. This 2D signal is called a FROG trace and is a quartic function of the unknown signal. Hereinafter, we consider SHG FROG but other types of nonlinearities exist for FROG measurements. A generalization of FROG in which two different unknown pulses gate each other in a nonlinear medium is called blind FROG. This method can be used to characterize simultaneously two signals [21], [23]. In this case, the measured data is referred to as a blind FROG trace and is quadratic in both signals. We refer to the problems of recovering a signal from its blind FROG trace and FROG trace as *bivariate phase retrieval* and *quartic phase retrieval*, respectively. Note that quartic phase retrieval is a special case of bivariate phase retrieval where both signals are equal. An illustration of the SHG FROG model is depicted in Figure II.1.

In bivariate phase retrieval we acquire, for each delay step m, the power spectrum of

$$\mathbf{y}_m[n] = \mathbf{x}_1[n]\,\mathbf{x}_2[n+mL], \tag{II.1}$$

where L determines the overlap factor between adjacent sections. We assume that $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^N$ are periodic, namely, $\mathbf{x}[i] = \mathbf{x}[N\ell + i]$ for all $\ell \in \mathbb{Z}$. The acquired data is given by

$$\mathbf{Z}[k,m] = |\mathbf{Y}[k,m]|^2, \tag{II.2}$$

where

$$\begin{aligned}\mathbf{Y}[k,m] &= (\mathbf{F}\mathbf{y}_m)[k] = \sum_{n=0}^{N-1} \mathbf{y}_m[n]\, e^{-2\pi jkn/N} \\ &= \sum_{n=0}^{N-1} \mathbf{x}_1[n]\,\mathbf{x}_2[n+mL] e^{-2\pi jkn/N},\end{aligned} \tag{II.3}$$

and $\mathbf{F}$ is the $N \times N$ DFT matrix. Quartic phase retrieval is the special case in which $\mathbf{x}_1 = \mathbf{x}_2$.

Current FROG reconstruction procedures [24], [25], [26] are based on 2D phase retrieval algorithms [2], [27]. One popular iterative algorithm is the principal components generalized projections (PCGP) method [28]. In each iteration, PCGP performs PCA (principal component analysis, see for instance [29]) on a data matrix constructed by a previous estimation. It is common to initialize the algorithm by a Gaussian pulse with random phases. A recent paper suggests to adopt ptychographic techniques where every power spectrum, measured at each delay, is treated separately as a 1D problem [30]. In Figure II.2 we present an example for the recovery of a signal from its noisy FROG trace using this algorithm.

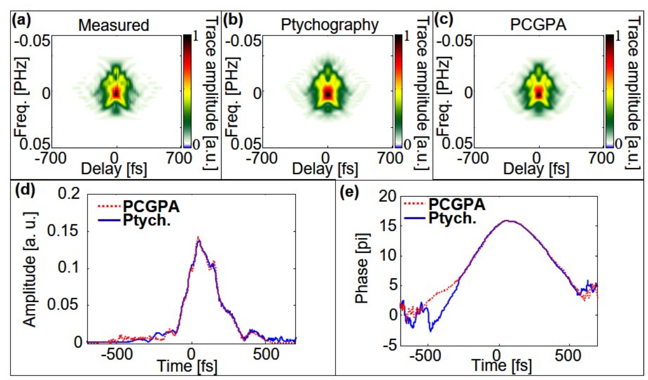

Figure II.2: Experimental example of a femtosecond (fs) pulse reconstruction by SHG-FROG. The experiment was conducted with a delay step of 3 fs and 512 delay points. Hence, the complete FROG trace consists of 512×512 data points. The laser pulse was produced by a typical ultrafast Ti-sapphire laser system (1KHz repetition rate, 2 Watt average power). (a) measured FROG trace (b) recovered trace by alternating projection algorithm for ptychography (Ptych.) proposed in [30] (c) recovered trace by the PCGPA algorithm [28] (d) recovered amplitudes by PCGPA and Ptych. (e) recovered phases by PCGPA and Ptych.

In the next section we present our main theoretical results. First, in Proposition 1 we identify the trivial ambiguities of blind FROG. Trivial ambiguities are the basic operations on the signals $\mathbf{x}_1, \mathbf{x}_2$ that do not change the blind FROG trace $\mathbf{Z}$. Then, we derive a uniqueness result for the mapping between the signals and their blind FROG trace. Particularly, suppose we can measure the power spectra of the unknown signals in addition to the blind FROG trace. We exploit recent advances in the theory of phase retrieval [11] and prove that in this case almost all band-limited signals are determined uniquely, up to trivial ambiguities. This result holds trivially for FROG as well. The proof is based on the observation that given the signal's power spectrum, the problem can be reduced to standard phase retrieval where both the temporal and spectral magnitudes are known.

III. Uniqueness Result

This letter aims at examining under what conditions the measurements $\mathbf{Z}$ determine $\mathbf{x}_1$ and $\mathbf{x}_2$ uniquely. In some cases, there is no way to distinguish between two pairs of signals, by any method, as they result in the same measurements. The following proposition describes four *trivial ambiguities* of bivariate phase retrieval. The first three are similar to equivalent results in phase retrieval, see for instance [10]. The proof follows from basic properties of the Fourier transform and is given in the Appendix.

Proposition 1. *Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^N$ and let $\mathbf{y}_m[n] := \mathbf{x}_1[n]\mathbf{x}_2[n+mL]$ for some fixed L. Then, the following signals have the same phaseless bivariate measurements $\mathbf{Z}[m,k]$ as $\mathbf{x}_1, \mathbf{x}_2$:*

1) *multiplication by global phases $\mathbf{x}_1 e^{j\psi_1}, \mathbf{x}_2 e^{j\psi_2}$ for some $\psi_1, \psi_2 \in \mathbb{R}$,*
2) *the shifted signal*
$$\mathbf{x}_1[n-n_0]\mathbf{x}_2[n-n_0+mL] = \mathbf{y}_m[n-n_0]$$
for some $n_0 \in \mathbb{Z}$,
3) *the conjugated and reflected signal*
$$\overline{\mathbf{x}_1[-n]} \cdot \overline{\mathbf{x}_2[-n+mL]} = \overline{\mathbf{y}_m[-n]},$$
4) *modulation, $\mathbf{x}_1[n]e^{-2\pi jk_0 n/N}$, $\mathbf{x}_2[n]e^{2\pi jk_0 n/N}$ for some $k_0 \in \mathbb{Z}$.*

Assume that one of the signals is band-limited and that we have access to the power spectrum of the underlying signals $|\mathbf{F}\mathbf{x}_1|^2$, $|\mathbf{F}\mathbf{x}_2|^2$ as well as the blind FROG trace $\mathbf{Z}[m,k]$ of (II.2). In ultra-short pulse characterization experiments the signals are indeed band-limited [31] and the power spectrum of the pulse under investigation is often available, or it can be easily measured by a spectrometer, which is already integrated in any FROG device. Inspired by [19], we show that in this case, the bivariate problem can be reduced to a standard (monovariate) phase retrieval problem where both the temporal and the spectral magnitudes are known. Consequently, we derive the following result which is proved in the next section.

Theorem 2. *Let $L = 1$, and let $\hat{\mathbf{x}}_1 := \mathbf{F}\mathbf{x}_1$ and $\hat{\mathbf{x}}_2 := \mathbf{F}\mathbf{x}_1$ be the Fourier transforms of $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Assume that $\hat{\mathbf{x}}_1$ has at least $\lceil (N-1)/2 \rceil$ consecutive zeros (e.g. band-limited signal). Then, almost all signals[1] are determined uniquely, up to trivial ambiguities, from the measurements $\mathbf{Z}[m,k]$ and the knowledge of $|\hat{\mathbf{x}}_1|$ and $|\hat{\mathbf{x}}_2|$. By trivial ambiguities we mean that $\mathbf{x}_1$ and $\mathbf{x}_2$ are determined up to global phase, time shift and conjugate reflection.*

Corollary 3. *The same result holds for quartic phase retrieval in which $\mathbf{x}_1 = \mathbf{x}_2$. This model fits the FROG setup.*

Proof. The proof follows the proof technique of Theorem 2 with $\mathbf{x}_1 = \mathbf{x}_2$. $\square$

[1]By almost all signals, we mean that there may be a set of measure zero for which the theorem does not hold.

IV. Proof of Theorem 2

The proof is based on the reduction of bivariate phase retrieval to a series of monovariate phase retrieval problems in which both temporal and spectral magnitudes are known [19]. The latter problem is well-posed for almost all signals.

Let
$$\mathbf{x}_1[n] = \frac{1}{N}\sum_{\ell=0}^{N-1} \hat{\mathbf{x}}_1[\ell] e^{2\pi j\ell n/N},$$
$$\mathbf{x}_2[n] = \frac{1}{N}\sum_{\ell=0}^{N-1} \hat{\mathbf{x}}_2[\ell] e^{2\pi j\ell n/N},$$
and
$$\delta[n] := \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have
$$\begin{aligned}
\mathbf{Y}[k,m] &= \sum_{n=0}^{N-1} \mathbf{x}_1[n]\mathbf{x}_2[n+m] e^{-2\pi jkn/N} \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\left(\sum_{\ell_1=0}^{N-1}\hat{\mathbf{x}}_1[\ell_1]e^{2\pi j\ell_1 n/N}\right) \\
&\quad \left(\sum_{\ell_2=0}^{N-1}\hat{\mathbf{x}}_2[\ell_2]e^{2\pi jm\ell_2/N}e^{2\pi j\ell_2 n/N}\right)e^{-2\pi jkn/N} \\
&= \frac{1}{N^2}\sum_{\ell_1=0}^{N-1}\sum_{\ell_2=0}^{N-1}\hat{\mathbf{x}}_1[\ell_1]\hat{\mathbf{x}}_2[\ell_2]e^{2\pi jm\ell_2/N} \\
&\quad \underbrace{\sum_{n=0}^{N-1} e^{-2\pi j(k-\ell_1-\ell_2)n/N}}_{=N\delta[k-\ell_1-\ell_2]} \\
&= \frac{1}{N}\sum_{\ell=0}^{N-1}\hat{\mathbf{x}}_1[k-\ell]\hat{\mathbf{x}}_2[\ell]e^{2\pi jm\ell/N}.
\end{aligned}$$

Let us denote $\hat{\mathbf{x}}_i[\ell] = |\hat{\mathbf{x}}_i[\ell]| e^{j\boldsymbol{\phi}_i[\ell]}$ for $i = 1,2$, $\mathbf{I}[k,\ell] = \frac{1}{N}|\hat{\mathbf{x}}_1[k-\ell]||\hat{\mathbf{x}}_2[\ell]|$ and $\mathbf{P}[k,\ell] = \boldsymbol{\phi}_1[k-\ell] + \boldsymbol{\phi}_2[\ell]$. Then[2],
$$\mathbf{Y}[k,-m] = \sum_{\ell=0}^{N-1}\mathbf{I}[k,\ell]e^{j\mathbf{P}[k,\ell]}e^{-2\pi jm\ell/N}.$$

By assumption, $|\hat{\mathbf{x}}_1|$ and $|\hat{\mathbf{x}}_2|$ are known and therefore $\mathbf{I}[k,\ell]$ is known as well. Moreover, note that by assumption, for any fixed k, $\mathbf{I}[k,\ell]$ has at least $\lceil (N-1)/2 \rceil$ consecutive zeros. Our problem is then reduced to that of recovering the signal $\mathbf{S}[k,\ell] := \mathbf{I}[k,\ell]e^{j\mathbf{P}[k,\ell]}$ from the knowledge of $\mathbf{Z}[k,-m]$ and $\mathbf{I}[k,\ell]$. For fixed k, this is a standard phase retrieval problem with respect to the second variable where the temporal magnitudes are known. To proceed, we state the finite-discrete version of Theorem 3.4 from [11]:

Lemma 4. *Let $t \in [0,\ldots,N-1]\setminus\{(N-1)/2\}$ and let $\mathbf{u} \in \mathbb{C}^N$ be such that $\mathbf{u}$ has at least $\lceil (N-1)/2 \rceil$ consecutive zeros. Then, almost every complex signal $\mathbf{u}$ is determined*

[2]Recall that all indices should be considered as modulo N. Hence, $\mathbf{Y}[k,-m]$ is just a reordering of $\mathbf{Y}[k,m]$.

uniquely from the magnitude of its Fourier transform and $|\mathbf{u}[N-1-t]|$ *up to to global phase.*

Lemma 4 implies that $\mathbf{Z}[k,-m]$ and $\mathbf{I}[k,\ell]$ determine, for fixed k, almost all $\mathbf{P}[k,\ell]$ up to global phase. So, for all k, $\mathbf{P}[k,\ell]$ is determined up to an arbitrary function $\boldsymbol{\psi}[k]$. We note that while Lemma 4 requires only one sample of $\mathbf{I}[k,\ell]$ to determine $\mathbf{S}[k,\ell]$ uniquely, $\mathbf{I}[k,\ell]$ does not determine $|\hat{\mathbf{x}}_1|$ and $|\hat{\mathbf{x}}_2|$ uniquely. For this reason, we need the full power spectrum of the signals in addition to the blind FROG trace.

Next, we will show that

$$\begin{aligned}\tilde{\mathbf{P}}[k,\ell] &= \mathbf{P}[k,\ell]+\boldsymbol{\psi}[k] \\ &= \boldsymbol{\phi}_1[k-\ell]+\boldsymbol{\phi}_2[\ell]+\boldsymbol{\psi}[k],\end{aligned} \tag{IV.1}$$

determines $\boldsymbol{\phi}_1,\boldsymbol{\phi}_2$ and $\boldsymbol{\psi}$ up to affine functions. Note that generally (IV.1) may include additional terms of $2\pi s[k,\ell]$ for some integers $s[k,\ell]\in\mathbb{Z}$. However, phase wrapping is physically meaningless since it will not change the light pulse [21, Section 2].

The relation (IV.1) can be written using matrix notation. Let $\tilde{\mathbf{P}}_{vec}\in\mathbb{R}^{N^2}$ be a column stacked version of $\tilde{\mathbf{P}}$ and let

$$\mathbf{v} := \begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\\ \boldsymbol{\psi}\end{bmatrix}\in\mathbb{R}^{3N}.$$

Then we obtain the over-determined linear system

$$\tilde{\mathbf{P}}_{vec}=\mathbf{A}\mathbf{v}, \tag{IV.2}$$

where $\mathbf{A}\in\mathbb{R}^{N^2\times 3N}$ is the matrix that relates $\mathbf{v}$ and $\tilde{\mathbf{P}}_{vec}$ according to (IV.1).

We aim at identifying the null space of the linear operator $\mathbf{A}$. To this end, suppose that there exists another triplet $\tilde{\boldsymbol{\phi}}_1,\tilde{\boldsymbol{\phi}}_2,\tilde{\boldsymbol{\psi}}$ that solves the linear system, i.e.

$$\tilde{\mathbf{P}}[k,\ell]=\tilde{\boldsymbol{\phi}}_1[k-\ell]+\tilde{\boldsymbol{\phi}}_2[\ell]+\tilde{\boldsymbol{\psi}}[k],$$

for all k and ℓ. Let us denote the difference functions by $\mathbf{d}_1 := \boldsymbol{\phi}_1-\tilde{\boldsymbol{\phi}}_1, \mathbf{d}_2 := \boldsymbol{\phi}_2-\tilde{\boldsymbol{\phi}}_2$ and $\mathbf{d}_3 := \boldsymbol{\psi}-\tilde{\boldsymbol{\psi}}$. Then, we can directly conclude that for all k,ℓ we have

$$\mathbf{d}_1[k-\ell]+\mathbf{d}_2[\ell]+\mathbf{d}_3[k]=0. \tag{IV.3}$$

Particularly, for $k=0$ and $\ell=0$ we obtain the relations

$$\begin{aligned}\mathbf{d}_1[-\ell]+\mathbf{d}_2[\ell]+\mathbf{d}_3[0]=0,\\ \mathbf{d}_1[k]+\mathbf{d}_2[0]+\mathbf{d}_3[k]=0.\end{aligned} \tag{IV.4}$$

Plugging (IV.4) into (IV.3) (and replace $-\ell$ by ℓ) we have

$$\mathbf{d}_1[k+\ell]=\mathbf{d}_1[\ell]+\mathbf{d}_1[k]+\mathbf{d}_3[0]+\mathbf{d}_2[0].$$

Hence, we conclude that $\mathbf{d}_1$ is an affine function of the form $\mathbf{d}_1[k]=ak-\mathbf{d}_3[0]-\mathbf{d}_2[0]$ for some scalar a. We can also derive that $\mathbf{d}_2[k]=ak+\mathbf{d}_2[0]$ and $\mathbf{d}_3[k]=-ak+\mathbf{d}_3[0]$. This implies that the null space of $\mathbf{A}$ contains those affine functions. We can compute the phases by $\mathbf{v}=\mathbf{A}^\dagger\tilde{\mathbf{P}}_{vec}$, where $\mathbf{A}^\dagger$ is the Moore-Penrose pseudoinverse.

To complete the proof, we recall that $\boldsymbol{\phi}_i$, $i=1,2$, are the phases of the Fourier transforms of $\mathbf{x}_i$. As we can estimate the phases up to affine functions, we can only determine $\hat{\mathbf{x}}_i[k]=|\hat{\mathbf{x}}_i[k]|e^{j(\boldsymbol{\phi}_i[k]+c_1k+c_2)}$ for some constants c_1 and c_2. This unknown affine function reflects the global phase and the translation ambiguities. Specifically, the term e^{jc_1k} reflects translation by c_1 indices and the e^{jc_2} product by a global phase. The conjugate-reflectness ambiguity arises from the fact that both the blind FROG trace and the signals power spectrum are invariant to this property. This completes the proof.

V. Discussion

In this paper we analyzed the uniqueness of bivariate and quartic phase retrieval problems. Particularly, we proposed a uniqueness result showing that given the signals power spectrum, blind FROG trace determines almost all signals up to trivial ambiguities for $L=1$. Nevertheless, it was shown experimentally and numerically [30] that stable signal recovery is possible with $L>1$. It is therefore important to investigate the minimal number of measurements which can guarantee uniqueness for FROG and blind FROG.

It is worth noting different FROG nonlinearities. Two examples are third-harmonic generation FROG and polarization gating FROG. In these techniques, the measured signal is modeled as the power spectrum of $\mathbf{y}_m[n]=\mathbf{x}^2[n]\mathbf{x}[n-mL]$ and $\mathbf{y}_m[n]=\mathbf{x}[n]|\mathbf{x}[n-mL]|$, respectively [32], [20]. It is interesting to examine the uniqueness of these high polynomial degree phase retrieval problems in different FROG implementations. Another important application is the so called Frequency-Resolved Optical Gating for Complete Reconstruction of Attosecond Bursts (FROG CRAB), which is based on the photoionization of atoms by the attosecond field, in the presence of a dressing laser field. In this setup, the signal is modeled as the power spectrum of $\mathbf{y}_m[n]=\mathbf{x}_1[n]e^{j\mathbf{x}_2[n-mL]}$ [33].

Acknowledgement

We would like to thank Kishore Jaganathan for many insightful discussions, Oren Cohen for his advice on ultrafast laser pulse measurement methods and Robert Beinert for helpful discussions about [11].

Appendix

Proof of Proposition 1

The proof is based on basic properties of the DFT matrix. Recall that $\mathbf{y}_m[n] := \mathbf{x}_1[n]\mathbf{x}_2[n+mL]$.

1) Let $\psi_1,\psi_2\in\mathbb{R}$ and define $\mathbf{x}_1^\psi := \mathbf{x}_1e^{j\psi_1}$, $\mathbf{x}_2^\psi := \mathbf{x}_2e^{j\psi_2}$ and $\mathbf{y}_m^\psi[n] := \mathbf{x}_1^\psi[n]\mathbf{x}_2^\psi[n+mL]$. Hence, $\mathbf{y}_m^\psi=\mathbf{y}_me^{j(\psi_1+\psi_2)}$ and it is then clear that $\mathbf{Z}$ is independent of ψ_1,ψ_2.
2) Let $n_0\in\mathbb{Z}$ and define $\tilde{\mathbf{y}}_m[n] := \mathbf{y}_m[n-n_0]$. Then, by standard Fourier properties we get
$$\left(\mathbf{F}\tilde{\mathbf{y}}_m\right)[k]=\left(\mathbf{F}\mathbf{y}_m\right)[k]\,e^{-2\pi jkn_0/N},$$
and consequently $|\mathbf{F}\tilde{\mathbf{y}}_m|=|\mathbf{F}\mathbf{y}_m|$.
3) By standard Fourier properties we have $|\mathbf{F}\hat{\mathbf{y}}_m|=|\mathbf{F}\mathbf{y}_m|$.
4) Let $k_0\in\mathbb{Z}$ and define $\mathbf{x}_1^{k_0}[n] := \mathbf{x}_1[n]^{-2\pi jk_0n/N}$, $\mathbf{x}_2^{k_0}[n] := \mathbf{x}_2[n]^{2\pi jk_0n/N}$ and $\mathbf{y}_m^{k_0}[n] := \mathbf{x}_1^{k_0}[n]\mathbf{x}_2^{k_0}[n+mL]$. Then, $\mathbf{y}_m^{k_0}[n]=\mathbf{y}_m[n]e^{2\pi jmLk_0/N}$. According to the global phase ambiguity, $\mathbf{Z}$ is independent of k_0. This completes the proof.

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