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Phase retrieval: Stability and recovery guarantees<sup>☆</sup>Yonina C. Eldar<sup>a,\*</sup>, Shahar Mendelson<sup>b</sup><sup>a</sup> Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel<sup>b</sup> Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel

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## ABSTRACT

We consider stability and uniqueness in real phase retrieval problems over general input sets, when the data consists of random and noisy quadratic measurements of an unknown input  $x_0 \in \mathbb{R}^n$  that lies in a general set  $T$ . We study conditions under which  $x_0$  can be stably recovered from the measurements. In the noise-free setting we show that the number of measurements needed to ensure a unique and stable solution depends on the set  $T$  through its Gaussian mean-width, which can be computed explicitly for many sets of interest. In particular, for  $k$ -sparse inputs,  $O(k \log(n/k))$  measurements suffice, while if  $x_0$  is an arbitrary vector in  $\mathbb{R}^n$ ,  $O(n)$  measurements are sufficient.

In the noisy case, we show that if the empirical risk is bounded by a given, computable constant that depends only on statistical properties of the noise, the error with respect to the true input is bounded by the same Gaussian parameter (up to logarithmic factors). Therefore, the number of measurements required for stable recovery is the same as in the noise-free setting up to log factors.

It turns out that the complexity parameter for the quadratic problem is the same as the one used for analyzing stability in linear measurements under very general conditions. Thus, no substantial price has to be paid in terms of stability when there is no knowledge of the phase of the measurements.

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## 1. Introduction

Recently, there has been growing interest in recovering an input vector  $x_0 \in \mathbb{R}^n$  from quadratic measurements

$$y_i = |\langle a_i, x_0 \rangle|^2 + w_i, \quad i = 1, \dots, N. \quad (1.1)$$

Here, we focus on the case in which  $(a_i)_{i=1}^N$  are selected independently according to a random vector  $a$  on  $\mathbb{R}^n$ , and  $(w_i)_{i=1}^N$  are selected independently according to the noise  $w$ , and are assumed to be independent of  $(a_i)_{i=1}^N$ .

Since only the magnitude of  $\langle a_i, x_0 \rangle$  is measured, and not the phase (or the sign, in the real case), this family of problems is referred to as *phase retrieval*. These problems arise in many areas of optics, where the detector can only measure the magnitude of the received optical wave. Several applications of phase retrieval include X-ray crystallography, transmission electron microscopy and coherent diffractive imaging [39,20,19,46].

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Many algorithmic methods have been developed for phase recovery (see, e.g., [20]) which often rely on prior information on the signal, such as positivity or support constraints. One of the most popular techniques is based on alternating projections, such as the Gerchberg and Saxton [16] and Fienup [15] iterations. To circumvent the difficulties associated with convergence of alternating projections, more recently, phase retrieval problems have been treated using semidefinite relaxation, and low-rank matrix recovery ideas [7,41]. Another possible approach that potentially leads to robust solutions is to assume that the input signal  $x_0$  is sparse, namely, that it contains only a few non-zeros values in an appropriate basis expansion. Both the semidefinite relaxation [41,21,37] and greedy recovery methods [36,5,40] can be extended to phase retrieval of sparse inputs.

Despite the vast interest in phase retrieval, there has been little theoretical work on the fundamental limits of this problem, which is the focus of this article. One question in this context is to estimate the number of measurements that are needed in order to ensure robust recovery of the input  $x_0$  – and regardless of the specific recovery method used. Several recent works treat this problem; most, study the case in which  $x_0$  is a general input, namely, there is no sparsity (or other) constraints on  $x_0$ . For example, in this case, with probability one,  $N = 4n - 2$  randomized equations are sufficient for recovery using a brute force (intractable) method, when there is no noise [2]. However, it is not clear, even in that restricted scenario, whether a stable recovery method exists with this number of measurements. In [8,9] the authors consider the case in which  $a_i$  are real or complex vectors that are either uniform on the sphere of radius  $\sqrt{n}$ , or iid zero-mean Gaussian vectors with unit variance. Under these assumptions they show that on the order of  $n$  measurements are needed in order to recover a generic  $x_0$  (and while using a semidefinite relaxation approach). In the presence of noise, it is shown in [8] that one can find an estimate  $\hat{x}$  satisfying

$$\|\hat{x} - e^{i\phi} x\|_2 \leq C_0 \min\left(\|x\|_2, \frac{\|w\|_1}{N\|x\|_2}\right), \quad (1.2)$$

for some  $\phi$ , where  $C_0$  is a constant and  $w$  is the noise vector that is assumed to be bounded so that  $\|w\|_1$  is finite.

The article [27] treats the case in which the input  $x_0$  is of norm one and  $k$ -sparse, and  $a_i$  are independent, zero-mean normal vectors. It shows that if  $N$  is on the order of  $k^2 \log n$ , then recovery is possible (by using a sparse semidefinite relaxation approach).

Here, we treat the real case and random measurements, using reasonable ensembles. The first question addressed is that of stable uniqueness, namely, identifying conditions under which a unique solution can be found in a stable way. Though the results presented here apply to arbitrary sets  $T \subset \mathbb{R}^n$ , the examples we consider are  $T = \mathbb{R}^n$ , and the class of  $k$ -sparse vectors. For the latter,  $O(k \log(n/k))$  measurements suffice for stability. This result is better by a factor of  $k$  than the estimate from [27]. Also, when  $x_0$  can be any vector in  $\mathbb{R}^n$ ,  $O(n)$  measurements suffice, which is also the bound derived in [8].

It turns out that the same complexity parameter, the Gaussian mean-width, captures both linear and quadratic problems. This observation will be discussed in Section 5. It implies that in a rather general sense, the number of measurements required for stable recovery in the quadratic setting, is of the same order of magnitude as the one needed to ensure stability under linear sampling.

The second main result of this article deals with the noisy phase retrieval problem; more specifically, recovery from noisy measurements of the form (1.1), generated by  $x_0 \in T$ . A straightforward approach is to select  $\hat{x}$  that minimizes the empirical risk, but since this leads to a nonconvex problem, finding its global solution is in general not possible. Nonetheless, one can show that if the empirical risk of  $\hat{x}$  is bounded by a given, computable constant (and that depends only on statistical properties of the noise), then  $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$  may be controlled using the Gaussian mean-width of the set. In particular, for reasonable noise levels, in the case of  $k$ -sparse vectors on the Euclidean sphere  $S^{n-1}$ , one can guarantee stable recovery from  $O(k \log(n/k)(\log^2 k + \log^2 \log(n/k)))$  noisy measurements, and when  $x_0$  can be any vector in  $S^{n-1}$ ,  $O(n \log^2 n)$  noisy measurements are sufficient. An exact formulation of both main results is presented in the next section.

A conclusion that could have practical importance is that although the squared error for nonlinear measurements as in (1.1) cannot be minimized directly, it is sufficient to find a point for which the empirical error is bounded by a known constant. Thus, one may use any desired recovery algorithm and check whether the solution  $\hat{x}$  satisfies the bound. For this purpose, methods such as those developed in [40] are advantageous since they allow for arbitrary initial points. As different initializations lead to different choices of  $\hat{x}$ , the algorithm can be used several times until an appropriate value of  $\hat{x}$  is found. The theoretical analysis ensures that such an  $\hat{x}$  is sufficiently close to  $x_0$  or to  $-x_0$  if enough measurements are used.

The remainder of the article is organized as follows. The problem and main results are formulated in Section 2. Stability results in the noise-free setting are developed in Section 3, while the noisy setting is treated in Section 4. In Section 5 the relation between the results in the quadratic case and those in the linear setting is discussed.

Throughout the article we use the following notation. All absolute constants (that is, fixed positive numbers) are denoted by  $c_1, c_2$ , etc. Their values may change from line to line. The expectation is denoted by  $\mathbb{E}$ , and if the probability space is a product space  $(\Omega \times \Omega', \mu \otimes \mu')$ , then  $\mathbb{E}_\mu$  and  $\mathbb{E}_{\mu'}$  are the conditional expectations. In the context of an empirical process,  $P_N f$  denotes the empirical mean of  $f$  while  $Pf$  is its expectation. If  $X$  is a random variable, then  $\|X\|_{L_p} = (\mathbb{E}|X|^p)^{1/p}$ . If  $x \in \mathbb{R}^n$ ,  $\| \|_p$  denotes its  $\ell_p$  norm.  $\ell_p^n$  is the normed space  $(\mathbb{R}^n, \| \|_p)$ , the corresponding unit ball is  $B_p^n$  and  $S^{n-1}$  is the Euclidean sphere in  $\mathbb{R}^n$ .

The relation  $a \sim b$  means that  $a$  is equal to  $b$  up to absolute multiplicative constants, i.e., that there are positive numbers  $c$  and  $C$ , independent of  $a, b$  or any other parameters of the problem, for which  $ca \leq b \leq Ca$ . The inequality  $a \lesssim b$  means that  $a \leq Cb$  for some constant  $C$ . We use  $a \lesssim_{L,\gamma} b$  to denote the fact that the constant  $C$  depends only on  $L$  and  $\gamma$ .

**2. Problem formulation and main results**

Suppose that one is given measurements  $y_i$  as in (1.1). Let  $s$  be a vector in  $\mathbb{R}^N$ , denote by  $\phi(s)$  the length- $N$  vector with elements  $|s_i|^2$  and put  $Ax = ((a_i, x))_{i=1}^N$ . With this notation, (1.1) can be written as

$$y = \phi(Ax) + w. \tag{2.1}$$

Our goal is to study conditions under which stable recovery is possible irrespective of the specific recovery method used, and to develop guarantees that ensure that empirical minimization or approximate empirical minimization lead to an estimate  $\hat{x}$  that is close to  $x$  in a squared-error sense.

**2.1. Assumptions on  $x_0$  and  $a$**

We assume throughout that  $x_0$  lies in a subset  $T$  of  $\mathbb{R}^n$ , which can be arbitrary. It is natural to expect that the number of measurements needed for stable recovery or for noisy recovery depends on the set  $T$ , though the way in which it depends on  $T$  is not obvious. Identifying the correct complexity parameter of  $T$  is one of the main goals of this work.

The assumption on the measurement vectors  $a_i$  is that they are independent, and distributed according to a probability measure  $\mu$  on  $\mathbb{R}^n$  that is isotropic and  $L$ -subgaussian [26,45,10]:

**Definition 2.1.** Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  and let  $a$  be distributed according to  $\mu$ . The measure  $\mu$  is isotropic if for every  $t \in \mathbb{R}^n$ ,  $\mathbb{E}|(a, t)|^2 = \|t\|_2^2$ . It is  $L$ -subgaussian if for every  $t \in \mathbb{R}^n$  and every  $u \geq 1$ ,  $\Pr(|(a, t)| \geq Lu \|t\|_{L_2}) \leq 2 \exp(-u^2/2)$ .

Among the examples of isotropic,  $L$ -subgaussian measures on  $\mathbb{R}^n$  for a constant  $L$  that is independent of the dimension  $n$  are the standard Gaussian measure, the uniform measure on  $\{-1, 1\}^n$  and the volume measure on the “correct” multiple of the unit ball of  $\ell_p^n$  for  $2 \leq p \leq \infty$  (that is, the volume measure on  $c_n n^{1/p} B_p^n$ , where  $c_n \sim 1$ , see, e.g., [4]). Also, it is standard to verify that if  $X$  is a zero-mean, variance 1 random variable that satisfies  $\Pr(|X| \geq Lu) \leq 2 \exp(-u^2/2)$ , then a vector of independent copies of  $X$  is isotropic and  $cL$  subgaussian, for a suitable absolute constant  $c$ .

**Definition 2.2.** A class of functions  $F$  on a probability space  $(\Omega, \mu)$  is  $L$ -subgaussian if for every  $f, h \in F \cup \{0\}$  and every  $t \geq 1$

$$\Pr(|f - h| \geq tL \|f - h\|_{L_2}) \leq 2 \exp(-t^2/2).$$

Clearly, if  $\mu$  is an  $L$ -subgaussian measure on  $\mathbb{R}^n$  then every class of linear functionals on  $\mathbb{R}^n$  is  $L$ -subgaussian.

**2.2. Stability results**

The noise-free setting is studied in Section 3. Since one is given only the absolute values of  $Ax_0$ , it is impossible to distinguish  $x_0$  and  $-x_0$ . Therefore, uniqueness will always be up to the sign of  $x_0$ . If  $\phi(Ax_0)$  is an invertible stable mapping, then it is natural to expect that when  $s \neq t$  and  $s \neq -t$ ,  $\phi(As)$  is far enough from  $\phi(At)$  in some sense.

**Definition 2.3.** The mapping  $\phi(Ax)$  is stable with a constant  $C$  in a set  $T$  if for every  $s, t \in T$ ,

$$\|\phi(At) - \phi(As)\|_1 \geq C \|s - t\|_2 \|s + t\|_2. \tag{2.2}$$

Note that stability in a set is a much stronger property than invertibility. Indeed, for the latter it suffices that if  $s \neq \pm t$  then  $\|\phi(At) - \phi(As)\|_1 > 0$ , but without any quantitative estimate on the difference.

The  $\ell_1$  norm, used on the left-hand side, is the natural way of measuring distances for the quadratic function  $\phi$ , if one wishes to compare the results with the linear case, in which the  $\ell_2$  distance is used (see Section 5 for more details). Using the  $\ell_1$  distance also has a technical advantage, as it simplifies the analysis considerably. Distances based on other  $\ell_p$  norms lead to processes that are harder to control, since higher powers emphasize the “unbounded” or “peaky” parts of a random variable, and make concentration around the mean much harder.

To formulate the stability result, Let  $(g_i)_{i=1}^n$  be independent, standard Gaussian variables. The Gaussian mean-width of  $T \subset \mathbb{R}^n$  is defined by

$$\ell(T) = \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n g_i t_i \right|, \tag{2.3}$$

and set

$$d(T) = \sup_{t \in T} \|t\|_2. \quad (2.4)$$

Geometrically,  $\ell(T)$  measures the best correlation (or width) of  $T$  in a random direction generated by the random vector  $G = (g_1, \dots, g_n)$ . This parameter appears in many different areas of mathematics, and is also essential in the study of compressed sensing problems (see, for example [10] for a survey on such results). We refer the reader to the books [26,44, 38,35] for more information on this parameter and for methods of computing it. For example, it is well known that  $\ell(T)$  can be bounded from above and below (with a possible  $\sqrt{\log n}$  gap between the upper and lower bounds) using the  $\ell_2$  covering numbers of  $T$ .

Consider the two sets

$$\begin{aligned} T_- &= \left\{ \frac{t-s}{\|t-s\|_2} : t, s \in T, t \neq s \right\}, \\ T_+ &= \left\{ \frac{t+s}{\|t+s\|_2} : t, s \in T, t \neq -s \right\}. \end{aligned} \quad (2.5)$$

Let

$$E = \max \left\{ \mathbb{E} \sup_{v \in T_-} \sum_{i=1}^n g_i v_i, \mathbb{E} \sup_{w \in T_+} \sum_{i=1}^n g_i w_i \right\} \quad (2.6)$$

and put

$$\rho_{T,N} = \frac{E}{\sqrt{N}} + \frac{E^2}{N}. \quad (2.7)$$

Finally, for every  $v, w \in \mathbb{R}^n$ , set

$$\kappa(v, w) = \mathbb{E} \left| \langle a, v / \|v\|_2 \rangle \langle a, w / \|w\|_2 \rangle \right|. \quad (2.8)$$

Using these definitions one can state the main result in the noise-free case:

**Theorem 2.4.** *For every  $L \geq 1$  there exist constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  for which the following holds. Let  $\mu$  be an isotropic,  $L$ -subgaussian measure. Then, for  $u \geq c_1$ , with probability at least  $1 - 2 \exp(-c_2 u^2 \min\{N, E^2\})$ , for every  $s, t \in T$ ,*

$$\|\phi(As) - \phi(At)\|_1 \geq \|s - t\|_2 \|s + t\|_2 (\kappa(s - t, s + t) - c_3 u^3 \rho_{T,N}).$$

To put Theorem 2.4 in the right perspective, one has to obtain lower bounds on  $\kappa(s - t, s + t)$  and upper bounds on  $\rho_{T,N}$ . Since the latter depends on the number of measurements  $N$ , its behavior provides insight into the number of measurements that are needed for stability.

The value of  $\kappa(v, w)$  may be bounded using several methods, as will be explained in Section 3.2.2. One natural example in which  $\inf_{v, w \in S^{n-1}} \kappa(v, w)$  is bounded from below, is when  $a$  satisfies a small ball assumption, namely, that for every  $t \in \mathbb{R}^n$  and every  $\varepsilon > 0$ ,

$$\Pr(|\langle a, t \rangle| \leq \|t\|_2 \varepsilon) \leq c\varepsilon. \quad (2.9)$$

It turns out that if (2.9) holds, then  $\inf_{v, w \in S^{n-1}} \kappa(v, w) \geq c_1$ , and  $c_1$  depends only on the constant  $c$  in (2.9). This assumption is satisfied for a large family of measures, such as the Gaussian measure on  $\mathbb{R}^n$  (see Section 3.2.2 for further details).

As for  $\rho_{T,N}$ , one may show, for example, that if  $T$  is the set of  $k$ -sparse vectors in  $\mathbb{R}^n$ , then  $\rho_{T,N} \lesssim \sqrt{k \log(en/k)/N}$ . Hence, under (2.9), and since  $\inf_{v, w \in S^{n-1}} \kappa(v, w) \geq c_1$ , it suffices to select  $N$  large enough to ensure that  $c_2 u^3 \rho_{T,N} \leq c_1/2$  to obtain a stability result. This leads to the following estimate:

**Corollary 2.5.** *For every  $L, c > 0$  there exist constants  $c_1, c_2, c_3$  and  $c_4$  that depend only on  $L$  and  $c$  for which the following holds. Let  $T$  be the set of  $k$ -sparse vectors in  $\mathbb{R}^n$ , set  $\mu$  to be an isotropic,  $L$ -subgaussian measure, and assume that  $a$  is distributed according to  $\mu$ . If  $a$  satisfies (2.9) with constant  $c$ , then for  $u > c_1$  and  $N \geq c_2 u^3 k \log(en/k)$ , with probability at least  $1 - 2 \exp(-c_3 u^2 k \log(en/k))$ , for every  $s, t \in T$*

$$\|\phi(As) - \phi(At)\|_1 \geq c_4 \|s - t\|_2 \|s + t\|_2.$$

In particular, the result is true for a random Gaussian matrix  $A$ , and  $c_1, c_2, c_3$  and  $c_4$  are absolute constants.

Since the analysis in the case of linear measurements uses very similar machinery (see [31] and Section 5), it should come as no surprise that the same complexity parameter appears, and leads to similar estimates. For example, the number of measurements for which stable recovery is guaranteed in the quadratic and linear settings is the same up to multiplicative constants – at least for ensembles that have a well-behaved  $\inf_{v, w \in S^{n-1}} \kappa(v, w)$ .

In Section 3.2 we study other choices of  $T$ , and the number of measurements needed in order to guarantee stability.

### 2.3. Noisy recovery results

Section 4 is devoted to the case in which the measurements are contaminated by iid noise. The goal is to find a point  $\hat{x}$  for which  $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$  is small, using the data  $(a_i, y_i)_{i=1}^N$ , and the fact that  $y$  is generated according to (1.1) for some  $x_0 \in T$ .

A natural approach is to recover  $x_0$  from  $y$  by minimizing the empirical risk:

$$\min_x \ell_x = \min_x \frac{1}{N} \sum_{i=1}^N |y_i - \langle a_i, x \rangle|^2|^p, \tag{2.10}$$

for some  $1 < p \leq 2$  (which will be taken to be very close to 1). The objective in (2.10) is not convex, and therefore it is not clear how to find its minimizer. Fortunately, it is possible to show that in order to find an estimate  $\hat{x}$  close to  $x_0$  one does not need to strictly minimize (2.10). Instead, it is sufficient to find a point  $\hat{x}$  for which the empirical risk of  $\hat{x}$  is small enough.

Suppose that  $w$  is  $L$ -subgaussian and that  $\|w\|_{L_2} = \sigma$ . For a given  $1 < p \leq 2$ , and  $u \geq 1$  (which will later on govern the probability estimates), one produces  $\hat{x}$  satisfying

$$\frac{1}{N} \sum_{i=1}^N |\langle a_i, \hat{x} \rangle|^2 - y_i|^p \leq \mathbb{E}|w|^p + uL^p \frac{\sigma^p}{\sqrt{N}}. \tag{2.11}$$

It follows that, with high probability, such a point  $\hat{x}$  is close to either  $x_0$  or to  $-x_0$ .

**Theorem 2.6.** *For every  $\kappa > 0$  and every  $L \geq 1$  there exists constants  $c_1, c_2, c_3$  and  $c_4$  that depend only on  $L$  and  $\kappa$  for which the following holds. Let  $a$  be distributed according to an isotropic,  $L$ -subgaussian measure, assume that  $T \subset \mathbb{R}^n$  is a bounded set and that  $\kappa_T \geq \kappa$  where  $\kappa_T = \inf_{s, t \in T} \kappa(s, t)$ . Let  $w$  be  $L$ -subgaussian, set*

$$\beta_N = \max\{c_1((1 + \sigma + d^2(T)) \log N + d^2(T)E^2), e^2\},$$

and put

$$p = 1 + 1/\log \beta_N.$$

If  $\hat{x}$  satisfies (2.11) for  $c_2 \leq u \leq N$ , then with probability at least  $1 - 2 \exp(-c_3 u^{1/3})$ , for  $\rho = \rho_{T, N}$  and  $d = d(T)$ ,

$$\|x_0 - \hat{x}\|_2 \|x_0 + \hat{x}\|_2 \leq c_4 u \max\left\{(\sigma + d + 1)\rho \log \beta_N, \frac{\sigma}{N^{1/4}} \sqrt{\log \beta_N}\right\}.$$

We point out that the results in Section 4 are stated in a slightly stronger form using the decay properties of the noise, though under our assumptions, the two formulations are, in fact, equivalent.

Section 4.4 is devoted to some implications of Theorem 2.6. In particular, it follows that for  $k$ -sparse vectors on the sphere, stable recovery is possible from  $O(k \log(n/k)(\log k + \log \log(en/k))^2)$  noisy measurements (this estimate is off only by a  $\log k$  factor from the optimal estimate in the linear case). Also, if  $x_0$  can be any vector in  $S^{n-1}$ , then the theorem implies that  $O(n \log n)$  noisy measurements suffice.

### 2.4. Method of analysis

The method used in the proof of both main results is based on properties of the empirical process indexed by  $\{fh: f \in F, h \in H\}$ . Although the estimates involved are true in a far more general setting, for the sake of simplicity they are formulated only in the context required here. We refer the reader to [29,30] for the more general statement and precise results, and to [25] for applications of these results to similar problems in statistics (e.g., regression).

Here,  $F$  and  $H$  are classes of linear functionals or of absolute values of linear functionals on  $\mathbb{R}^n$ , which is endowed with an isotropic,  $L$ -subgaussian probability measure  $\mu$ . In Theorem 2.7, for the stability result  $F = \{\langle t, \cdot \rangle: t \in T_+\}$  and  $H = \{\langle t, \cdot \rangle: t \in T_-\}$ , while in the noisy case,  $F = \{\langle t - x_0, \cdot \rangle: t \in T\}$  and  $H = \{\langle t + x_0, \cdot \rangle: t \in T\}$ . In both scenarios, the two indexing sets are denoted by  $T_1, T_2 \subset \mathbb{R}^n$ .

**Theorem 2.7.** (See [30].) For every  $L \geq 1$  there are constants  $c_1, c_2, c_3$  and  $c_4$  that depend only on  $L$  and for which the following hold. Let  $T_1, T_2 \subset \mathbb{R}^n$  of cardinality at least 2 and set  $F$  and  $H$  to be the corresponding classes. Assume without loss of generality that  $\ell(T_1)/d(T_1) \geq \ell(T_2)/d(T_2)$ . Then, for every  $u \geq c_1$ , with probability at least

$$1 - 2 \exp(-c_2 u^2 \min\{N, (\ell(T_1)/d(T_1))^2\}),$$

$$\sup_{f \in F, h \in H} \left| \frac{1}{N} \sum_{i=1}^N f(a_i)h(a_i) - \mathbb{E}fh \right| \leq c_3 u^3 \left( d(T_2) \frac{\ell(T_1)}{\sqrt{N}} + \frac{\ell^2(T_1)}{N} \right).$$

If  $(\varepsilon_i)_{i=1}^N$  are independent, symmetric  $\{-1, 1\}$ -valued random variables that are also independent of  $(a_i)_{i=1}^N$ , then with the same probability estimate (relative to the product measure  $(\varepsilon \otimes \mu)^N$ ),

$$\sup_{f \in F, h \in H} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i f(a_i)h(a_i) \right| \leq c_3 u^3 \left( d(T_2) \frac{\ell(T_1)}{\sqrt{N}} + \frac{\ell^2(T_1)}{N} \right).$$

In particular, for every  $q \geq 2$ ,

$$\left\| \sup_{f \in F, h \in H} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i f(a_i)h(a_i) \right| \right\|_{L_q} \leq c_4 q^{3/2} \left( d(T_2) \frac{\ell(T_1)}{\sqrt{N}} + \frac{\ell^2(T_1)}{N} \right).$$

### 3. Stability results

In this section we present the proof of Theorem 2.4, followed by estimates on the values of  $\kappa(v, w)$  and  $\rho_{T,N}$  appearing in the theorem.

#### 3.1. Proof of Theorem 2.4

Observe that

$$\|\phi(At) - \phi(As)\|_1 = \sum_{i=1}^N \left| |\langle a_i, t \rangle|^2 - |\langle a_i, s \rangle|^2 \right| = \sum_{i=1}^N |\langle a_i, s-t \rangle \langle a_i, s+t \rangle|. \tag{3.1}$$

Therefore, to establish the desired stability result, it suffices to show that

$$\inf_{\{s,t \in T, s \neq \pm t\}} \frac{\|\phi(At) - \phi(As)\|_1}{\|s-t\|_2 \|s+t\|_2} \geq C,$$

i.e., that

$$\inf_{\{s,t \in T, s \neq \pm t\}} z_{t,s} \geq \frac{C}{N}, \tag{3.2}$$

where

$$z_{t,s} = \frac{1}{N} \sum_{i=1}^N \left| \left\langle a_i, \frac{s-t}{\|s-t\|_2} \right\rangle \left\langle a_i, \frac{s+t}{\|s+t\|_2} \right\rangle \right|. \tag{3.3}$$

Since  $\kappa(s-t, s+t) = \mathbb{E}z_{t,s}$ , if  $\kappa(s-t, s+t)$  is very small, then a random selection of  $a_i$  is unlikely to lead to (3.2). Therefore, a reasonable pre-requisite for a stability result is that  $\inf_{s \neq \pm t, s, t \in T} \kappa(s-t, s+t)$  is bounded away from zero. Indeed, with this assumption, one may obtain the following stability result.

**Proposition 3.1.** For every  $L \geq 1$  there exist constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  for which the following holds. Let  $\mu$  be an isotropic,  $L$ -subgaussian measure on  $\mathbb{R}^n$  and set  $a$  to be a random vector distributed according to  $\mu$ . Then, for  $u \geq c_1$ , with probability at least  $1 - 2 \exp(-c_2 u^2 \min\{N, E^2\})$ , for every  $s, t \in T$ ,

$$z_{s,t} \geq (\kappa(s-t, s+t) - c_3 u^3 \rho_{T,N}) \|s-t\|_2 \|s+t\|_2.$$

**Proof.** Observe that

$$\sup |z_{t,s} - \kappa(s-t, s+t)| = \sup_{v \in T_+, w \in T_-} \left| \frac{1}{N} \sum_{i=1}^N \langle a_i, v \rangle \langle a_i, w \rangle - \mathbb{E} \langle a_i, v \rangle \langle a_i, w \rangle \right|.$$

By Theorem 2.7 for  $F = \{|\langle v, \cdot \rangle| : v \in T_+\}$  and  $H = \{|\langle w, \cdot \rangle| : w \in T_-\}$ , it follows that if  $N \geq c_1 E^2$  and  $u \geq c_2$ , then with probability at least  $1 - 2 \exp(-c_3 u^2 E^2)$ ,

$$\sup_{v \in T_-, w \in T_+} \left| \frac{1}{N} \sum_{i=1}^N |\langle a_i, v \rangle \langle a_i, w \rangle| - \mathbb{E} |\langle a_i, v \rangle \langle a_i, w \rangle| \right| \leq c_4 u^3 \rho_{T,N}. \tag{3.4}$$

The claim now follows immediately from the definition of  $z_{s,t}$  and of  $\kappa(s - t, s + t)$ .  $\square$

### 3.2. Computing $\kappa$ and $\rho_{T,N}$

#### 3.2.1. Bounding $\rho_{T,N}$

It is well known that if  $T \subset \mathbb{R}^n$  then  $\ell(T)$  (and therefore,  $\rho_{T,N}$  as well) is determined by the Euclidean metric structure of  $T$ . This is the outcome of the celebrated majorizing measures/generic chaining theory (see the books [26,12,44] for a detailed exposition on this topic). In the examples we present here, the following estimate, which is, in general, suboptimal, suffices.

**Definition 3.2.** Let  $(T, d)$  be a compact metric space. For every  $\varepsilon > 0$ , let  $N(T, d, \varepsilon)$  be the smallest number of open balls of radius  $\varepsilon$  needed to cover  $T$ . The numbers  $N(T, d, \varepsilon)$  are called the  $\varepsilon$ -covering numbers of  $T$  relative to the metric  $d$ .

Given  $T \subset \mathbb{R}^n$ , set  $N(T, \varepsilon) = N(T, \|\cdot\|_2, \varepsilon)$ , i.e., the covering numbers relative to the Euclidean metric.

**Proposition 3.3.** There exist absolute constants  $c$  and  $C$  for which the following holds. If  $T \subset \mathbb{R}^n$  then

$$c \sup_{\varepsilon > 0} \varepsilon \sqrt{\log N(T, \varepsilon)} \leq \ell(T) \leq C \int_0^{d(T)} \sqrt{\log N(T, \varepsilon)} d\varepsilon.$$

The upper bound is due to Dudley [13] and the lower to Sudakov [43]. The proof of both bounds may be found, for example, in [26,38,12].

It is straightforward to verify that the gap between the upper and lower bounds in Proposition 3.3 is at most  $\sim \sqrt{\log n}$ , and in all the examples we study below, the resulting upper estimate is sharp.

#### 3.2.2. Bounding $\kappa$

Here, we present two simple methods for bounding  $\inf_{v,w \in S^{n-1}} \kappa(v, w)$  from below. These methods are not the only possibilities by which one may obtain such a bound; rather, they serve as an indication that the assumption on  $\kappa$  is less restrictive than may appear at first glance.

Recall the small ball assumption: for every  $t \in \mathbb{R}^n$  and every  $\varepsilon > 0$ ,  $\Pr(|\langle a, t \rangle| \leq \varepsilon \|t\|_2) \leq c\varepsilon$ .

**Lemma 3.4.** If  $a$  satisfies the small ball assumption with constant  $c$  then

$$\inf_{v,w \in S^{n-1}} \kappa(v, w) \geq \kappa,$$

where  $\kappa$  depends only on  $c$ .

**Proof.** Consider  $\varepsilon$  for which  $c\varepsilon \leq 1/4$ . Then, for every  $v \in S^{n-1}$ , there is an event of measure at least  $3/4$  on which  $|\langle v, a \rangle| \geq \varepsilon$ . Hence, for two fixed vectors  $v, w \in S^{n-1}$ ,

$$\Pr(\{|\langle v, a \rangle| \geq \varepsilon\} \cap \{|\langle w, a \rangle| \geq \varepsilon\}) \geq 1/2,$$

and thus  $\mathbb{E}|\langle v, a \rangle \langle w, a \rangle| \geq \varepsilon^2/2$ .  $\square$

The small ball assumption is true in many cases. The simplest example is the standard Gaussian measure on  $\mathbb{R}^n$ . Indeed, if  $a = (g_1, \dots, g_n)$  then  $|\langle a, v \rangle|$  is distributed as  $\|v\|_2 |g|$  and the small ball property follows immediately by applying the  $L_\infty$  estimate on the density of  $g$ .

A more general example is based on the notion of log-concavity. A measure  $\mu$  on  $\mathbb{R}^n$  is called log-concave if for every nonempty, Borel measurable sets  $A, B \subset \mathbb{R}^n$ , and any  $0 \leq \lambda \leq 1$ ,  $\mu(\lambda A + (1 - \lambda)B) \geq \mu^\lambda(A) \mu^{1-\lambda}(B)$ , where  $\lambda A + (1 - \lambda)B = \{\lambda a + (1 - \lambda)b : a \in A, b \in B\}$ . It is well known that  $\mu$  is a log-concave measure if and only if it has a density of the form  $\exp(\phi)$  for a concave function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ .

The following lemma is standard (see e.g. [17,6,34]).



**Lemma 3.5.** *There exists an absolute constant  $c$  for which the following holds. Let  $a$  be distributed according to an isotropic, symmetric, log-concave measure. Then for every  $\theta \in S^{n-1}$ ,  $\langle a, \theta \rangle$  is distributed according to an isotropic, symmetric, log-concave measure on  $\mathbb{R}$ . Also, if  $f_\theta$  is the density of  $\langle a, \theta \rangle$  then  $\|f_\theta\|_\infty \leq c$ .*

The desired small ball estimate clearly follows from the lemma, since

$$\Pr(|\langle a, \theta \rangle| \leq \varepsilon) = \int_{-\varepsilon}^{\varepsilon} f_\theta(t) dt \leq 2c\varepsilon.$$

Among the family of log-concave measures are volume measures on convex symmetric bodies (i.e. measures that have a constant density on the body and zero outside the body). Moreover, it can be shown (see, e.g., [17]), that for every convex body  $K \subset \mathbb{R}^n$  there is an invertible linear operator  $A$  for which the volume measure on  $AK$  is also isotropic.

Another example of log-concave measures on  $\mathbb{R}^n$  are product measures of log-concave measures on  $\mathbb{R}$ . If  $X$  is a real valued, symmetric, log-concave random variable (i.e. with a log-concave density) with variance one, and  $X_1, \dots, X_n$  are iid copies of  $X$ , then  $a = (X_1, \dots, X_n)$  is an isotropic log-concave measure on  $\mathbb{R}^n$ . Standard examples for log-concave measures on  $\mathbb{R}$  are those with a density proportional to  $\exp(-c_p|t|^p)$  for  $p \geq 1$ .

The second method for obtaining a lower bound on  $\kappa$ , and which we only outline, is based on a Paley–Zygmund argument.

**Corollary 3.6.** *Let  $X$  be a symmetric, variance 1 random variable, with a finite  $L_{2q}$  moment for some  $q > 2$ . If  $a = (X_1, \dots, X_n)$  then*

$$\inf_{v, w \in S^{n-1}} \kappa(v, w) \geq c(\mathbb{E}X^4 - 1)^{1/2},$$

where  $c$  depends on  $q$  and on  $\|X\|_{L_{2q}}$ .

Observe that the two assumptions are not very restrictive, since  $a$  is assumed to be isotropic and  $L$ -subgaussian. Hence, if  $a = (X_1, \dots, X_n)$ , then  $\|X\|_{L_q} \leq Lq$  for every  $q \geq 2$  (see Section 4.1). Also note that for any random variable  $X$ ,  $\mathbb{E}X^4 \geq (\mathbb{E}X^2)^2 = 1$ , so that the square root is well defined.

The proof of the Corollary relies on the Paley–Zygmund lemma.

**Lemma 3.7.** (See [11].) *Let  $Z$  be a random variable, set  $0 < p < q$  and put  $c_{p,q} = \|Z\|_{L_p} / \|Z\|_{L_q}$ . Then, for every  $0 \leq \lambda \leq 1$ ,*

$$\Pr(|Z| > \lambda \|Z\|_{L_p}) \geq [(1 - \lambda^p) c_{p,q}^p]^{q/q-p}.$$

In particular,  $\mathbb{E}|Z| \geq c_1 \|Z\|_{L_p}$ , where  $c_1$  depends only on  $p, q$  and  $c_{p,q}$ .

Fix  $p = 2$  and  $q > 2$ . Assume that  $(X_i)_{i=1}^n$  are independent copies of a symmetric, variance 1 random variable and set  $a = (X_1, \dots, X_n)$ . If  $v, w \in S^{n-1}$ , then a straightforward computation shows that

$$\begin{aligned} \mathbb{E}|\langle a, v \rangle \langle a, w \rangle|^2 &= \sum_{i \neq j} v_i^2 w_j^2 + 2 \sum_{i \neq j} (v_i w_i)(v_j w_j) + \mathbb{E}X^4 \sum_{i=1}^n v_i^2 w_i^2 \\ &= \sum_{i \neq j} (v_i^2 w_j^2 + (v_i w_i)(v_j w_j)) + \langle v, w \rangle^2 + (\mathbb{E}X^4 - 1) \sum_{i=1}^n v_i^2 w_i^2. \end{aligned} \tag{3.5}$$

Using the fact that  $\|v\|_2 = \|w\|_2 = 1$ , (3.5) reduces to

$$\mathbb{E}|\langle a, v \rangle \langle a, w \rangle|^2 = 1 + 2\langle v, w \rangle^2 - 2 \sum_{i=1}^n v_i^2 w_i^2 + (\mathbb{E}X^4 - 1) \sum_{i=1}^n v_i^2 w_i^2. \tag{3.6}$$

Consider two cases. First, if  $\sum_{i=1}^n v_i^2 w_i^2 \leq 1/10$ , and since  $\mathbb{E}X^4 \geq (\mathbb{E}X^2)^2 = 1$ , then it follows from (3.6) that

$$\mathbb{E}|\langle a, v \rangle \langle a, w \rangle|^2 \geq 1/2.$$

On the other hand, if the reverse inequality holds, then using

$$\sum_{i \neq j} (v_i^2 w_j^2 + (v_i w_i)(v_j w_j)) + \langle v, w \rangle^2 = \sum_{i > j} (v_i w_j + v_j w_i)^2 + \langle v, w \rangle^2 \geq 0,$$



and applying (3.5),

$$\mathbb{E}|\langle a, v \rangle \langle a, w \rangle|^2 \geq (\mathbb{E}X^4 - 1) \sum_{i=1}^n v_i^2 w_i^2 \geq (\mathbb{E}X^4 - 1)/10.$$

**Proof of Corollary 3.6.** Assume that  $X \in L_{2q}$  for some  $q > 2$ . Observe that if  $v \in S^{n-1}$  then for every  $2 \leq r \leq 2q$ ,  $\|\langle a, v \rangle\|_{L_r} \leq c_r \|X\|_{L_r}$ . Indeed, by a Rosenthal type inequality (see, e.g., [11, Section 1.5]),

$$\left\| \sum_{i=1}^n v_i X_i \right\|_{L_r} \leq c_r \max \left\{ \left( \sum_{i=1}^n v_i^2 \mathbb{E}X_i^2 \right)^{1/2}, \left( \sum_{i=1}^n v_i^r \mathbb{E}|X_i|^r \right)^{1/r} \right\}.$$

Since  $\|X\|_{L_r} \geq \|X\|_{L_2}$  and  $\|v\|_r \leq \|v\|_2 = 1$ , the claim follows.

Therefore,  $\sup_{v \in S^{n-1}} \|\langle a, v \rangle\|_{L_{2q}} \leq c_q \|X\|_{L_{2q}}$ , and thus,

$$\|\langle a, v \rangle \langle a, w \rangle\|_{L_q} \leq \|\langle a, v \rangle\|_{L_{2q}} \|\langle a, w \rangle\|_{L_{2q}} \lesssim_q \|X\|_{L_{2q}}^2.$$

Let  $Z = \langle a, v \rangle \langle a, w \rangle$ . Using the notation of Lemma 3.7,  $c_{2,q} \gtrsim (\mathbb{E}|X|^4 - 1)^{1/2} / \|X\|_{L_{2q}}^2$ . Hence, for every  $v, w \in S^{n-1}$ ,  $\mathbb{E}|\langle a, v \rangle \langle a, w \rangle| \geq c(\mathbb{E}|X|^4 - 1)^{1/2}$ , where  $c$  depends only on  $q$  and on  $\|X\|_{L_{2q}}$ , as claimed.  $\square$

Observe that if  $\mathbb{E}X^4 = 1$ , then it is possible that  $\mathbb{E}|\langle a, v \rangle \langle a, w \rangle| = 0$ , even for  $v, w$  of the specific form one would like to control – namely,  $v = (s + t)/\|s + t\|_2$  and  $w = (s - t)/\|s - t\|_2$  for  $s \neq \pm t$ . Indeed, let  $X$  be a symmetric,  $\{-1, 1\}$ -valued random variable (and in particular, it is  $L$ -subgaussian as well). Let  $(e_i)_{i=1}^n$  be the standard basis in  $\mathbb{R}^n$  and set  $s = e_1$ ,  $t = e_2$ . It is straightforward to verify that in this case,  $\mathbb{E}|\langle a, v \rangle \langle a, w \rangle| = 0$ , and therefore, the assumption on  $\mathbb{E}X^4$  cannot be relaxed.

### 3.3. Examples

We now consider a few special cases in which Theorem 2.4 can be applied. To that end, explicit expressions for  $\rho_{T,N}$  are required for the sets of interest.

#### 3.3.1. Entire space $T = \mathbb{R}^n$

If  $T = \mathbb{R}^n$  then  $T_+ = T_- = S^{n-1}$ . Therefore,

$$E = \mathbb{E} \sup_{x \in S^{n-1}} \sum_{i=1}^n g_i x_i = \mathbb{E} \left( \sum_{i=1}^n g_i^2 \right)^{1/2} \sim \sqrt{n},$$

implying that

$$\rho_{\mathbb{R}^n, N} \lesssim \left( \sqrt{\frac{n}{N}} + \frac{n}{N} \right).$$

**Corollary 3.8.** For every  $L \geq 1$  there are constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  and for which the following holds. If  $\inf_{v, w \in S^{n-1}} \kappa(v, w) \geq \kappa$ ,  $u \geq c_1$  and  $N \geq c_2 u^3 n / \kappa^2$ , then with probability at least  $1 - 2 \exp(-c_3 u^2 n)$ , for every  $s, t \in \mathbb{R}^n$ ,

$$\|\phi(As) - \phi(At)\|_1 \geq \frac{\kappa}{2} \|s - t\|_2 \|s + t\|_2.$$

The corollary follows from the fact that with this choice of  $N$ ,  $cu^3 \rho_{T,N} \leq \kappa/2$ .

When  $\kappa$  is given by a constant, independent of the dimension  $n$ , Corollary 3.8 implies that it is sufficient to choose  $N \sim n$  to ensure stable recovery with high probability.

#### 3.3.2. Sparse vectors

Next, let  $T = S_k$ , the set of  $k$ -sparse vectors in  $\mathbb{R}^n$ . Denote  $U_k = \{x \in S^{n-1} : \|x\|_0 \leq k\}$  and observe that  $T_+, T_- \subset U_{2k}$ . Therefore,

$$E = \mathbb{E} \sup_{x \in U_{2k}} \sum_{i=1}^n g_i x_i = \mathbb{E} \left( \sum_{i=1}^{2k} (g_i^*)^2 \right)^{1/2},$$

where  $(v_i^*)_{i=1}^n$  is a monotone rearrangement of  $(|v_i|)_{i=1}^n$ . It is standard to verify (see, e.g., [18]) that there is an absolute constant  $c$  satisfying that for every  $1 \leq k \leq n/4$ ,

$$\mathbb{E} \left( \sum_{i=1}^{2k} (g_i^*)^2 \right)^{1/2} \leq c \sqrt{k \log(en/k)}.$$

Therefore,

$$\rho_{S_k, N} \lesssim \left( \sqrt{\frac{k \log(en/k)}{N}} + \frac{k \log(en/k)}{N} \right).$$

**Corollary 3.9.** For every  $L \geq 1$  there are constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  and for which the following holds. If  $\inf_{v, w \in U_k} \kappa(v, w) \geq \kappa, u \geq c_1$  and  $N \geq c_2 u^3 k \log(en/k) / \kappa^2$ , then with probability at least  $1 - 2 \exp(-c_3 u^2 k \log(en/k))$ , for every  $s, t \in S_k$ ,

$$\|\phi(As) - \phi(At)\|_1 \geq \frac{\kappa}{2} \|s - t\|_2 \|s + t\|_2.$$

When  $\kappa$  is an absolute constant, Corollary 3.9 implies that it is sufficient to choose  $N \sim k \log(en/k)$  to ensure stable recovery with high probability.

3.3.3. Finite set

Assume now that  $T$  is a finite set. Then  $T_+, T_- \subset S^{n-1}$  are of cardinality at most  $|T|^2$ . A straightforward application of the union bound to each random variable  $\sum_{i=1}^n v_i g_i$  shows that if  $V \subset \mathbb{R}^n$  is a finite set, then

$$\mathbb{E} \sup_{v \in V} \sum_{i=1}^n g_i v_i \lesssim \sqrt{\log |V|} d(V).$$

Therefore,  $E \lesssim \sqrt{\log |T|^2} \sim \sqrt{\log |T|}$ , implying that

$$\rho_{T, N} \lesssim \left( \sqrt{\frac{\log |T|}{N}} + \frac{\log |T|}{N} \right).$$

**Corollary 3.10.** For every  $L \geq 1$  there are constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  and for which the following holds. If  $\inf_{v, w \in T_+} \kappa(v, w) \geq \kappa, u \geq c_1$  and  $N \geq c_2 u^3 \log |T| / \kappa^2$ , then with probability at least  $1 - 2 \exp(-c_3 u^2 \log |T|)$ , for every  $s, t \in T$ ,

$$\|\phi(As) - \phi(At)\|_1 \geq \frac{\kappa}{2} \|s - t\|_2 \|s + t\|_2.$$

In this case, with constant  $\kappa, N \sim \log |T|$  measurements ensure stable recovery with high probability.

3.3.4. Block sparse vectors

Let  $T = S_k^d$  be the set of block sparse vectors of size  $d$ . Let  $(I_\ell), \ell = 1, \dots, n/d$  be a decomposition of  $\{1, \dots, n\}$  to disjoint blocks of cardinality  $d$ , and set  $W_k$  to be the vectors in the unit sphere, supported on at most  $k$  blocks. Then  $T_+, T_- \subset W_{2k}$ , and it remains to estimate

$$E = \mathbb{E} \sup_{v \in W_{2k}} \sum_{i=1}^n g_i v_i.$$

**Lemma 3.11.** There exist absolute constants  $c_1$  and  $c_2$  for which the following holds. For every  $0 < \varepsilon < 1/2$ ,

$$\log N(W_k, \varepsilon) \leq c_1 (k \log(en/(dk)) + dk \log(5/\varepsilon)).$$

Therefore,

$$\mathbb{E} \sup_{v \in V} \sum_{i=1}^n g_i v_i \leq c_2 \sqrt{k} (\sqrt{\log(en/(dk))} + \sqrt{d}).$$

**Proof.** Let  $I_J = \{i \in I_j, j \in J\}$  and observe that

$$W_k = \bigcup_{\{J \subset \{1, \dots, n\}: |J|=k\}} S^{I_J},$$

where for every  $I \subset \{1, \dots, n\}$ ,  $S^I$  is the Euclidean sphere on the coordinates  $I$ . Clearly, there are at most  $\binom{n}{k}$  such subsets  $J$ . Using a standard volumetric estimate (see, e.g., [38,10]), for every fixed set  $J$  and every  $\varepsilon < 1/2$ , one needs at most  $(5/\varepsilon)^{d|J|} = (5/\varepsilon)^{dk}$  Euclidean balls of radius  $\varepsilon$  to cover  $S^{I_J}$ . Therefore, for every  $0 < \varepsilon < 1/2$ ,

$$\log N(W_k, \varepsilon B_2^n) \lesssim k \log(en/(dk)) + dk \log(5/\varepsilon),$$

as claimed.

The second part of the claim is an immediate consequence of Proposition 3.3 and the fact that  $N(T, \varepsilon)$  is a decreasing function of  $\varepsilon$ . □

**Corollary 3.12.** For every  $L \geq 1$  there are constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  and for which the following holds. If  $\inf_{v, w \in W_k} \kappa(v, w) \geq \kappa, u \geq c_1$  and  $N \geq c_2 u^3 (k \log(en/(dk)) + dk) / \kappa^2$ , then with probability at least  $1 - 2 \exp(-c_3 u (k \log(en/(dk)) + dk))$ , for every  $s, t \in S_k^d$ ,

$$\|\phi(As) - \phi(At)\|_1 \geq \frac{\kappa}{2} \|s - t\|_2 \|s + t\|_2.$$

When  $\kappa$  is constant we conclude that  $N \sim k(\log(en/(kd)) + d)$  measurements are needed for stability. This result is consistent with that of [14] which shows that the same value  $N$  ensures that a random Gaussian matrix satisfies the block restricted-isometry constant.

#### 4. Noisy measurements

Next, consider the phase retrieval problem in the presence of noise. The goal is to find an estimate  $\hat{x}$  of the true signal  $x_0$  that is close to  $x_0$  (or  $-x_0$ ) in a squared-error sense.

Suppose that

$$y_i = |\langle a_i, x_0 \rangle|^2 + w_i, \quad i = 1, \dots, N \tag{4.1}$$

for some  $x_0 \in T$ . Let  $a$  be an isotropic,  $L$ -subgaussian random vector and assume that the noise  $w$  is independent of  $a$ , symmetric, and of reasonable decay properties, that will be specified in Assumption 4.1 below.

**Question 4.1.** Given  $(a_i, y_i)_{i=1}^N$ , combined with the information that the noisy data  $y_i$  is generated by a point  $x_0 \in T$  via (4.1), is it possible to produce an estimate  $\hat{x} \in T$  for which  $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$  is small?

**Definition 4.2.** Given  $T \subset \mathbb{R}^n$  and an integer  $N$ , the procedure  $\hat{x}$  is an  $\varepsilon$ -recovery procedure with confidence parameter  $\delta$ , if for every  $x_0 \in T$ , with probability at least  $1 - \delta$  over  $N$ -samples  $(a_i, y_i)_{i=1}^N$  generated according to (4.1),

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 < \varepsilon.$$

Note that the error is measured by the product  $\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2$ , since it is impossible to distinguish between  $x_0$  and  $-x_0$ .

The answer to Question 4.1 is affirmative, and one may obtain quantitative estimates on  $\varepsilon$  and  $\delta$  in Definition 4.2 when  $\hat{x}$  is selected to have a well-behaved empirical risk, as will be explained below.

##### 4.1. Preliminaries: $\psi_\alpha$ random variables

Throughout this section it is assumed that the noise  $w$  decays quickly. The rate of decay is quantified using the notion of  $\psi_\alpha$  random variables (see [26,45,10] as general references for properties of  $\psi_\alpha$  random variables).

**Definition 4.3.** Let  $X$  be a random variable. For  $1 \leq \alpha \leq 2$  let

$$\|X\|_{\psi_\alpha} = \inf\{C > 0: \mathbb{E} \exp(|X/C|^\alpha) \leq 2\},$$

and denote by  $L_{\psi_\alpha}$  the set of random variables for which  $\|X\|_{\psi_\alpha} < \infty$ .

The  $\psi_\alpha$  norm can be characterized using information on the tail of  $X$ . Indeed, there exists an absolute constant  $c$ , for which, if  $t \geq 1$ , then  $Pr(|X| \geq t) \leq 2 \exp(-ct^\alpha / \|X\|_{\psi_\alpha}^\alpha)$ . The reverse direction is also true, that is, if  $Pr(|X| \geq t) \leq 2 \exp(-t^\alpha / A^\alpha)$ , then  $\|X\|_{\psi_\alpha} \leq c_1 A$  for an absolute constant  $c_1$ .

It is well known that  $\|\cdot\|_{\psi_\alpha}$  is a norm on  $L_{\psi_\alpha}$ , and that

$$\|X\|_{\psi_\alpha} \sim \sup_{p \geq 1} \frac{\|X\|_{L_p}}{p^{1/\alpha}}.$$

Therefore, if  $p \geq 1$ , then

$$\|X\|_{L_p} \lesssim \|X\|_{\psi_\alpha} p^{1/\alpha}. \tag{4.2}$$

In the language of the previous section,  $X$  is  $L$ -subgaussian if and only if  $\|X\|_{\psi_2} \leq cL\|X\|_{L_2}$ . Since the  $\psi_\alpha$  norms have a natural hierarchy, it follows that if  $X$  is  $L$ -subgaussian then

$$\|X\|_{L_2} \lesssim \|X\|_{\psi_1} \leq \|X\|_{\psi_2} \leq cL\|X\|_{L_2}.$$

Therefore, if  $X$  is  $L$ -subgaussian and mean-zero then  $\|X\|_{\psi_2} \sim_L \sigma_X$ , where  $\sigma_X$  is the standard deviation of  $X$ .

A straightforward application of the tail behavior of a  $\psi_\alpha$  random variable implies that if  $X_1, \dots, X_N$  are independent copies of  $X$  and  $t \geq 1$ , then

$$Pr\left(\max_{i \leq N} |X_i| \geq t \log^{1/\alpha} N \|X\|_{\psi_\alpha}\right) \leq 2 \exp(-c_2 t^{1/\alpha});$$

hence,

$$\| \max_{1 \leq i \leq N} X_i \|_{\psi_\alpha} \leq c_3 \|X\|_{\psi_\alpha} \log^{1/\alpha} N. \tag{4.3}$$

From the definition of the  $\psi_\alpha$  norm it is evident that if  $\alpha = \beta/q$  then

$$\| |X|^q \|_{\psi_\alpha} = \|X\|_{\psi_\beta}^q, \tag{4.4}$$

and in particular,  $X \in L_{\psi_\beta}$  for  $\beta > 1$  if and only if  $|X|^\beta \in L_{\psi_1}$ .

Although there are versions of the following theorem (and of Definition 4.3) for any  $\alpha > 0$ , for the sake of simplicity, we shall restrict ourselves to the case  $\alpha = 1$ , which is the setting needed in the proofs below.

**Theorem 4.4.** *There exists an absolute constant  $c_1$  for which the following holds. If  $X \in L_{\psi_1}$  and  $X_1, \dots, X_N$  are independent copies of  $X$ , then for every  $t > 0$ ,*

$$Pr\left(\left|\frac{1}{N} \sum_{i=1}^N X_i - \mathbb{E}X\right| > t \|X\|_{\psi_1}\right) \leq 2 \exp(-cN \min\{t^2, t\}).$$

Combining Theorem 4.4 and (4.4) leads to the following corollary:

**Corollary 4.5.** *Let  $p > 1$  and assume that  $w$  is a random variable for which  $|w|^p \in L_{\psi_1}$  (or  $w \in L_{\psi_p}$ ). Then, with probability at least  $1 - 2 \exp(-ct)$  for  $0 < t < N$ ,*

$$\left| \frac{1}{N} \sum_{i=1}^N |w_i|^p - \mathbb{E}|w|^p \right| \leq \|w\|_{\psi_p}^p \sqrt{\frac{t}{N}}.$$

The corollary follows immediately from Theorem 4.4 by taking  $t' = \sqrt{t/N}$  for  $0 < t < N$ , and since  $\| |w|^p \|_{\psi_1} = \|w\|_{\psi_p}^p$ .

We will also be interested in decay properties of the random variable  $\sup_{t \in T} |\langle X, t \rangle|$  for a set  $T \subset \mathbb{R}^n$ . If  $\mu$  is an isotropic,  $L$ -subgaussian measure on  $\mathbb{R}^n$ , then one has the following (see, e.g., [29]).

**Theorem 4.6.** *For every  $L > 1$  there exist constants  $c_1, c_2, c_3$  and  $c_4$  that depend only on  $L$  and for which the following holds. If  $u \geq c_1$ , then with probability at least  $1 - 2 \exp(-c_2 u \log N)$ ,*

$$\max_{1 \leq i \leq N} \sup_{t \in T} | \langle a_i, t \rangle |^2 \leq c_3 u (\ell^2(T) + d^2(T) \log N),$$

where  $\ell(T)$  and  $d(T)$  are defined by (2.3) and (2.4).

In particular,

$$\| \max_{1 \leq i \leq N} \sup_{t \in T} | \langle a_i, t \rangle | \|_{\psi_1} \leq c_4 (\ell^2(T) + d^2(T) \log N).$$

4.2. The recovery approach

The assumptions we make throughout this section are as follows:

**Assumption 4.1.** Assume that  $T \subset \mathbb{R}^n$  is a bounded set, that  $a$  is an isotropic and  $L$ -subgaussian random vector, and that the noise  $w$  is a symmetric,  $\psi_2$  random variable that is independent of  $a$ .

Recall that the goal is to find an estimate  $\hat{x}$  of  $x_0$  that is close to  $x_0$  or to  $-x_0$ . Given the measurements  $(y_i)_{i=1}^N$ , a reasonable approach is to seek a value of  $x$  that minimizes the empirical risk function:

$$P_N \ell_x = \frac{1}{N} \sum_{i=1}^N | \langle a_i, x \rangle^2 - y_i |^p, \tag{4.5}$$

for some  $p$  in the regime  $1 < p \leq 2$  that is close to 1; the exact choice of  $p$  will become clear later on. Note that for every  $x \in T$ ,

$$\ell_x = | \langle a, x \rangle^2 - y |^p = | \langle a, x - x_0 \rangle \langle a, x + x_0 \rangle - w |^p. \tag{4.6}$$

Since the empirical average  $P_N \ell_x$  is not a convex function of  $x$ , it is impossible in general to find a value of  $x$  that minimizes it. Furthermore, given a candidate solution, it is not clear how to check whether indeed it is a minimizer. Luckily, for our purposes, one does not need an exact minimizer. Instead, in order to bound the estimation error, it is sufficient to find a value of  $x$  for which the empirical risk is small enough, as incorporated in Definition 4.7 below. This provides a concrete way of checking whether a candidate point is valid: all that has to be done is to substitute it into the bound. To find such a point, one may use any algorithm for phase retrieval and check whether the resulting solution satisfies the bound. To that end, techniques that depend on the initial starting point could prove useful; such methods can be started from several different points, and in that way, if a particular solution does not satisfy the bound then the algorithm may be used again, but from a different starting point. Eventually, with high probability, a point satisfying the bound will be obtained. One algorithm of this form is the GESPAR method developed in [40].

**Definition 4.7.** Let  $1 < p \leq 2$  be given, and choose a value of  $u \geq 1$ . Given the data  $(a_i, y_i)_{i=1}^N$ ,  $\hat{x} \in T$  is called a good estimate if it satisfies that

$$\frac{1}{N} \sum_{i=1}^N | \langle a_i, \hat{x} \rangle^2 - y_i |^p \leq \mathbb{E} |w|^p + u \frac{\|w\|_{\psi_2}^p}{\sqrt{N}}. \tag{4.7}$$

The parameter  $u$  tunes the probability estimate, and for the moment is of secondary importance. The exact choice of  $p$  will be specified in Theorem 4.9.

To motivate Definition 4.7, observe that the bound in (4.7) is independent of the data, and depends only on the number of measurements  $N$  and on the noise properties. This choice of  $\hat{x}$  is a modified empirical risk minimization – modified in two ways. First, instead of minimizing the loss functional  $P_N \ell_x$ , the search is for an empirical feasible point. In fact, the significance of a minimizer of the empirical loss functional is that it is also a minimizer of

$$\frac{1}{N} \sum_{i=1}^N | \langle a_i, x \rangle^2 - y_i |^p - \frac{1}{N} \sum_{i=1}^N |w_i|^p. \tag{4.8}$$

Let

$$\mathcal{L}_x = \ell_x - \ell_{x_0} = | \langle a, x - x_0 \rangle \langle a, x + x_0 \rangle - w |^p - |w|^p, \tag{4.9}$$

be the excess risk functional. It is evident that a minimizer of (4.8) is simply a minimizer of the empirical excess risk  $P_N \mathcal{L}_x$ . Since  $x_0$  is a candidate in this minimization problem, the empirical excess risk of the minimizer is non-positive. Thus, the choice of a minimizer allows one to identify a point  $x$  for which  $P_N \mathcal{L}_x \leq 0$ . The heart of the analysis of the problem is showing that such a point has a small conditional expectation  $\mathbb{E} \mathcal{L}_{\hat{x}}$ .

Unfortunately, it is impossible to estimate the empirical excess risk directly, as one does not have access to the sampled noise  $w_1, \dots, w_N$ , and therefore, nor to  $\frac{1}{N} \sum_{i=1}^N |w_i|^p$  – which is the reason for the second modification. By Assumption 4.1,  $w \in L_{\psi_2}$  and consequently  $|w|^p \in L_{\psi_1}$ . From Corollary 4.5, if  $u \leq N$ , then with probability at least  $1 - 2 \exp(-c_1 u^2)$ ,

$$\left| \frac{1}{N} \sum_{i=1}^N |w_i|^p - \mathbb{E} |w|^p \right| \leq u \frac{\|w\|_{\psi_2}^p}{\sqrt{N}}.$$

Thus, one may replace the empirical mean with  $\mathbb{E} |w|^p \pm u \|w\|_{\psi_2}^p / \sqrt{N}$ , leading to a small value of  $P_N \mathcal{L}_{\hat{x}}$ :

**Proposition 4.8.** *There exists an absolute constant  $c_1$  for which the following holds. Let  $\hat{x}$  be a point that satisfies (4.7). If  $0 \leq u \leq N$ , then with probability at least  $1 - 2 \exp(-c_1 u^2)$ ,  $P_N \mathcal{L}_{\hat{x}} \leq 2u \|w\|_{\psi_2}^p / \sqrt{N}$ .*

To see that there is always a point  $\hat{x}$  that satisfies (4.7), observe that for  $x_0$  and  $0 < u \leq N$ , with probability at least  $1 - 2 \exp(-cu^2)$

$$\frac{1}{N} \sum_{i=1}^N \left| \left| \langle a_i, x_0 \rangle \right|^2 - y_i \right|^p = \frac{1}{N} \sum_{i=1}^N |w_i|^p \leq \mathbb{E}|w|^p + u \frac{\|w\|_{\psi_2}^p}{\sqrt{N}}.$$

To formulate the main result of this section, let  $\kappa_T = \inf_{s,t \in T} \kappa(s, t)$  and recall that

$$E = \max\{\ell(T_+), \ell(T_-)\} \quad \text{and} \quad \rho_{T,N} = \frac{E}{\sqrt{N}} + \frac{E^2}{N}. \tag{4.10}$$

**Theorem 4.9.** *For every  $\kappa > 0$  and every  $L \geq 1$  there exists constants  $c_1, c_2 > 1$  and  $c_3, c_4$  that depend only on  $L$  and  $\kappa$  for which the following holds. Let  $a$  be distributed according to an isotropic,  $L$ -subgaussian measure, assume that  $T \subset \mathbb{R}^n$  is a bounded set and that  $\kappa_T \geq \kappa$ . Assume further that  $\|w\|_{\psi_2} < \infty$ . For every integer  $N$  set*

$$\beta_N = \max\{c_1((1 + \|w\|_{\psi_2} + d^2(T)) \log N + d^2(T)E^2), e^2\}$$

and let

$$p = 1 + 1/\log \beta_N.$$

If  $\hat{x}$  satisfies (4.7) for  $c_2 \leq u \leq N$ , then with probability at least  $1 - 2 \exp(-c_3 u^{1/3})$ , for  $\rho = \rho_{T,N}$  and  $d = d(T)$ ,

$$\|x_0 - \hat{x}\|_2 \|x_0 + \hat{x}\|_2 \leq c_4 u \max\left\{ (\|w\|_{\psi_2} + d) \rho \log \beta_N, \frac{\|w\|_{\psi_2}}{N^{1/4}} \sqrt{\log \beta_N} \right\} = (*).$$

In particular, if  $(*) < \varepsilon$ , then the choice of  $\hat{x}$  is an  $\varepsilon$ -recovery procedure with confidence parameter  $\delta = 2 \exp(-c_3 u^{1/3})$ .

As an example, consider the case in which  $T$  is the set of  $k$ -sparse vectors on the sphere. Thus,  $d(T) = 1$ , and by results from Section 3.2,  $E \sim (k \log(en/k))^{1/2}$ . If  $w$  is  $L$ -subgaussian with standard deviation  $\sigma$ , then  $\|w\|_{\psi_2} \lesssim_L \sigma$  and

$$\beta_N \sim_L (\sigma + 1) \log N + k \log(en/k).$$

Assume that  $\sigma \leq k$ , and thus

$$\log \beta_N \leq \log k + \log \log N + \log \log(en/k).$$

Let  $k \leq N \leq k^2 \log^2(en/k)$ . Since  $\log \beta_N > 1$ , it is straightforward to verify that

$$\rho \lesssim \sqrt{\frac{k \log(en/k)}{N}},$$

and that

$$\frac{\sigma}{N^{1/4}} \sqrt{\log \beta_N} \lesssim (1 + \sigma \log \beta_N) \sqrt{\frac{k \log(en/k)}{N}}.$$

Thus, by Theorem 4.9, for those values of  $N$ ,

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u (1 + \sigma \log \beta_N) \sqrt{\frac{k \log(en/k)}{N}}.$$

If  $N \geq k^2 \log^2(en/k)$ , then

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u (1 + \sigma \log \beta_N) \frac{1}{N^{1/4}}.$$

Therefore,  $\hat{x}$  is an  $\varepsilon$ -recovery procedure with confidence parameter  $\delta = 2 \exp(-cu^{1/3})$  if

$$N(\varepsilon) \gtrsim_{L,u} (1 + \sigma)^2 \log^2 \beta_N \max\{\varepsilon^{-2} k \log(en/k), \varepsilon^{-4}\}.$$

**Remark 4.10.** Since  $x_0, \hat{x} \in S^{n-1}$ , it follows that either  $\|\hat{x} - x_0\|_2 \geq 1$  or  $\|\hat{x} + x_0\|_2 \geq 1$ . Thus, if  $(*) < \varepsilon$  then either  $\|\hat{x} - x_0\|_2 \leq \varepsilon$  or  $\|\hat{x} + x_0\|_2 \leq \varepsilon$ .

In comparison, note that in the case of linear measurements, with high probability,

$$\|\hat{x} - x_0\|_2 \lesssim_L \sigma \sqrt{\frac{k \log(en/k)}{N}}$$

(see also the discussion in Section 5), which means that up to logarithmic factors and the dependence of  $\sigma$  (particularly, for small values of  $\sigma$ ), and as long as  $\varepsilon \gtrsim 1/\sqrt{k \log(en/k)}$ , the two estimates are of the same order of magnitude.

4.3. Proof of Theorem 4.9

The proof of the theorem requires several preliminary facts about empirical and Bernoulli processes. We refer the reader to [26,24] for more details on these processes.

Let  $(\Omega, \mu)$  be a probability space and set  $(X_i)_{i=1}^N$  to be independent variables, distributed according to  $\mu$ . Let  $\varepsilon_1, \dots, \varepsilon_N$  be independent, symmetric,  $\{-1, 1\}$ -valued random variables, that are independent of  $X_1, \dots, X_N$ .

The first result we require is the contraction inequality for Bernoulli processes.

**Theorem 4.11.** (See [26].) Let  $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be convex and increasing. Assume that  $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$  satisfy that  $\phi_i(0) = 0$  and have a Lipschitz constant at most  $A$ . Then, for any bounded  $T \subset \mathbb{R}^N$ ,

$$\mathbb{E}F\left(\frac{1}{2A} \sup_{t \in T} \left| \sum_{i=1}^N \varepsilon_i \phi_i(t_i) \right|\right) \leq \mathbb{E}F\left(\sup_{t \in T} \left| \sum_{i=1}^N \varepsilon_i t_i \right|\right).$$

The following symmetrization argument allows one to bound an empirical process using the Bernoulli process indexed by the random set  $\{(h(X_i))_{i=1}^N : h \in H\}$ .

**Theorem 4.12.** [45] If  $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is convex and increasing and  $\mathcal{H}$  is a class of functions, then

$$\mathbb{E}F\left(\sup_{h \in H} \left| \frac{1}{N} \sum_{i=1}^N h(X_i) - \mathbb{E}h \right|\right) \leq \mathbb{E}F\left(2 \sup_{h \in H} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i h(X_i) \right|\right).$$

Given a bounded  $T \subset \mathbb{R}^n$  and  $x_0 \in T$ , let  $h_x(a) = \langle x - x_0, a \rangle \langle x + x_0, a \rangle$  and recall that  $\mathcal{L}_x = |h_x(a) - w|^p - |w|^p$ . Theorems 4.11 and 4.12 will be used with the choices  $F(x) = |x|^q$  for  $q \geq 2$  and  $H = \{\mathcal{L}_x : x \in T\}$ .

**Lemma 4.13.** There exists an absolute constant  $c_1$  for which the following holds. Let  $1 < p < 2$ . If  $\|h_x\|_{L_p} \leq 2\|w\|_{L_p}/(p-1)^{1/2}$  then

$$\mathbb{E}\mathcal{L}_x \geq c_1 \kappa^2 p(p-1) \|x - x_0\|_2^2 \|x + x_0\|_2^2 / (\|w\|_{L_2}^{2-p});$$

and if  $\|h_x\|_{L_p} \geq 2\|w\|_{L_p}/(p-1)^{1/2}$ , then

$$\mathbb{E}\mathcal{L}_x \geq c_1 \kappa^p \|x - x_0\|_2^p \|x + x_0\|_2^p.$$

**Proof.** Since  $w$  is a symmetric random variable, it is distributed as  $\varepsilon w$ , where, as always,  $\varepsilon$  is a symmetric  $\{-1, 1\}$ -valued random variable, independent of  $w$  and of  $a$ . Therefore,

$$\begin{aligned} \mathbb{E}\mathcal{L}_x &= \mathbb{E}_{a \times w} \mathbb{E}_\varepsilon (|h_x(a) - \varepsilon w|^p - |w|^p) \\ &= \mathbb{E}_{a \times w} \left( \frac{1}{2} |w - h_x(a)|^p + \frac{1}{2} |w + h_x(a)|^p - |w|^p \right) \end{aligned} \tag{4.11}$$

$$= \frac{1}{2} (\|w - h_x\|_{L_p}^p + \|w + h_x\|_{L_p}^p) - \|w\|_{L_p}^p. \tag{4.12}$$

It is well known (see, e.g., [28,3]) that if  $1 < p \leq 2$ , then

$$\frac{1}{2} (\|w - h_x\|_{L_p}^p + \|w + h_x\|_{L_p}^p) \geq (\|w\|_{L_p}^2 + (p-1)\|h_x\|_{L_p}^2)^{p/2}. \tag{4.13}$$

Set  $a = \|w\|_{L_p}$  and  $b = \|h_x\|_{L_p}$ . Observe that

$$(a^2 + (p-1)b^2)^{p/2} - a^p = a^p ((1 + (p-1)(b/a)^2)^{p/2} - 1),$$



and that if  $0 < y < 4$  then  $(1 + y)^{p/2} - 1 \geq py/12$ . Indeed, since  $1 < p \leq 2$  and  $y > 0$ , it suffices to show that  $(1 + y)^{1/2} \geq y/6$ , which holds for  $0 < y < 4$ . Let

$$y = (p - 1)(b/a)^2 = (p - 1)(\|h_x\|_{L_p}/\|w\|_{L_p})^2,$$

and note that by the assumption on  $x \in T$ ,  $y \leq 4$ ; thus,

$$(\|w\|_{L_p}^2 + (p - 1)\|h_x\|_{L_p}^2)^{p/2} - \|w\|_{L_p}^p \gtrsim p(p - 1) \frac{\|h_x\|_{L_p}^2}{\|w\|_{L_p}^{2-p}}.$$

Recall that by the assumption on  $a$ , for every  $s, t \in T$  and  $p \geq 1$ ,

$$(\mathbb{E}|\langle a, t \rangle \langle a, s \rangle|^p)^{1/p} \geq \mathbb{E}|\langle a, t \rangle \langle a, s \rangle| \geq \kappa \|t\|_2 \|s\|_2,$$

and since  $\|w\|_{L_p} \leq \|w\|_{L_2}$ , it follows that

$$\mathbb{E}\mathcal{L}_x \geq \kappa^2 p(p - 1) \frac{\|x - x_0\|_2^2 \|x + x_0\|_2^2}{\|w\|_{L_2}^{2-p}}.$$

On the other hand, if  $y = (p - 1)(b/a)^2 \geq 4$ , that is, if  $\|h_x\|_{L_p}^2 \geq 4\|w\|_{L_p}^2/(p - 1)$ , then reversing the roles in (4.13),

$$\frac{1}{2}(\|w - h_x\|_{L_p}^p + \|w + h_x\|_{L_p}^p) \geq (\|h_x\|_{L_p}^2 + (p - 1)\|a\|_{L_p}^2)^{p/2} \geq \|h_x\|_{L_p}^p.$$

Therefore,

$$\begin{aligned} \mathbb{E}\mathcal{L}_x &\geq \|h_x\|_{L_p}^p - \|w\|_{L_p}^p \geq \frac{1}{2}\|h_x\|_{L_p}^p + \|w\|_{L_p}^p \left( \frac{1}{2} \left( \frac{2}{\sqrt{p-1}} \right)^p - 1 \right) \\ &\geq \frac{1}{2}\|h_x\|_{L_p}^p \gtrsim \kappa^p \|x - x_0\|_2^p \|x + x_0\|_2^p, \end{aligned}$$

provided that  $p \leq 2$ .  $\square$

For every  $r > 0$ , set

$$T_r = \{x \in T : r < \|x - x_0\|_2 \|x + x_0\|_2 \leq 2r\},$$

let

$$T_1 = \{x \in T : \|h_x\|_{L_p} \geq 2\|w\|_{L_p}/(p - 1)^{1/2}\},$$

and put  $T_2 = T \setminus T_1$ .

The key step in the proof is the following estimate on the supremum of the empirical process

$$x \rightarrow |P_N \mathcal{L}_x - P \mathcal{L}_x| = \left| \frac{1}{N} \sum_{i=1}^N \mathcal{L}_x(a_i, y_i) - \mathbb{E}\mathcal{L}_x \right|,$$

indexed by  $T_r$  and by  $T_1$ .

**Lemma 4.14.** *For every  $L \geq 1$ , there exist constants  $c_1$  and  $c_2$  that depend only on  $L$  for which the following holds. For every  $r > 0$ , with probability at least  $1 - 2 \exp(-c_1 u^{1/3})$ ,*

$$\sup_{x \in T_r} |P_N \mathcal{L}_x - P \mathcal{L}_x| \leq c_2 r \rho_{T,N}.$$

Also, with probability at least  $1 - 2 \exp(-c_1 u^{1/3})$ ,

$$\sup_{x \in T_1} |P_N \mathcal{L}_x - P \mathcal{L}_x| \leq c_2 d(T) \rho_{T,N}.$$

**Proof.** We will present a proof of the first part of the claim and omit the proof of the second one, as it follows an almost identical path to the proof of the first.

Fix  $r > 0$  and consider the empirical process  $x \rightarrow |P_N \mathcal{L}_x - P \mathcal{L}_x|$  indexed by  $T_r$ . Set  $q \geq 2$ . By the symmetrization theorem (Theorem 4.12) and the independence of  $a$  and  $w$ ,

$$\begin{aligned} \mathbb{E} \sup_{x \in T_r} |P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x|^q &\leq \mathbb{E} \mathbb{E}_\varepsilon \sup_{x \in T_r} \left| \frac{2}{N} \sum_{i=1}^N \varepsilon_i \mathcal{L}_x(a_i, w_i) \right|^q \\ &= \mathbb{E} \mathbb{E}_\varepsilon \sup_{x \in T_r} \left| \frac{2}{N} \sum_{i=1}^N \varepsilon_i (|\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle - w_i|^p - |w_i|^p) \right|^q. \end{aligned}$$

Let

$$D_{\infty, N} = 2 \max_{1 \leq i \leq N} \left( |w_i| + \sup_{x \in T_r} |\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle| \right),$$

and observe that for every realization of  $(w_i)_{i=1}^N$ , the functions  $y \rightarrow |y - w_i|^p - |w_i|^p$  vanish at 0 and are Lipschitz on  $[-b, b]$  with a constant  $p(b + |w_i|)^{p-1}$ . For  $b \leq \max_{1 \leq i \leq N} \sup_{x \in T_r} |\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle|$  this constant is proportional to  $D_{\infty, N}^{p-1}$ , as  $p \leq 2$ . Applying the contraction inequality (Theorem 4.11), conditioned on  $w_1, \dots, w_N$  and  $a_1, \dots, a_N$ ,

$$\begin{aligned} \mathbb{E}_\varepsilon \sup_{x \in T_r} \left| \sum_{i=1}^N \varepsilon_i (|\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle - w_i|^p - |w_i|^p) \right|^q &\leq c^q (D_{\infty, N}^{p-1})^q \mathbb{E}_\varepsilon \cdot \sup_{x \in T_r} \left| \sum_{i=1}^N \varepsilon_i \langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle \right|^q \\ &= c^q (D_{\infty, N}^{p-1})^q \mathbb{E}_\varepsilon \cdot \sup_{x \in T_r} (A(x) B_{T, N}(x))^q, \end{aligned}$$

where  $A(x) = \|x - x_0\|_2 \|x + x_0\|_2$  and

$$B_{T, N}(x) = \left| \sum_{i=1}^N \varepsilon_i \left\langle a_i, \frac{x - x_0}{\|x - x_0\|_2} \right\rangle \left\langle a_i, \frac{x + x_0}{\|x + x_0\|_2} \right\rangle \right|.$$

By the Cauchy-Schwarz inequality, and recalling that on  $T_r$ ,  $A(x) \leq 2r$ ,

$$\mathbb{E}_{a \times W} \mathbb{E}_\varepsilon \left[ (D_{\infty, N}^{p-1})^q \cdot \sup_{x \in T_r} (A(x) B_{T, N}(x))^q \right] \leq c^q r^q \|D_{\infty, N}\|_{L_{2q}}^q \cdot \left( \mathbb{E} \sup_{v \in T_+, u \in T_-} \left| \sum_{i=1}^N \varepsilon_i \langle a_i, v \rangle \langle a_i, u \rangle \right|^{2q} \right)^{1/2},$$

for a suitable absolute constant  $c$ . Setting

$$B_{T, N, q} = \left\| \sup_{v \in T_+, u \in T_-} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \langle a_i, v \rangle \langle a_i, u \rangle \right| \right\|_{L_{2q}},$$

it follows from Theorem 2.7 that

$$B_{T, N, q} \lesssim q^{3/2} \left( \frac{E}{\sqrt{N}} + \frac{E^2}{N} \right).$$

Next, by Theorem 4.6,

$$\begin{aligned} \left\| \max_{1 \leq i \leq N} \sup_{x \in T_r} |\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle| \right\|_{\psi_1} &\lesssim r \left\| \max_{1 \leq i \leq N} \sup_{x \in T_r} \left| \left\langle a_i, \frac{x - x_0}{\|x - x_0\|_2} \right\rangle \left\langle a_i, \frac{x + x_0}{\|x + x_0\|_2} \right\rangle \right| \right\|_{\psi_1} \\ &\lesssim_L r (E^2 + \log N). \end{aligned}$$

Hence, from (4.3),

$$\begin{aligned} \|D_{\infty, N}\|_{\psi_1} &\lesssim \left( \left\| \max_{1 \leq i \leq N} w_i \right\|_{\psi_1} + \left\| \max_{1 \leq i \leq N} \sup_{x \in T_r} |\langle a_i, x - x_0 \rangle \langle a_i, x + x_0 \rangle| \right\|_{\psi_1} \right) \\ &\lesssim ((\|w\|_{\psi_1} + r) \log N + rE^2) \lesssim \beta_N. \end{aligned}$$

Since  $p = 1 + 1/\log \beta_N$  and applying the moment characterization of the  $\psi_1$  norm (4.2), it is evident that

$$\|D_{\infty, N}^{p-1}\|_{L_{2q}} \lesssim q^{p-1}. \tag{4.14}$$

Indeed,

$$(\mathbb{E} D_{\infty, N}^{(p-1)2q})^{1/(p-1)2q} \lesssim (p-1)q \|D_{\infty, N}\|_{\psi_1},$$

and thus,

$$\|D_{\infty,N}^{p-1}\|_{L_{2q}} \lesssim ((p-1)q)^{p-1} \|D_{\infty,N}\|_{\psi_1}^{p-1} \lesssim ((p-1)q)^{p-1} \beta_N^{1/\log \beta_N} \lesssim q^{p-1}.$$

With the two estimates in place, it is evident that there exists a constant  $c_1$  that depends only on  $L$ , for which, for every  $q \geq 2$ ,

$$\left\| \sup_{x \in T_r} |P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x| \right\|_{L_q} \leq c_1 q^{3/2+(p-1)} \left( \frac{E}{\sqrt{N}} + \frac{E^2}{N} \right) \cdot r \leq c_1 q^3 r \rho_{T,N}.$$

Therefore, it is standard to show (see, e.g., [10] for a similar argument), that for  $u \geq 1$ , with probability at least  $1 - 2 \exp(-c_2 u^{1/3})$ ,

$$\sup_{x \in T} |P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x| \leq c_3 u r \rho_{T,N},$$

where  $c_2$  and  $c_3$  depend only on  $L$ .  $\square$

Next, set  $\sigma = \|w\|_{L_2}$ ; since  $w$  is  $L$ -subgaussian,  $\|w\|_{\psi_2} \sim_L \sigma$ . By the choice of  $p$ ,

$$\sigma^p = \sigma^{1+1/\log \beta_N} \leq \sigma^{1+1/\log(2+\sigma)} \leq e\sigma.$$

Therefore,

$$P_N \mathcal{L}_{\hat{x}} \leq 2u \frac{\|w\|_{\psi_2}^p}{\sqrt{N}} \lesssim_L u \frac{\sigma^p}{\sqrt{N}} \lesssim_L u \frac{\sigma}{\sqrt{N}}.$$

Finally, let

$$\rho_0^{2-p} = \alpha \sigma \rho_{T,N},$$

for a constant  $\alpha$  satisfying  $\alpha \sim_k 1/(p-1)$ .

**Corollary 4.15.** *There exist constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  and for which the following holds. If  $j_0 \geq c_1$ , then with probability at least  $1 - 2 \exp(-c_2 2^{j_0(2-p)})$ , for every  $x \in T_2$  with  $\|x - x_0\|_2 \|x + x_0\|_2 \geq 2^{j_0} \rho_0$ ,*

$$|P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x| \leq \frac{1}{2} \mathbb{E} \mathcal{L}_x.$$

**Proof.** Fix  $j_0 \geq c_1$  and  $j \geq j_0$ . Put  $u_j = 2^{j(2-p)}$  and set  $r_j = 2^j \rho_0$ . By Lemma 4.14 for  $T_{r_j}$ , with probability at least  $1 - 2 \exp(-c_1 u_j^{1/3}) = 1 - 2 \exp(-c_1 2^{j(2-p)/3})$ ,

$$\sup_{x \in T_{r_j}} |P_N \mathcal{L}_x - P \mathcal{L}_x| \leq c_2 2^{j(2-p)} r_j^p \rho_{T,N} \lesssim 2^{j(2-p)} (\|x - x_0\|_2 \|x + x_0\|_2)^p \rho_{T,N},$$

because on  $T_{r_j}$ ,  $\|x - x_0\|_2 \|x + x_0\|_2 \sim r_j$ .

Taking the union bound over these events, there is an event  $\Omega_0$  of probability at least

$$1 - 2 \sum_{j \geq j_0} \exp(-c_1 2^{j(2-p)/3}) \geq 1 - 2 \exp(-c_3 2^{j_0(2-p)/3}),$$

on which, for every  $j \geq j_0$  and every  $x \in T_j$ ,

$$|P_N \mathcal{L}_x - P \mathcal{L}_x| \lesssim 2^{j(2-p)} (\|x - x_0\|_2 \|x + x_0\|_2)^p \rho_{T,N}.$$

Fix a sample in  $\Omega_0$  and  $x \in T_{r_j} \cap T_2$  for  $j \geq j_0$ . By Lemma 4.13,

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \mathcal{L}_x - |P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x| \gtrsim \kappa^2 (p-1) \frac{\|x - x_0\|_2^2 \|x + x_0\|_2^2}{\sigma^{2-p}} - |P_N \mathcal{L}_x - \mathbb{E} \mathcal{L}_x| \\ & \gtrsim \kappa^2 (p-1) \frac{\|x - x_0\|_2^2 \|x + x_0\|_2^2}{\sigma^{2-p}} - \|x - x_0\|_2^p \|x + x_0\|_2^p 2^{j(2-p)} \rho_{T,N} \\ & \gtrsim (\|x - x_0\|_2 \|x + x_0\|_2)^p \cdot \left( \kappa^2 (p-1) \frac{(\|x - x_0\|_2 \|x + x_0\|_2)^{2-p}}{\sigma^{2-p}} - 2^{j(2-p)} \rho_{T,N} \right) > 0, \end{aligned}$$

provided that  $\alpha \sim_{\kappa} 1/(p-1)$ . Indeed,  $\rho_{T,N}^{2-p} \sim \rho_{T,N}$  and  $\sigma^{2-p} \sim \sigma$ . For this choice of  $\alpha$  and every  $x \in T_{r_j}$ ,

$$\begin{aligned} \kappa^2(p-1) \frac{(\|x-x_0\|_2 \|x+x_0\|_2)^{2-p}}{\sigma^{2-p}} &\gtrsim \kappa^2(p-1) \frac{r_j^{2-p}}{\sigma} = \kappa^2(p-1) \frac{2^{j(2-p)} \rho_0^{2-p}}{\sigma} \\ &\geq \kappa^2(p-1) \frac{2^{j(2-p)}}{\sigma} \cdot \alpha \sigma \rho_{T,N} \geq 2^{j(2-p)} \rho_{T,N}, \end{aligned}$$

completing the proof.  $\square$

Finally, fix  $1 \leq u \leq N$  let  $j_0$  satisfy that  $2^{j_0(2-p)} \sim u$ . Consider the event  $\Omega_1$  on which both

1.  $\sup_{x \in T_1} |P_N \mathcal{L}_x - P \mathcal{L}_x| \leq c_1 u d(T) \rho_{T,N}$  and
2.  $|P_N \mathcal{L}_x - P \mathcal{L}_x| \leq \frac{1}{2} \mathbb{E} \mathcal{L}_x$  for  $x \in T_2 \cap \{z: \|z-x_0\|_2 \|z+x_0\|_2 \geq 2^{j_0} \rho_0\}$ .

By the second part of Lemma 4.14 and Corollary 4.15,  $Pr(\Omega_1) \geq 1 - 2 \exp(-c_2 u^{1/3})$ .

Fix a sample in  $\Omega_1$ , let  $\hat{x}$  be the point selected by the feasible point procedure (4.7) and consider the two possibilities: either  $\hat{x} \in T_1$  or  $\hat{x} \in T_2$ .

If  $\hat{x} \in T_1$  then by Lemma 4.13 and Lemma 4.14,

$$\begin{aligned} \kappa^p \|x-x_0\|_2^p \|x+x_0\|_2^p &\lesssim \mathbb{E} \mathcal{L}_{\hat{x}} \leq P_N \mathcal{L}_{\hat{x}} + \sup_{x \in T_1} |P_N \mathcal{L}_x - P \mathcal{L}_x| \\ &\leq P_N \mathcal{L}_{\hat{x}} + c_1 u d(T) \rho_{T,N} \lesssim_L u \frac{\sigma^p}{\sqrt{N}} + u d(T) \rho_{T,N}. \end{aligned}$$

By the choice of  $p$ ,  $(d(T) \rho_{T,N})^{1/p} \sim d(T) \rho_{T,N}$  and since  $1 \lesssim u \leq N$ ,  $u^{1/p} \sim u$ . Thus,

$$\|x-x_0\|_2 \|x+x_0\|_2 \lesssim_{L,\kappa} u \left( \frac{\sigma}{N^{1/2p}} + d(T) \rho_{T,N} \right).$$

If  $\hat{x} \in T_2$  and  $\|\hat{x}-x_0\|_2 \|\hat{x}+x_0\|_2 \geq 2^{j_0} \rho_0$ , then by property (2) in the definition of  $\Omega_1$  and the choice of  $p$ ,

$$\mathbb{E} \mathcal{L}_{\hat{x}} \leq P_N \mathcal{L}_{\hat{x}} + |P_N \mathcal{L}_{\hat{x}} - \mathbb{E} \mathcal{L}_{\hat{x}}| \lesssim_L u \frac{\sigma}{\sqrt{N}} + \frac{1}{2} \mathbb{E} \mathcal{L}_{\hat{x}}.$$

Hence,  $\mathbb{E} \mathcal{L}_{\hat{x}} \lesssim_L u \sigma / \sqrt{N}$ . By Lemma 4.13 and since

$$\sigma^{2-p} \sim \sigma \quad \text{and} \quad \mathbb{E} \mathcal{L}_{\hat{x}} \gtrsim \frac{\kappa^2(p-1)}{\sigma} \|\hat{x}-x_0\|_2^2 \|\hat{x}+x_0\|_2^2,$$

it is evident that

$$\|\hat{x}-x_0\|_2 \|\hat{x}+x_0\|_2 \lesssim_{L,\kappa} \left( \frac{\sigma^2 u}{(p-1) \sqrt{N}} \right)^{1/2} = u^{1/2} \sigma \frac{\sqrt{\log \beta_N}}{N^{1/4}}.$$

Otherwise, if  $\hat{x} \in T_2$  and  $\|\hat{x}-x_0\|_2 \|\hat{x}+x_0\|_2 \leq 2^{j_0} \rho_0$ , then using the choice of  $p$ ,

$$\|\hat{x}-x_0\|_2 \|\hat{x}+x_0\|_2 \leq 2^{j_0} \rho_0 \lesssim_{\kappa} 2^{j_0} \sigma \rho_{T,N} / (p-1) \sim_{\kappa} u \sigma \rho_{T,N} \log \beta_N.$$

Therefore, on  $\Omega_1$ , since  $2^{j_0(2-p)} \sim u \geq 1$ ,

$$\|\hat{x}-x_0\|_2 \|\hat{x}+x_0\|_2 \lesssim_{L,\kappa} u \max \left\{ \left( \frac{\sigma}{N^{1/2p}} + d(T) \rho_{T,N} \right), \sigma \frac{\sqrt{\log \beta_N}}{N^{1/4}}, \sigma \rho_{T,N} \log \beta_N \right\}.$$

Finally, observe that  $\log \beta_N \geq 1$ , and thus,

$$\frac{\sigma}{N^{1/2p}} \leq \sqrt{\log \beta_N} \frac{\sigma}{N^{1/4}} \quad \text{and} \quad d(T) \rho_{T,N} \leq (\sigma + d(T)) \rho_{T,N} \log \beta_N,$$

and the claim follows.  $\square$

#### 4.4. Examples

Next, consider some of the examples presented in Section 3.2, this time, in the noisy setting. Other examples may be derived with similar arguments.

In all the examples below  $T \subset S^{n-1}$  and so  $d(T) = 1$ . Since  $w$  is symmetric and  $L$ -subgaussian,  $\|w\|_{\psi_1} \leq \|w\|_{\psi_2} \lesssim L\sigma$ , where, as always,  $\sigma$  is the noise variance. Also, since  $1 < p \leq 2$ ,  $\|w\|_{\psi_1}^p = \|w\|_{\psi_p}^p \lesssim (L\sigma)^p$ .

4.4.1. The unit sphere  $T = S^{n-1}$

If  $T = S^{n-1}$  then  $E \sim \sqrt{n}$ . Thus,

$$\rho_{T,N} \lesssim \left( \sqrt{\frac{n}{N}} + \frac{n}{N} \right) \lesssim \sqrt{\frac{n}{N}}$$

and

$$\beta_N \sim (\sigma + 1) \log N + n.$$

Suppose that  $\sigma \leq \sqrt{n}$  and that  $n \lesssim N \lesssim n^2$ . Then, by [Theorem 4.9](#), with probability at least  $1 - 2 \exp(-cu^{1/3})$ ,

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u(1 + \sigma \log n) \sqrt{\frac{n}{N}}, \quad (4.15)$$

and if  $N \geq n^2$ , then

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u \sigma \frac{(\log n + \log \log N)^{1/2}}{N^{1/4}}.$$

Therefore, if  $\sigma \sim 1$ , one may ensure  $\varepsilon$ -recovery with confidence parameter  $\delta$  if

$$N(\varepsilon) \gtrsim_{L,\kappa,\delta} \max\{\varepsilon^{-2} n \log^2 n, \varepsilon^{-4} (\log^2 n + \log^2(\log N))\}.$$

4.4.2. Block sparse vectors

The norm-one block-sparse setting can be treated in a similar manner, leading to the following corollary.

**Corollary 4.16.** For every  $L \geq 1$  and  $\kappa > 0$  there exist constants  $c_1, c_2, c_3$  that depend only on  $L$  and  $\kappa$  and for which the following holds. Assume that  $\sigma \leq \sqrt{n}$  and let  $T$  be the set of  $k$ -block sparse vectors of length  $d$  on the sphere. Then,

$$\beta_N \lesssim_L (1 + \sigma) \log N + k(d + \log(en/dk)).$$

If

$$(k \log(en/dk) + dk) \lesssim N \lesssim (k \log(en/dk) + dk)^2,$$

and  $c_2 \leq u \leq N$ , then with probability at least  $1 - 2 \exp(-c_3 u^{1/3})$ ,

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u(1 + \sigma) \sqrt{\frac{k(\log(en/dk) + d)}{N}} \cdot \log \beta_N.$$

Furthermore, if  $N \geq k(d + \log(en/dk))^2$ , then

$$\|\hat{x} - x_0\|_2 \|\hat{x} + x_0\|_2 \lesssim_{L,\kappa} u \sigma \frac{\sqrt{\log \beta_N}}{N^{1/4}}.$$

Thus, if  $\sigma \sim 1$ , and setting

$$\log m = \max\{\log k, \log d, \log \log(en/dk), \log \log N\},$$

$\hat{x}$  is an  $\varepsilon$ -recovery procedure with confidence parameter  $\delta \geq \exp(-cN^{1/3})$  when

$$N(\varepsilon) \gtrsim_{L,\kappa,\delta} \log^2 m \max\{\varepsilon^{-2} k(d + \log(en/dk)), \varepsilon^{-4}\}.$$

## 5. Connection with results on linear estimation

The methods used throughout this article are very similar in nature to the analogous techniques used in the setting of linear measurements. Both stability and noisy recovery are well understood in the linear case, and in a sharp way, as explained below.

First, consider the question of stability. Suppose that the measurements are given by  $y = Ax$  for some  $N \times n$  matrix  $A$ . In the linear setting, a natural notion of stability in a set  $T \subset \mathbb{R}^n$  is that for every  $s, t \in T$ ,

$$\|At - As\|_2 \geq C \|t - s\|_2, \quad (5.1)$$

where  $C$  is a positive constant.

Note that here the  $\ell_2$  norm is used in the left-hand side, rather than the  $\ell_1$  norm. An  $\ell_2$  stability result is superior to an  $\ell_1$  estimate, because the  $\ell_2$  norm is smaller. It is natural to compare an  $\ell_2$  stability result in the linear case to the  $\ell_1$  stability result for quadratic measurements established here.

Stability in a set  $T$  for a random operator ensemble depends on the way in which a typical operator in the ensemble acts on the set

$$T_- = \left\{ \frac{t-s}{\|t-s\|_2} : t \neq s, t \in T \right\} \subset S^{n-1}.$$

Indeed, because  $a$  is distributed according to an isotropic measure, for every  $z \in S^{n-1}$ ,  $\mathbb{E}|\langle a, z \rangle|^2 = 1$ . Thus, stability on  $T$  is equivalent to an estimate on

$$\sup_{z \in T_-} \left| \frac{1}{N} \sum_{i=1}^N |\langle a_i, z \rangle|^2 - 1 \right|, \tag{5.2}$$

which is strictly smaller than 1. With this in mind, the stability constant of  $\mathbb{R}^n$  is a lower bound on the smallest singular value of a typical operator from the given random ensemble.

The study of the process (5.2), both for  $T_- = S^{n-1}$  and for an arbitrary subset of the sphere has been extensive in recent years. A good starting point for the interested reader would be [22,31] for subgaussian ensembles, [1,29] for log-concave ensembles, and [42,32,33] for ensembles with heavy tails.

In the context of this article, subgaussian ensembles, the best estimate on (5.2) follows from Theorem 2.7, applied to the class  $F = H = \{\langle v, \cdot \rangle, v \in T_-\}$ . Moreover, in [30] it was shown that under very mild assumptions on the set  $T_-$ , the following estimate is sharp.

**Theorem 5.1.** *For every  $L \geq 1$  there exist constants  $c_1, c_2$  and  $c_3$  that depend only on  $L$  for which the following holds. If  $T \subset \mathbb{R}^n$  and  $a$  is distributed according to an isotropic,  $L$ -subgaussian measure, then for  $u \geq c_1$ , with probability at least  $1 - 2 \exp(-c_2 u^2 \ell^2(T_-))$ , for every  $s, t \in T$ ,*

$$\|As - At\|_2 \geq \|s - t\|_2 / \sqrt{2},$$

provided that  $N \geq c_3 u^3 \ell^2(T_-)$ .

**Proof.** Since

$$\frac{\|At - As\|_2^2}{\|t - s\|_2^2} = \sum_{i=1}^N |\langle a_i, z \rangle|^2,$$

for  $z = (t-s)/\|t-s\|_2 \in T_-$ , and setting  $z_{s,t} = N^{-1} \sum_{i=1}^N |\langle a_i, z \rangle|^2$ , it suffices to bound  $\inf_{s,t \in T, s \neq t} z_{s,t}$  from below. Recall that  $a$  is isotropic and thus  $\mathbb{E}z_{s,t} = \mathbb{E}|\langle a, z \rangle|^2 = 1$ . Applying Theorem 2.7 for  $N \gtrsim_L u^6 \ell^2(T_-)$  and recalling that  $T_- \subset S^{n-1}$ , it follows that with probability at least  $1 - 2 \exp(-cu^2 \ell^2(T_-))$ ,

$$\sup_{z \in T_-} \left| \frac{1}{N} \sum_{i=1}^N \langle z, a \rangle^2 - 1 \right| \lesssim_L u^3 \left( \frac{\ell(T_-)}{\sqrt{N}} + \frac{\ell^2(T_-)}{N} \right) \leq 1/2.$$

On that event, for every  $s \neq t$ ,

$$\|As - At\|_2^2 = \|A(s-t)\|_2^2 \geq \|s-t\|_2^2 / 2,$$

as claimed.  $\square$

Observe that the same complexity parameter appears in both the linear case and in the “quadratic” stability result – the Gaussian complexity of a “projection” of  $T - T$  onto the sphere (and, of course, the  $T + T$  component does not appear). In all the examples we presented in this note,  $T + T$  and  $T - T$  have essentially (or exactly) the same complexity, and thus the stability estimates in the linear case coincide with quadratic bounds, as will be the case for any  $T \subset \mathbb{R}^n$  with a similar property. Thus, in these cases, there is essentially no loss in requiring stability over quadratic measurements rather than with respect to linear ones.

The noisy recovery problem in the linear case is much simpler, since the resulting empirical process is well behaved even if one uses the squared loss functional. The advantage in considering a squared loss is that one has the benefit of the required convexity “for free”. With this objective, noisy recovery becomes a linear regression problem in  $\mathbb{R}^n$ , indexed by  $T$ . This is a well studied topic in statistics. We refer the reader to [23] for relatively recent results related to this question.

The best results to-date on linear regression that take into account the complexity of the indexing set  $T$  can be found in [25]. One may show that these estimates are sharp under very mild assumptions on  $T$ , and it turns out that these

assumptions are satisfied in the examples that have been presented here. Unfortunately, the methods required to prove the optimality of the bounds are rather involved, and we will not explore this issue here. Rather, we refer the reader to [25], in which the linear case is explored.

## References

- [1] R. Adamczak, A. Litvak, A. Pajor, N. Tomczak-Jaegermann, Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles, *J. Amer. Math. Soc.* 23 (2010) 535–561.
- [2] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, *Appl. Comput. Harmon. Anal.* 20 (3) (2006) 345–356.
- [3] K. Ball, E.A. Carlen, E.H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, *Invent. Math.* 115 (1994) 463–482.
- [4] F. Barthe, O. Guédon, S. Mendelson, A. Naor, A probabilistic approach to the geometry of the  $\ell_p^n$  ball, *Ann. Probab.* 33 (2005) 480–513.
- [5] A. Beck, Y.C. Eldar, Sparsity constrained nonlinear optimization: Optimality conditions and algorithms, arXiv:1203.4580v1.
- [6] S.G. Bobkov, Isoperimetric and analytic inequalities for log-concave probability measures, *Ann. Probab.* 27 (1999) 1903–1921.
- [7] E.J. Candes, Y.C. Eldar, T. Strohmer, V. Voroninski, Phase retrieval via matrix completion, *SIAM J. Imaging Sci.* 6 (1) (Feb. 2013) 199–225.
- [8] E.J. Candes, X. Li, Solving quadratic equations via phaselift when there are about as many equations as unknowns, August 2012, ArXiv e-prints.
- [9] E.J. Candes, T. Strohmer, V. Voroninski, Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming, *Comm. Pure Appl. Math.* (2012), in press.
- [10] D. Chafa, O. Guédon, G. Lecue, A. Pajor, Interactions between compressed sensing, Random matrices and high-dimensional geometry, *Panor. Synth. - Soc. Math. Fr.* (2013), in press.
- [11] V. de la Peña, E. Giné, Decoupling: From Dependence to Independence, Probability and Its Applications, Springer, 1999.
- [12] R.M. Dudley, Uniform Central Limit Theorems, Cambridge Stud. Adv. Math., vol. 63, Cambridge University Press, 1999.
- [13] R.M. Dudley, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, *J. Funct. Anal.* 1 (1967) 290–330.
- [14] Y.C. Eldar, M. Mishali, Robust recovery of signals from a structured union of subspaces, *IEEE Trans. Inform. Theory* 55 (11) (2009) 5302–5316.
- [15] J.R. Fienup, Phase retrieval algorithms: A comparison, *Appl. Opt.* 21 (15) (1982) 2758–2769.
- [16] R.W. Gerchberg, W.O. Saxton, Phase retrieval by iterated projections, *Optik* 35 (1972).
- [17] A. Giannopoulos, Notes on isotropic convex bodies, available at <http://users.uoa.gr/~apgiannop/>.
- [18] Y. Gordon, A. Litvak, S. Mendelson, A. Pajor, Gaussian averages of interpolated bodies, *J. Approx. Theory* 149 (2008) 59–73.
- [19] R.W. Harrison, Phase problem in crystallography, *J. Opt. Soc. Amer. A* 10 (1993) 1045–1055.
- [20] N.E. Hurt, Phase Retrieval and Zero Crossings: Mathematical Methods in Image Reconstruction, vol. 52, Springer, 2001.
- [21] K. Jaganathan, S. Oymak, B. Hassibi, Recovery of sparse 1-D signals from the magnitudes of their Fourier transform, arXiv:1206.1405v1.
- [22] B. Klartag, S. Mendelson, Empirical processes and random projections, *J. Funct. Anal.* 225 (2005) 229–245.
- [23] V. Koltchinskii, Oracle Inequalities in Empirical Risk. Minimization and Sparse Recovery Problems, Lecture Notes in Math., vol. 2033, Springer, 2011.
- [24] S. Kwapiień, W.A. Woyczyński, Random Series and Stochastic Integrals: Single and Multiple, Birkhäuser, 1992.
- [25] G. Lecué, S. Mendelson, Learning subgaussian classes: Upper and minimax bounds, available at <http://arxiv.org/pdf/1305.4825.pdf>.
- [26] M. Ledoux, M. Talagrand, Probability in Banach Spaces, *Ergeb. Math. Grenzgeb.* (3), Springer-Verlag, 1991.
- [27] X. Li, V. Voroninski, Sparse signal recovery from quadratic measurements via convex programming, arXiv:1209.4785.
- [28] J. Matoušek, The uniform-convexity inequality for  $\ell_p$ -norms, <http://www.cims.nyu.edu/~naor/homepage%20files/bcl.pdf>.
- [29] S. Mendelson, Empirical processes with a bounded  $\psi_1$  diameter, *Geom. Funct. Anal.* 20 (2010) 988–1027.
- [30] S. Mendelson, On the geometry of subgaussian coordinate projections, available at <http://www.maths.anu.edu.au/~mendelso/publications.htm>.
- [31] S. Mendelson, A. Pajor, N. Tomczak-Jaegermann, Reconstruction and subgaussian operators, geometric and functional analysis, *Geom. Funct. Anal.* 17 (2007) 1248–1282.
- [32] S. Mendelson, G. Paouris, On generic chaining and the smallest singular values of random matrices with heavy tails, *J. Funct. Anal.* 262 (2012) 3775–3811.
- [33] S. Mendelson, G. Paouris, On the singular values of random matrices, *J. Eur. Math. Soc. (JEMS)* (2013), in press.
- [34] V.D. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoid of the unit ball of a normed  $n$ -dimensional space, in: Springer Lecture Notes, vol. 1376, 1989, pp. 64–104.
- [35] V.D. Milman, G. Schechtman, Asymptotic Theory of Finite Dimensional Normed Spaces, Lecture Notes in Math., vol. 1200, Springer, 1986.
- [36] M.L. Moravec, J.K. Romberg, R.G. Baraniuk, Compressive phase retrieval, in: Optical Engineering + Applications, International Society for Optics and Photonics, 2007, p. 670120.
- [37] H. Ohlsson, A.Y. Yang, R. Dong, S.S. Sastry, Compressive phase retrieval from squared output measurements via semidefinite programming, arXiv:1111.6323v3.
- [38] G. Pisier, The Volume of Convex Bodies and Banach Space Geometry, Cambridge University Press, 1989.
- [39] H.M. Quiney, Coherent diffractive imaging using short wavelength light sources: A tutorial review, *J. Modern Opt.* 57 (2010) 1109–1149.
- [40] Y. Shechtman, A. Beck, Y.C. Eldar, GESPAR: Efficient phase retrieval of sparse signals, <http://arxiv.org/pdf/1301.1018.pdf>, Jan. 2013.
- [41] Y. Shechtman, Y.C. Eldar, A. Szameit, M. Segev, Sparsity based sub-wavelength imaging with partially incoherent light via quadratic compressed sensing, *Opt. Express* 19 (16) (2011) 14807–14822.
- [42] N. Srivastava, R. Vershynin, Covariance estimation for distributions with  $2 + \epsilon$  moments, *Ann. Probab.* (2013), in press.
- [43] V.N. Sudakov, Gaussian processes and measures of solid angles in Hilbert space, *Sov. Math. Dokl.* 12 (1971) 412–415.
- [44] M. Talagrand, The Generic Chaining, Springer, 2005.
- [45] A.W. Van der Vaart, J.A. Wellner, Weak Convergence and Empirical Processes, Springer-Verlag, 1996.
- [46] A. Walther, The question of phase retrieval in optics, *Opt. Acta* 10 (1963) 41–49.