# ON THE IDENTIFICATION OF PARAMETRIC UNDERSPREAD LINEAR SYSTEMS

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**ABSTRACT**

Identification of time-varying linear systems, which introduce both time-shifts (delays) and frequency-shifts (Doppler-shifts), is a central task in many engineering applications. This paper studies the problem of identification of underspread linear systems (ULSs), defined as time-varying linear systems whose responses lie within a unit-area region in the delay–Doppler space, by probing them with a known input signal. The main contribution of the paper is that it characterizes conditions on the bandwidth and temporal support of the input signal that ensure identification of ULSs described by a finite set of delays and Doppler-shifts, and referred to as parametric ULSs, from single observations. In particular, the paper establishes that sufficiently-underspread parametric linear systems are identifiable as long as the time–bandwidth product of the input signal is proportional to the square of the total number of delay–Doppler pairs in the system. In addition, the paper describes a procedure that enables identification of parametric ULSs from an input train of pulses in polynomial time by exploiting recent results on sub-Nyquist sampling for time delay estimation and classical results on recovery of frequencies from a sum of complex exponentials.

## 1. INTRODUCTION

Identification of time-varying linear systems, which introduce both time-shifts (*delays*) and frequency-shifts (*Doppler-shifts*) to the input signal, is one of the central tasks in applications such as wireless communications and radar target detection. Mathematically, identification of a time-varying linear system $\mathcal{H}$ involves probing it with a *single* known input signal $x(t)$ and identifying $\mathcal{H}$ by analyzing the *single* system output $\mathcal{H}(x(t))$, as illustrated in Fig. 1. Kailath was the first to recognize that the identifiability of a time-varying linear system $\mathcal{H}$ from a single observation is directly tied to the area of the region $\mathcal{R}$ in the delay–Doppler space that contains $\mathcal{H}(\delta(t))$ [1]. Kailath's seminal work in [1] laid the foundations for the future works of Bello [2], Kozek and Pfander [3], and Pfander and Walnut [4], which establish the nonidentifiability of *overspread* linear systems—defined as systems with $\texttt{area}(\mathcal{R}) > 1$—and provide constructive proofs for the identifiability of *underspread* linear systems—defined as systems with $\texttt{area}(\mathcal{R}) < 1$.[1]

In this paper, we study the problem of identification of underspread linear systems (ULSs) whose responses can be described by a finite set of delays and Doppler-shifts. That is,

$$\mathcal{H}(x(t)) = \sum_{k=1}^{K} \alpha_k x(t-\tau_k) e^{j2\pi\nu_k t} \tag{1}$$

[1]It is still an open question as to whether *critically-spread* linear systems, which correspond to $\texttt{area}(\mathcal{R}) = 1$, are identifiable or nonidentifiable [4]; see [3] for a partial answer to this question when $\mathcal{R}$ is a rectangular region.

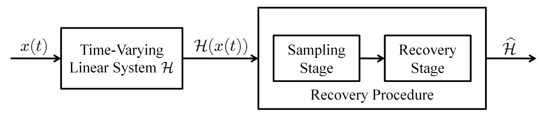



**Fig. 1**. Schematic representation of identification of a time-varying linear system $\mathcal{H}$ by probing it with a known input signal. Characterization of an identification scheme involves specification of the input probe, $x(t)$, and the accompanying sampling and recovery stages.

where $(\tau_k, \nu_k)$ denotes a delay–Doppler pair and $\alpha_k \in \mathbb{C}$ is the complex attenuation factor associated with $(\tau_k, \nu_k)$. Unlike most of the existing work in the literature, however, our goal in this paper is to explicitly characterize conditions on the bandwidth and temporal support of the input signal that ensure identification of such ULSs, henceforth referred to as *parametric* ULSs, from single observations. Specifically, note that the constructive proofs provided in [1–4] are for the identification of *arbitrary* ULSs. None of these results therefore shed any light on the bandwidth and temporal support of the input signal needed to ensure identification of parametric ULSs. On the contrary, the constructive proofs of [1–4] require use of input signals with infinite bandwidth and temporal support.

In contrast, this paper uses a constructive proof to establish that sufficiently underspread parametric linear systems are identifiable as long as the time–bandwidth product of the input signal is proportional to square of the total number of delay–Doppler pairs, $K$, in the system. Equally importantly, as part of our constructive proof, we specify the nature of the input signal (a finite train of pulses) and the structure of a corresponding polynomial-time recovery procedure that enable identification of parametric ULSs. The key developments in the paper leverage recent results on sub-Nyquist sampling for time-delay estimation [5] and classical results on direction-of-arrival estimation [6–8]. The connection to sub-Nyquist sampling in this regard can be understood by noting that the sub-Nyquist sampling results of [5] enable recovery of the delays associated with a parametric ULS using a small-bandwidth input signal. Further, the "train-of-pulses" nature of the input signal combined with results on recovery of frequencies from a sum of complex exponentials [9] allow recovery of the Doppler-shifts and attenuation factors using an input signal of small temporal support.

Several works in the past have considered identification of specialized versions of parametric ULSs. Specifically, [10–14] treat parametric ULSs whose delays and Doppler-shifts lie on a quantized grid in the delay–Doppler space. On the other hand, [15] considers the case in which there are no more than two Doppler-shifts associated with the same delay. The proposed recovery procedures in [15] also have exponential complexity, since they require exhaustive searches in a $K$-dimensional space, and stable initializations of these procedures stipulate that the system output be observed by an $M$-element antenna array with $M \gtrapprox K$. Finally, while the in-

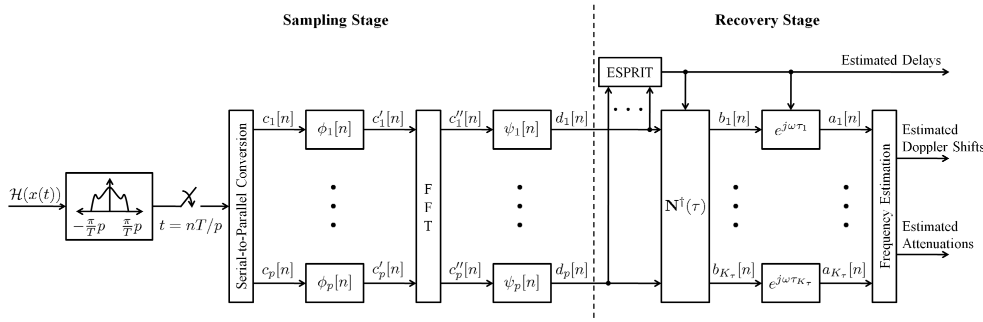



**Fig. 2**. Schematic representation of the polynomial-time recovery procedure proposed in the paper for identification of parametric underspread linear systems from single observations.

sights of [10–14] can be extended to arbitrary parametric ULSs by taking infinitesimally-fine quantization of the delay–Doppler space, this will require input signals with infinite bandwidth and temporal support. In contrast, our ability to avoid quantization of the delay–Doppler space is due to the fact that we treat the system-identification problem directly in the analog domain.

## 2. PROBLEM FORMULATION AND MAIN RESULTS

In this section, we formalize the problem of identification of parametric ULSs and state main results of the paper. We begin by first expressing the response of a parametric ULS $\mathcal{H}$ comprising of $K$ delay–Doppler pairs [cf. (1)] in terms of $K_\tau \leq K$ *distinct* delays[2]

$$\mathcal{H}(x(t)) = \sum_{i=1}^{K_\tau} \sum_{j=1}^{K_{\nu,i}} \alpha_{ij} x(t-\tau_i) e^{j2\pi\nu_{ij}t} \tag{2}$$

where $\nu_{ij}$ denotes the $j$th Doppler-shift associated with the $i$th distinct delay $\tau_i$, $\alpha_{ij} \in \mathbb{C}$ denotes the attenuation factor associated with the delay–Doppler pair $(\tau_i, \nu_{ij})$, and $K \stackrel{def}{=} \sum_{i=1}^{K_\tau} K_{\nu,i}$. Throughout the rest of the paper, we use $\boldsymbol{\tau} \stackrel{def}{=} \{\tau_i,\ i = 1, \ldots, K_\tau\}$ to denote the set of $K_\tau$ distinct delays associated with $\mathcal{H}$. The first main assumption that we make here concerns the footprint of $\mathcal{H}$ in the delay–Doppler space.

- **[A1]** The response $\mathcal{H}(\delta(t))$ of $\mathcal{H}$ lies within a rectangular region: $(\tau_i, \nu_{ij}) \in [0, \tau_{max}] \times [-\nu_{max}/2, \nu_{max}/2]$. This is indeed the case in many engineering applications (see, e.g., [12,14]), and the parameters $\tau_{max}$ and $\nu_{max}$ are termed as the *delay spread* and the *Doppler spread* of the system, respectively.

Next, we use $\mathcal{T}$ and $\mathcal{W}$ to denote the temporal support and the two-sided bandwidth of the known input signal $x(t)$ used to probe $\mathcal{H}$, respectively. We propose using input signals that correspond to a finite train of pulses:

$$x(t) = \sum_{n=0}^{N-1} x_n g(t - nT),\ 0 \leq t \leq \mathcal{T} \tag{3}$$

where $g(t)$ is a prototype pulse of bandwidth $\mathcal{W}$ that is (essentially) temporally supported on $[0, T]$ and is assumed to have unit energy ($\int |g(t)|^2 dt = 1$), and $\{x_n \in \mathbb{C}\}$ is an $N$-length probing sequence.

[2] Note that (1) and (2) are equivalent in terms of the mathematical characterization; nevertheless, we choose to express $\mathcal{H}(x(t))$ as in (2) so as to facilitate the forthcoming analysis.

The parameter $N$ is proportional to the time–bandwidth product of $x(t)$, which roughly defines the number of temporal degrees of freedom available for estimating $\mathcal{H}$: $N = \mathcal{T}/T \propto \mathcal{TW}$.[3] The final two assumptions that we make in this paper concern the relationship between the delay spread and the Doppler spread of $\mathcal{H}$ and the temporal support and bandwidth of $g(t)$.

- **[A2]** The delay spread of $\mathcal{H}$ is strictly smaller than the temporal support of $g(t)$ (in other words, $\tau_{max} < T$).
- **[A3]** The Doppler spread of $\mathcal{H}$ is much smaller than the bandwidth of $g(t)$ (in other words, $\nu_{max} \ll \mathcal{W}$).

Note that, since $\mathcal{W} \propto 1/T$, **[A3]** equivalently imposes that $\nu_{max}T \ll 1$; in words, this assumption states that the temporal scale of variations in the system response is large relative to the temporal scale of variations in $x(t)$. It is worth pointing out here that linear systems that are sufficiently underspread in the sense that $\tau_{max}\nu_{max} \ll 1$ can always be made to satisfy **[A2]** and **[A3]** for any given budget of the time–bandwidth product.

We are now ready to summarize the key findings of this paper concerning identification of parametric ULSs.

**Theorem 1.** *Suppose that $\mathcal{H}$ is a parametric ULS that is completely described by a total of $K = \sum_{i=1}^{K_\tau} K_{\nu,i}$ triplets $(\tau_i, \nu_{ij}, \alpha_{ij})$. Then, irrespective of the distribution of $\{(\tau_i, \nu_{ij})\}$ within the delay–Doppler space, the polynomial-time recovery procedure depicted in Fig. 2 with samples taken at $\{t = 2n\pi/\mathcal{W}\}$ uniquely identifies $\mathcal{H}$ from a single observation $\mathcal{H}(x(t))$ as long as **[A1]–[A3]** are satisfied, the probing sequence $\{x_n\}$ remains bounded away from zero in the sense that $|x_n| > 0\ \forall\ n = 0, \ldots, N-1$, and the time–bandwidth product of the input $x(t)$ satisfies the condition*

$$\mathcal{TW} \geq 8\pi K_\tau K_{\nu,max} \tag{4}$$

*where $K_{\nu,max} \stackrel{def}{=} \max_i K_{\nu,i}$ is the maximum number of Doppler-shifts associated with any one of the distinct delays. Further, the time–bandwidth product of $x(t)$ is guaranteed to satisfy* (4) *as long as $\mathcal{TW} \geq 2\pi(K+1)^2$.*

For the sake of brevity, we limit ourselves in the following to describing the polynomial-time recovery procedure used for identification of $\mathcal{H}$. We refer the reader to [16] for the accompanying conditions on $x(t)$ needed to guarantee identification of $\mathcal{H}$ using the proposed procedure, which in turn lead to a formal proof of Theorem 1 in [16].

[3] Recall that the temporal support and the bandwidth of an arbitrary pulse $g(t)$ are related to each other as $\mathcal{W} \propto 1/T$.

## 3. POLYNOMIAL-TIME IDENTIFICATION OF PARAMETRIC UNDERSPREAD LINEAR SYSTEMS

In this section, we characterize the polynomial-time recovery procedure proposed in the paper for identification of $\mathcal{H}$. In order to facilitate understanding of the proposed algorithm, shown in Fig. 2, we conceptually partition the procedure into two stages: the sampling stage and the recovery stage. Before describing these two stages in detail, we first make use of (2) and (3) to rewrite the output of $\mathcal{H}$ as

$$\mathcal{H}(x(t)) \approx \sum_{i=1}^{K_\tau} \sum_{n=0}^{N-1} a_i[n] g(t - \tau_i - nT) \tag{5}$$

where the sequences $\{a_i[n]\}, i = 1, \ldots, K_\tau$, are defined as

$$a_i[n] \stackrel{def}{=} \sum_{j=1}^{K_{\nu,i}} \alpha_{ij} x_n e^{j2\pi\nu_{ij} nT},\ n = 0, \ldots, N-1 \tag{6}$$

and (5) follows from the assumption that $\nu_{max} T \ll 1$, which implies that $e^{j2\pi\nu_{ij} t} \approx e^{j2\pi\nu_{ij} nT}$ for all $t \in [(n-1)T, nT)$.

### 3.1. The Sampling Stage

The sampling stage of our recovery method first passes the system output $\mathcal{H}(x(t))$ through a low-pass filter (LPF) whose impulse response is given by $s^*(-t)$ and then uniformly samples the output of this LPF at times $\{t = nT/p\}$. Here, we only require that the frequency response, $S^*(\omega)$, of the LPF is nonzero in the spectral band $\mathcal{F}$, defined as $\mathcal{F} \stackrel{def}{=} \left[-\frac{\pi}{T}p, \frac{\pi}{T}p\right]$, while $S^*(\omega)$ is zero for frequencies $\omega \notin \mathcal{F}$. The other condition that we have is that the parameter $p$ is even and satisfies the inequality $p \geq 2K_\tau$.

The sampling stage afterwards periodically splits the sampled sequence at the output of the LPF, which is generated at a rate of $p/T$, into $p$ slower sequences $\{c_\ell[n]\}$ at a rate of $1/T$ each using a serial-to-parallel converter. Next, we define two sets of digital filters $\{\phi_\ell[n], 1 \leq \ell \leq p\}$ and $\{\psi_\ell[n], 1 \leq \ell \leq p\}$ as follows:

$$\phi_\ell[n] \stackrel{def}{=} \text{IDTFT}\left\{\left[\sqrt{p}(-1)^{\ell-1} e^{j\omega(\ell-1)T/p}\right]^{-1}\right\}[n], \quad \text{and} \tag{7}$$

$$\psi_\ell[n] \stackrel{def}{=} \text{IDTFT}\Bigg\{\Big[\frac{1}{T} S^*\big(\omega + \frac{2\pi}{T}(\ell - p/2 - 1)\big) \times G\big(\omega + \frac{2\pi}{T}(\ell - p/2 - 1)\big)\Big]^{-1}\Bigg\}[n]. \tag{8}$$

Here, IDTFT denotes the inverse *discrete-time Fourier transform* (DTFT) operation and $G(\omega)$ denotes the frequency response of the prototype pulse $g(t)$. The next step in the sampling stage involves filtering the (sub)sequences $\{c_\ell[n]\}$ using the set of filters $\{\phi_\ell[n]\}$. This is followed by an application of the *fast Fourier transform* (FFT) to the outputs of the filters $\{\phi_\ell[n]\}$. The final step in the sampling stage involves filtering the resulting sequences using the set of filters $\{\psi_\ell[n]\}$ to get sequences $\{d_\ell[n]\}$; see Fig. 2 for a detailed schematic representation of the sampling stage.

### 3.2. The Recovery Stage

By defining a vector $\mathbf{d}[n]$ as the $p$-length vector whose $\ell$th element is $d_\ell[n]$, we have established in [16] that

$$\mathbf{d}[n] = \mathbf{N}(\boldsymbol{\tau})\, \mathbf{b}[n], \quad n \in \mathbb{Z}. \tag{9}$$

Here, $\mathbf{N}(\boldsymbol{\tau})$ is a $p \times K_\tau$ Vandermonde matrix whose $(m,i)$th element is given by $\mathbf{N}_{mi}(\boldsymbol{\tau}) \stackrel{def}{=} e^{-j\frac{2\pi}{T}(m-p/2-1)\tau_i}$. On the other hand, the elements of $\mathbf{b}[n]$ are discrete-time sequences that are inverse DTFT of the elements of $\mathbf{b}\left(e^{j\omega T}\right) \stackrel{def}{=} \mathbf{D}\left(e^{j\omega T}, \boldsymbol{\tau}\right) \mathbf{a}\left(e^{j\omega T}\right)$, where $\mathbf{D}\left(e^{j\omega T}, \boldsymbol{\tau}\right)$ is a $K_\tau \times K_\tau$ diagonal matrix whose $i$th diagonal element is given by $e^{-j\omega\tau_i}$ and $\mathbf{a}\left(e^{j\omega T}\right)$ is a $K_\tau$-length vector whose $i$th element is $A_i\left(e^{j\omega T}\right)$, the DTFT of $a_i[n]$ [cf. (6)].

Note that (9) can be viewed as an infinite ensemble of modified measurement vectors in which each element corresponds to a distinct matrix $\mathbf{N}(\boldsymbol{\tau})$ that, in turn, depends on the set of (distinct) delays $\boldsymbol{\tau}$. Linear measurement models of the form (9) have been studied extensively in the literature on DOA estimation. One specific class of methods that has proven to be quite useful in this area are the so-called *subspace methods* [6–8]. Consequently, our approach in the following is to first use subspace methods in order to recover the set $\boldsymbol{\tau}$ from $\mathbf{d}[n]$. Afterwards, since the Moore–Penrose pseudoinverse $\mathbf{N}^\dagger(\boldsymbol{\tau})$ of $\mathbf{N}(\boldsymbol{\tau})$ is a left inverse of $\mathbf{N}(\boldsymbol{\tau})$ because of the assumption that $p \geq 2K_\tau$, we recover the vector $\mathbf{a}\left(e^{j\omega T}\right)$ from the measurement vector $\mathbf{d}[n]$ as

$$\mathbf{a}\left(e^{j\omega T}\right) = \mathbf{D}^{-1}\left(e^{j\omega T}, \boldsymbol{\tau}\right) \mathbf{N}^\dagger(\boldsymbol{\tau})\, \mathbf{d}\left(e^{j\omega T}\right). \tag{10}$$

Finally, we recover the Doppler-shifts and the attenuation factors from $\mathbf{a}\left(e^{j\omega T}\right)$ by making another use of the subspace methods.

#### 3.2.1. *Recovery of the Delays*

We propose to recover $\boldsymbol{\tau}$ from $\mathbf{d}[n]$ using the following method that is based on the well-known ESPRIT algorithm [7] together with an additional smoothing stage [17].

(i) Construct the matrix $\overline{\mathbf{R}}_{dd} = \frac{1}{M}\sum_{m=1}^{M}\sum_{n\in\mathbb{Z}} \mathbf{d}_m[n]\mathbf{d}_m^H[n]$, where $\mathbf{d}_m$ is a $M = p/2$ length subvector that is given by $\mathbf{d}_m[n] = \begin{bmatrix} d_m[n] & d_{m+1}[n] & \ldots & d_{m+M}[n] \end{bmatrix}^T$.

(ii) Recover $K_\tau$ as the rank of $\overline{\mathbf{R}}_{dd}$.

(iii) Perform *singular value decomposition* of $\overline{\mathbf{R}}_{dd}$ and construct $\mathbf{E}_s$ consisting of the $K_\tau$ singular vectors corresponding to the $K_\tau$ nonzero singular values of $\overline{\mathbf{R}}_{dd}$ as its columns.

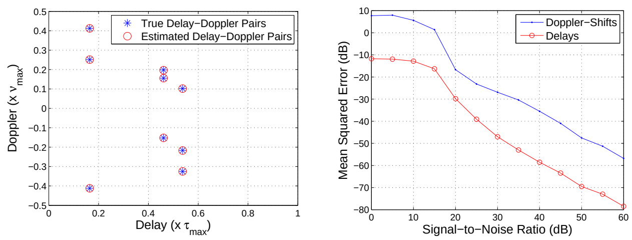
(iv) Compute $\boldsymbol{\Phi} = \mathbf{E}_{\mathbf{s}\downarrow}^\dagger \mathbf{E}_{\mathbf{s}\uparrow}$, where $\mathbf{E}_{s\uparrow}$ and $\mathbf{E}_{s\downarrow}$ are obtained by removing the first and the last row of $\mathbf{E}_s$, respectively.

(v) Compute the eigenvalues of $\boldsymbol{\Phi}$, $\lambda_i, i = 1, 2, \ldots, K_\tau$.

(vi) Recover the unknown delays as follows: $\tau_i = -\frac{T}{2\pi}\arg(\lambda_i)$.

#### 3.2.2. *Recovery of the Doppler-Shifts and Attenuation Factors*

Once the unknown delays are recovered, we can also easily recover the vectors $\mathbf{a}[n]$ from (10). Next, define for each delay $\tau_i$, the set of corresponding Doppler-shifts $\boldsymbol{\nu}_i$ as $\boldsymbol{\nu}_i \stackrel{def}{=} \{\nu_{ij},\ j = 1, \ldots, K_{\nu,i}\}$ and recall that the $i$th element of $\mathbf{a}[n]$ is given by (6). We can therefore write the $N$-length sequence $\{a_i[n]\}$ for each index $i$ in the matrix–vector form $\mathbf{a}_i = \mathbf{X}\mathbf{R}(\boldsymbol{\nu}_i)\boldsymbol{\alpha}_i$, where $\mathbf{a}_i$ is a length-$N$ vector whose $n$th element is $a_i[n]$, $\mathbf{X}$ is an $N \times N$ diagonal matrix whose $n$th diagonal element is given by $x_n$, $\mathbf{R}(\boldsymbol{\nu}_i)$ is an $N \times K_{\nu,i}$ Vandermonde matrix with $(n,j)$th element $e^{j2\pi\nu_{ij}nT}$, and $\boldsymbol{\alpha}_i$ is length-$K_{\nu,i}$ vector with $j$th element $\alpha_{ij}$. Now since the sequence $\{x_n\}$ is completely determined by the input signal $x(t)$, $\mathbf{X}$ can be inverted under the assumption that the probing sequence $\{x_n\}$ satisfies $|x_n| > 0\ \forall\, n = 0, \ldots, N-1$ and we therefore obtain




(a) (b)

**Fig. 3**. (a) An example illustrating the ability of the proposed recovery procedure to accurately identify parametric underspread linear systems. (b) An example illustrating the ability of the proposed recovery procedure to perform robustly in the presence of noise.

$\tilde{\mathbf{a}}_i = \mathbf{R}(\boldsymbol{\nu}_i)\boldsymbol{\alpha}_i$, where we have that $\tilde{\mathbf{a}}_i \stackrel{def}{=} \mathbf{X}^{-1}\boldsymbol{a}_i$. It now follows from a simple inspection of the elements of $\tilde{\mathbf{a}}_i$ that

$$\tilde{a}_i[n] = \sum_{j=1}^{K_{\nu,i}} \alpha_{ij} e^{j2\pi\nu_{ij}nT}, \quad 0 \leq n \leq N-1. \tag{11}$$

The recovery of the Doppler-shifts from the sequences $\{\tilde{a}_i[n]\}$ is now equivalent to the problem of recovering distinct frequencies from a (weighted) sum of complex exponentials. In our case, for each fixed index $i$, the frequency of the $j$th exponential is given by $\omega_{ij} = 2\pi\nu_{ij}nT$ and its amplitude is given by $\alpha_{ij}$.

Fortunately, the problem of recovering frequencies from a sum of complex exponentials has been studied extensively in the literature and various strategies exist for solving this problem. One of these techniques that has gained interest recently, especially in the literature on finite rate of innovation [18], is the *annihilating-filter* method. The annihilating-filter approach, in contrast to some of the other techniques, allows the recovery of the frequencies associated with the $i$th index even at the critical value of $N = 2K_{\nu,i}$. On the other hand, subspace methods [6–8] are generally more robust to noise but also require more than $2K_{\nu,i}$ samples. In summary, we conclude that there are a number of methods in the literature that can be used for recovery of the Doppler-shifts from (11) depending upon the temporal degrees of freedom $N$ available for identification of $\mathcal{H}$. In particular, if one is faced with the condition that $N = 2K_{\nu,i}$ for any one of the indices then the annihilating filter should be used.

Finally, under the assumption that the Doppler-shifts for each index $i$ have been recovered using any one of the subspace methods, the attenuation factors $\{\alpha_{ij}\}$ associated with each of the delays $\tau_i$ can simply be determined as $\boldsymbol{\alpha}_i = \mathbf{R}^\dagger(\boldsymbol{\nu}_i)\tilde{\mathbf{a}}_i, i = 1, \dots, K_\tau$, since $\mathbf{R}^\dagger(\boldsymbol{\nu}_i)\mathbf{R}(\boldsymbol{\nu}_i) = \mathbf{I}$ because of the requirement $N \geq 2K_{\nu,i}$.

## 4. CONCLUSION

In this paper, we have revisited the problem of identification of parametric underspread linear systems and presented a polynomial-time recovery procedure that enables identification of such systems as long as the time–bandwidth product of the input signal is proportional to the square of the total number of delay–Doppler pairs (cf. Theorem 1 and [16]). Extensive simulation results reported in [16] confirm that—as long as the time–bandwidth product of the input signal satisfies the requisite conditions—the proposed recovery procedure is quite robust to noise and other implementation issues (also, see Fig. 3(a) and Fig. 3(b)). This makes our algorithm extremely useful for application areas in which the system performance depends critically on the ability to resolve closely spaced delay–Doppler pairs. In particular, as elaborated in [16], our method for identifying parametric underspread linear systems can be used for super-resolution target detection using radar. We conclude this paper by referring the reader to [16] for a formal proof of Theorem 1, an extensive discussion of its relationship with existing work, and its application in super-resolution radar.

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