# Phase Retrieval with Masks using Convex Optimization

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***Abstract*—Signal recovery from the magnitude of the Fourier transform, or equivalently, from the autocorrelation, is a classical problem known as phase retrieval. Due to the absence of phase information, some form of additional information is required in order to be able to uniquely identify the underlying signal. In this work, we consider the problem of phase retrieval using masks. Due to our interest in developing robust algorithms with theoretical guarantees, we explore a convex optimization-based framework. In this work, we show that *two* specific masks (each mask provides $2n$ Fourier magnitude measurements) or *five* specific masks (each mask provides $n$ Fourier magnitude measurements) are sufficient for a convex relaxation of the phase retrieval problem to provably recover almost all signals (up to global phase). We also show that the recovery is stable in the presence of measurement noise. This is a *significant* improvement over the existing results, which require $O(\log^2 n)$ random masks (each mask provides $n$ Fourier magnitude measurements) in order to guarantee unique recovery (up to global phase). Numerical experiments complement our theoretical analysis and show interesting trends, which we hope to explain in a future publication.**

***Index Terms*—phase retrieval, masked signals, autocorrelation, semidefinite programming, convex optimization**

## I. Introduction

In many physical measurement systems such as X-ray crystallography [1], optics [2], astronomical imaging [3], speech recognition [4], etc, the magnitude square of the Fourier transform is the measurable quantity. Recovering a signal from its Fourier transform magnitude, or equivalently, its autocorrelation, is classically known as phase retrieval.

This problem has attracted a lot of attention from researchers over the last few decades and a wide variety of techniques have been developed (see [6] for a comprehensive summary of classical approaches, a more recent survey can be found in [7]). However, the search for robust algorithms with provable recovery guarantees is still ongoing.

The mapping from signals to their Fourier transform magnitude is not one-to-one. In order to overcome this issue, researchers have tried various methods which can be broadly classified into two categories:

*(i) Prior information*: In some applications, it is possible to have prior information on the signal. The set of locations where the signal can be non-zero, for instance, considerably improves the performance of classical algorithms ( [5], [6]). Recently, sparsity as a prior has been explored by various researchers ( [8]–[12]).

*(ii) Additional intensity measurements*: In some applications, it is possible to obtain additional intensity measurements. Common approaches include the use of modulated light beams and masks right after the sample ( [13], [14]). In some applications, it is natural to define the Short-Time Fourier Transform. The idea of achieving uniqueness using overlapping short-time sections has been explored by various researchers ( [15]–[17]).

Recently, there has been a lot of interest in convex optimization-based approaches to solve the phase retrieval problem ( [18]–[20]). In this work, we consider the convex formulation of the problem of phase retrieval with masks. This setup was recently explored in ( [21]–[23]). Candes et al. [21] showed that $O(\log^4 n)$ random masks chosen from a particular distribution are sufficient to provably recover the signal (throughout this work, whenever we speak of recovery, we mean up to a global phase). Gross et al. [22] tightened the analysis and showed that $O(\log^2 n)$ random masks chosen from a particular distribution are sufficient to provably recover the signal. While these results are theoretically very exciting, random masks (chosen from distributions) are very difficult to construct in practice. Also, while these results are not very far off from the optimum number of *random masks* order-wise, the numerical constants involved are very high.

In this work, in contrast, we consider *specific masks* instead of random masks. We show that *two* specific simple masks (each mask provides $2n$ measurements) or $five$ specific simple masks (each mask provides $n$ measurements) are sufficient for the convex formulation to *provably* and *stably* recover almost all signals.

This paper is organized as follows. In Section 2, we mathematically set up the phase retrieval problem using masks and develop a convex formulation. Section 3 contains the unique recovery/ stability theorems for the convex formulation with specific masks. In Section 4, we present the results of various numerical experiments. Section 5 concludes the paper. The proofs of various theorems are provided in the Appendix.

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## II. Convex Formulation of Phase Retrieval

Let $\mathbf{x} = (x[0], x[1], ..., x[n-1])^T$ be a complex-valued signal of length $n$. For $1 \leq l \leq L$, let $\mathbf{D}_l$ be a diagonal matrix (corresponding to $l^{th}$ mask) with diagonal entries $\{d_l[0], d_l[1], ..., d_l[n-1]\}$, where $L$ is the number of masks used. If $\mathbf{F}$ is the $n$-DFT matrix and $\mathbf{y}_l$ is the Fourier transform of $(\mathbf{D}_l\mathbf{x})$, then the phase retrieval problem using masks can be stated as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && |\mathbf{y}_l| = |\mathbf{F}\mathbf{D}_l\mathbf{x}| : \quad 1 \leq l \leq L. \end{aligned} \tag{1}$$

Since the magnitude-square of the Fourier transform and circular autocorrelation are Fourier pairs, (1) can be equivalently stated as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && \mathbf{b}_l = (\mathbf{D}_l\mathbf{x}) \circledast (\tilde{\mathbf{D}_l\mathbf{x}}) : \quad 1 \leq l \leq L, \end{aligned} \tag{2}$$

where $(\tilde{\mathbf{D}_l\mathbf{x}})$ is the conjugate-flipped version of $(\mathbf{D}_l\mathbf{x})$, $\circledast$ is the circular convolution operator and $\mathbf{b}_l = (b_l[0], b_l[1], ..., b_l[n-1])^T$ is the circular autocorrelation of $(\mathbf{D}_l\mathbf{x})$.

Researchers have observed that zero padding the signal $\mathbf{x}$ with $n$ zeros and considering the $2n$-DFT greatly reduces the number of solutions (this is possible to do in most instances of the phase retrieval problem). In this setup, the phase retrieval problem using masks can be stated as

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && \mathbf{a}_l = (\mathbf{D}_l\mathbf{x}) \star (\tilde{\mathbf{D}_l\mathbf{x}}) : \quad 1 \leq l \leq L, \end{aligned} \tag{3}$$

where $(\tilde{\mathbf{D}_l\mathbf{x}})$ is the conjugate-flipped version of $(\mathbf{D}_l\mathbf{x})$, $\star$ is the convolution operator and $\mathbf{a}_l = (a_l[0], a_l[1], ..., a_l[n-1])^T$ is the autocorrelation of $(\mathbf{D}_l\mathbf{x})$.

**Theorem II.1.** *Consider any arbitrary signal $\mathbf{x} \in \mathbb{C}^n$. For almost all $\mathbf{x}$, the feasible set of (3) is unique (and hence $\mathbf{x}$ can be recovered) up to a global phase if measurements are taken using two masks defined by $\mathbf{D}_\alpha$ and $\mathbf{D}_\beta$ which satisfy*

*(i) $d_\alpha[i] \neq 0$ or $d_\beta[i] \neq 0$ for each $0 \leq i \leq n-1$*

*(ii) $d_\alpha[i]d_\beta[i] \neq 0$ for some $0 \leq i \leq n-1$.*

*Proof:* The proof of this theorem involves a technique called *dimension counting*, and is omitted in this paper. The first condition is not surprising. If $d_\alpha[i] = 0$ and $d_\beta[i] = 0$ for some $0 \leq i \leq n-1$, then no information about $x[i]$ is obtained and hence $x[i]$ cannot be recovered. The second condition, that both the masks should collect information about $x[i]$ for some $0 \leq i \leq n-1$, can be used to argue (using dimension counting) that the set of signals for which (3) does not have a unique feasible point has a dimension strictly less than the dimension of the set of all signals. The interested reader is referred to ( [17], [26]) for details. ■

Hence, in principle, two masks (each providing $2n$ Fourier magnitude measurements) are sufficient in order to be able to uniquely recover almost all signals (up to a global phase). Next, we shift our attention to recovery algorithms.

Problems (2) and (3) have quadratic constraints. A technique, popularly known as *lifting*, has enjoyed success in solving some quadratically-constrained problems (for example, see [24]). The steps can be summarized as follows: (i) embed the problem in a higher dimensional space using the transformation $\mathbf{X} = \mathbf{x}\mathbf{x}^\star$, a process which converts the problem of recovering a signal with quadratic constraints into a problem of recovering a rank-one matrix with affine constraints (ii) relax the rank-one constraint to obtain a convex program.

Using this technique, the convex program to solve (2) can be written as

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{subject to} && \mathcal{B}(\mathbf{X}) = \mathbf{b} \\ &&& \mathbf{X} \succcurlyeq 0, \end{aligned} \tag{4}$$

where $\mathcal{B}(\mathbf{X}) = \mathbf{b}$ is the set of affine constraints $\{b_l[i] = \sum_{j=0}^{n-1} d_l[j]d_l[j+i]X_{j,j+i} : 0 \leq i \leq n-1, 1 \leq l \leq L\}$ (the indices are defined modulo $n$).

Similarly, the convex program to solve (3) can be written as

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{subject to} && \mathcal{A}(\mathbf{X}) = \mathbf{a} \\ &&& \mathbf{X} \succcurlyeq 0, \end{aligned} \tag{5}$$

where $\mathcal{A}(\mathbf{X}) = \mathbf{a}$ is the set of affine constraints $\{a_l[i] = \sum_{j=0}^{n-1-i} d_l[j]d_l[j+i]X_{j,j+i} : 0 \leq i \leq n-1, 1 \leq l \leq L\}$.

## III. Mask Design for Stable Recovery

In this section, we first describe the masks which reduce the feasible set of (4) and (5) to a unique point in the noiseless setup. We then consider the noisy setup, reformulate (4) and (5) and show that these masks ensure stable recovery in the presence of measurement noise.

Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be diagonal matrices with diagonal entries

$$d_1[i] = 1 \qquad 0 \leq i \leq n-1 \tag{6}$$

$$d_2[i] = \begin{cases} 0 & i = 0 \\ 1 & 1 \leq i \leq n-1. \end{cases} \tag{7}$$

There is a simple combinatorial recovery algorithm for this particular choice of masks. The measurements obtained using the masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ are

$$a_1[i] = \sum_{j=0}^{n-1-i} x[j]x^\star[j+i] \quad \& \quad a_2[i] = \sum_{j=1}^{n-1-i} x[j]x^\star[j+i]. \tag{8}$$

for $0 \leq i \leq n-1$. Since $a_1[0] - a_2[0] = x[0]x^\star[0]$, we can infer $x[0]$ up to a phase. Using $a_1[i] - a_2[i] = x[0]x^\star[i]$ for $1 \leq i \leq n-1$, we can infer the entire signal $\mathbf{x}$ up to a global phase. However, this method of recovery is unstable in the presence of measurement noise as it does not optimally make use of the available measurements. Hence, we consider a convex relaxation-based recovery algorithm.

**Theorem III.1.** *Consider any arbitrary signal* $\mathbf{x} \in \mathbb{C}^n$ *such that* $x[0] \neq 0$*. Suppose measurements are taken with the masks defined by* $\mathbf{D}_1$ *and* $\mathbf{D}_2$*, the convex program (5) has a unique feasible point, namely,* $\mathbf{x}\mathbf{x}^\star$*, and hence* $\mathbf{x}$ *can be uniquely recovered (up to a global phase).*

Let $\mathbf{D}_3, \mathbf{D}_4, \mathbf{D}_5$, $\mathbf{D}_6$ and $\mathbf{D}_7$ be diagonal matrices with diagonal entries

$$d_3[i] = \begin{cases} 1 & 0 \leq i \leq n/2 - 1 \\ 0 & n/2 \leq i \leq n-1 \end{cases} \tag{9}$$

$$d_4[i] = \begin{cases} 0 & i = 0 \\ 1 & 1 \leq i \leq n/2 - 1 \\ 0 & n/2 \leq i \leq n-1 \end{cases} \tag{10}$$

$$d_5[i] = \begin{cases} 0 & 0 \leq i \leq n/2 \\ 1 & n/2 + 1 \leq i \leq n-1 \end{cases} \tag{11}$$

$$d_6[i] = \begin{cases} 0 & 0 \leq i \leq n/2 - 1 \\ 1 & n/2 \leq i \leq n-1. \end{cases} \tag{12}$$

$$d_7[i] = \begin{cases} 0 & 0 \leq i \leq n/4 - 1 \\ 1 & n/4 \leq i \leq 3n/4 - 1 \\ 0 & 3n/4 \leq i \leq n-1. \end{cases} \tag{13}$$

**Theorem III.2.** *Consider any arbitrary signal* $\mathbf{x} \in \mathbb{C}^n$ *such that* $x[0], x[n/2-1], x[n/2] \neq 0$*. Suppose measurements are taken with the masks defined by* $\mathbf{D}_3, \mathbf{D}_4, \mathbf{D}_5$*,* $\mathbf{D}_6$ *and* $\mathbf{D}_7$*, the convex program (4) has a unique feasible point, namely,* $\mathbf{x}\mathbf{x}^\star$*, and hence* $\mathbf{x}$ *can be uniquely recovered (up to a global phase).*

In the real world, measurements are contaminated by noise. Suppose the noise corresponding to each measurement is bounded by $\epsilon$. We consider the solution to

$$\begin{aligned} \text{minimize} \quad & trace(\mathbf{X}) \\ \text{subject to} \quad & ||\mathcal{B}(\mathbf{X}) - \mathbf{b}||_\infty \leq \epsilon \\ & \mathbf{X} \succcurlyeq 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \text{minimize} \quad & trace(\mathbf{X}) \\ \text{subject to} \quad & ||\mathcal{A}(\mathbf{X}) - \mathbf{a}||_\infty \leq \epsilon \\ & \mathbf{X} \succcurlyeq 0. \end{aligned} \tag{15}$$

**Theorem III.3.** *Consider any arbitrary signal* $\mathbf{x} \in \mathbb{C}^n$ *such that* $||\mathbf{x}||_1 \leq \beta$ *and* $|x[0]| \geq \gamma > 0$ *for some* $\beta, \gamma$*. Suppose measurements are taken with the masks defined by* $\mathbf{D}_1$ *and* $\mathbf{D}_2$*, the solution to the convex program (15)* $\hat{\mathbf{X}}$ *obeys*

$$||\hat{\mathbf{X}} - \mathbf{x}\mathbf{x}^\star||_2 \leq C_0(\beta, \gamma)\epsilon. \tag{16}$$

*for some numerical constant* $C_0(\beta, \gamma)$*.*

The proofs of Theorems III.1, III.2 and III.3 are provided in the Appendix.

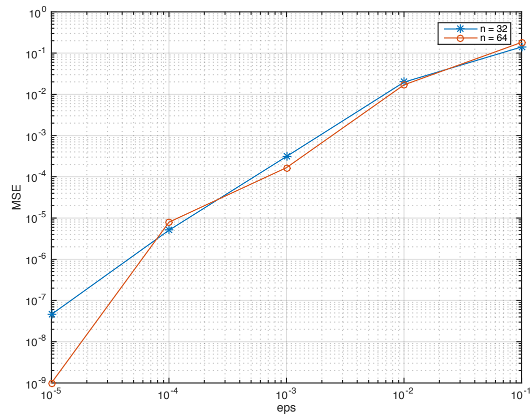


Fig. 1. Performance of convex program (15) with masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ for various choices of $n$ and $\epsilon$

## IV. Numerical Experiments

In this section, we demonstrate the effectiveness of the proposed masks using numerical simulations. We consider signal lengths $n = 32$ and $n = 64$. The signal values are chosen from an i.i.d. Gaussian distribution, and $\gamma = 0.1$ is used.

In the first set of simulations, we evaluate the stability of recovery using (15) with masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ in the noisy setup. For various values of measurement noise bounds $\epsilon$, we plot the values of the reconstruction mean square error, defined by

$$MSE = \frac{||\hat{\mathbf{X}} - \mathbf{x}\mathbf{x}^\star||_2^2}{||\mathbf{x}\mathbf{x}^\star||_2^2}.$$

Figure 1 contains the results of this set of simulations, the stability of recovery in the presence of measurement noise can be clearly seen.

Let $\mathbf{D}_8$ be a diagonal matrix with diagonal entries

$$d_8[i] = \begin{cases} 0 & 0 \leq i \leq t-1 \\ 1 & t \leq i \leq n-1 \end{cases} \tag{17}$$

In the second set of experiments, we evaluate the performance of (5) with masks defined by $\mathbf{D}_1$ and $\mathbf{D}_8$ in the noiseless setup. For various values of $t$, we plot the probability of successful recovery of the underlying signal.

Figure 2 contains the results of this set of simulations. We see that for $t$ greater than $\sim n/2$, the program fails to recover the underlying signal. For $t$ less than $\sim n/2$, the program recovers the underlying signal with very high probability. We hope to provide a theoretical analysis of this *phase transition* in a future publication.

When the first $t$ diagonal entries of $\mathbf{D}_8$ are chosen from a random distribution instead of 0, the program does not *recover the underlying signal with high probability* if $t > 1$. This

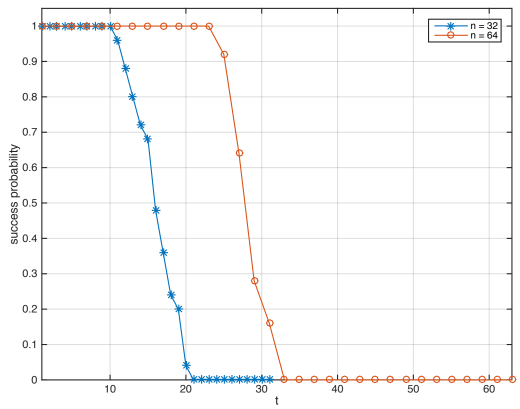


Fig. 2. Performance of convex program (5) with masks defined by $\mathbf{D}_1$ and $\mathbf{D}_8$ for various choices of $n$ and $t$

suggests that only *particular* sets of two masks have the ability to recover signals with high probability. A characterization of this set would be a very interesting topic for future study.

## V. Conclusion

In this work, we studied the problem of phase retrieval with masks using a convex optimization-based framework. We showed that two specific masks (each mask provides $2n$ Fourier magnitude measurements) or five specific masks (each mask provides $n$ Fourier magnitude measurements) are sufficient for the convex formulation to provably and stably recover the underlying signal in the presence of measurement noise. Numerical simulations verify the theory and suggest directions for future work.

## VI. Appendix

### A. Proof of Theorem III.1

The affine set of measurements $\mathcal{A}(\mathbf{X}) = \mathbf{a}$ obtained with the masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ are

$$a_1[i] = \sum_{j=0}^{n-1-i} X_{j,j+i} \quad \& \quad a_2[i] = \sum_{j=1}^{n-1-i} X_{j,j+i}. \tag{18}$$

From a matrix sensing perspective, these set of measurements fix (i) the entries of the first row and column of $\mathbf{X}$ (can be seen by subtracting $\mathbf{a}_2$ from $\mathbf{a}_1$) (ii) the sum along the $i^{th}$ off-diagonal of $\mathbf{X}$ excluding the first row and column for each $i$ (can be seen as measurements due to $\mathbf{a}_2$). We will show the following: if $\mathbf{x}\mathbf{x}^\star$ satisfies (18), then it is the only positive semidefinite matrix which satisfies (18).

Let $T$ be the set of symmetric matrices of the form

$$T = \{\mathbf{X} = \mathbf{x}\mathbf{w}^\star + \mathbf{w}\mathbf{x}^\star : \mathbf{w} \in \mathbb{C}^n\}$$

and $T^\perp$ be its orthogonal complement. $T$ can be interpreted as the tangent space at $\mathbf{x}\mathbf{x}^\star$ to the manifold of symmetric matrices of rank 1. Influenced by [18], we use $\mathbf{X}_T$ and $\mathbf{X}_{T^\perp}$ to denote the projection of a matrix $\mathbf{X}$ onto the subspaces $T$ and $T^\perp$ respectively.

Standard duality arguments in semidefinite programming show that sufficient conditions for $\mathbf{x}\mathbf{x}^\star$ to be the unique optimizer to (5) is:

(i) *Condition 1*: $\mathbf{X} \in T \quad \& \quad \mathcal{A}(\mathbf{X}) = 0 \Rightarrow \mathbf{X} = 0$
(ii) *Condition 2*: There exists a *dual certificate* $\mathbf{W}$ in the range space of $\mathcal{A}^\star$ obeying:
- $\mathbf{W}\mathbf{x} = 0$
- $rank(\mathbf{W}) = n - 1$
- $\mathbf{W} \succcurlyeq 0$

The proof of this is straightforward, and can be found in any standard reference on semidefinite programming (for example, see [25]).

First, we will show that the measurement operator $\mathcal{A}$ obtained with the masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ satisfies *Condition 1*.

The set of constraints $\mathcal{A}(\mathbf{X}) = 0$ fix the entries of the first row and column of $\mathbf{X}$ to 0, i.e.,
$X[0, i] = X[i, 0] = 0 : \quad 0 \leq i \leq n - 1.$

Since $\mathbf{X} \in T$, we can write $\mathbf{X} = \mathbf{x}\mathbf{w}^\star + \mathbf{w}\mathbf{x}^\star$ for some $\mathbf{w} = (w[0], w[1], ..., w[n-1])^T$. We have
$x[0]w[0] = 0 \Rightarrow w[0] = 0$
$x[0]w[i] + w[0]x[i] = 0 : 1 \leq i \leq n-1 \Rightarrow w[i] = 0 : 1 \leq i \leq n - 1.$

Hence, $\mathbf{w} = 0$, which implies $\mathbf{X} = 0$.

Next, we will show that *Condition 2* is satisfied.

The range space of $\mathcal{A}^\star$ obtained with the masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ is the set of all symmetric matrices whose principal submatrix obtained by removing the first row and column has Toeplitz structure (this can be easily seen by writing the dual of (5)). Suppose $\mathbf{z} = -(x[1], x[2], ..., x[n-1])^T / x[0]$ (well defined if $x[0] \neq 0$) and $\mathbf{I}_{n-1}$ is the identity matrix of size $n - 1$. Consider the following matrix:

$$\mathbf{W} = \begin{bmatrix} \mathbf{z}^\star\mathbf{z} & \mathbf{z}^\star \\ \mathbf{z} & \mathbf{I}_{n-1} \end{bmatrix} \tag{19}$$

$\mathbf{W}$ is in the range space of $\mathcal{A}^\star$ as $\mathbf{I}_{n-1}$ has Toeplitz structure. Also, $\mathbf{W}\mathbf{x} = 0$ as $\mathbf{z}x[0] + (x[1], x[2], ..., x[n-1])^T = 0$. By writing out the characteristic equation, it is straightforward to see that the eigenvalues of $\mathbf{W}$ are $\{1 + ||\mathbf{z}||^2, 1, 1, .....1, 0\}$. Hence, $rank(\mathbf{W}) = n - 1$ and $\mathbf{W} \succcurlyeq 0$. This completes the proof.

### B. Proof of Theorem III.2

Consider the set of measurements obtained with the masks defined by $\mathbf{D}_3$ and $\mathbf{D}_4$. Since both these masks are zero throughout the region $n/2 \leq i \leq n - 1$, the circular autocorrelation and autocorrelation measurements are identical for $0 \leq i \leq n/2 - 1$. These set of measurements correspond to the setup of Theorem III.1 with $n$ replaced by $n/2$. Hence, if $x[0] \neq 0$, $X[i, j]$ in the region $0 \leq i, j \leq n/2 - 1$ can be uniquely recovered.

Similarly, using the measurements obtained with the masks defined by $\mathbf{D}_5$ and $\mathbf{D}_6$, $X[i,j]$ in the region $n/2 \leq i,j \leq n-1$ can be uniquely recovered if $x[n/2] \neq 0$.

The measurements obtained with the mask defined by $\mathbf{D}_7$ recovers the precise value of $X[n/2-1, n/2]$. If $X[n/2-1, n/2] \neq 0$, given all the entries that have been determined till now, it is straightforward to see that $\mathbf{x}\mathbf{x}^\star$ is the only feasible positive semidefinite completion [27]. Hence, $\mathbf{x}$ can be uniquely identified up to a global phase.

### C. Proof of Theorem III.3

Let $\hat{\mathbf{X}} = \mathbf{x}\mathbf{x}^\star + \mathbf{H}$ be the optimizer of (15). Since both $\mathbf{x}\mathbf{x}^\star$ and $\hat{\mathbf{X}}$ are feasible, we have $||\mathcal{A}(\mathbf{H})||_\infty = ||\mathcal{A}(\hat{\mathbf{X}} - \mathbf{X})||_\infty \leq ||\mathcal{A}(\hat{\mathbf{X}}) - \mathbf{a}||_\infty + ||\mathcal{A}(\mathbf{x}\mathbf{x}^\star) - \mathbf{a}||_\infty \leq 2\epsilon$. Also, note that $\mathbf{H}_{T^\perp} \succcurlyeq 0$.

Since the measurement operator $\mathcal{A}$ obtained with the masks defined by $\mathbf{D}_1$ and $\mathbf{D}_2$ fixes, for each $i$, the sum along the $i^{th}$ off-diagonal and the sum along the $i^{th}$ off-diagonal excluding the first row and column, we have

$\left|\sum_{j\geq 1} H[j,j]\right| \leq 2\epsilon$

$|H[0,i]| \leq \left|\sum_{j\geq 0} H[j,i+j]\right| + \left|\sum_{j\geq 1} |H[j,i+j]\right| \leq 4\epsilon : 0 \leq i \leq n-1$.

First, we bound $||\mathbf{H}_{T^\perp}||_2$ using the dual certificate (19):

$\langle \mathbf{W}, \mathbf{H} \rangle = \sum_i \sum_j W[i,j]H[i,j] \leq |\mathbf{z}^T\mathbf{z}||H[0,0]| + 2\sum_{j\geq 1} |z[j-1]||H[0,j]| + \left|\sum_{j\geq 1} H[j,j]\right| \leq c_1(\beta,\gamma)\epsilon$.

for some constant $c_1(\beta,\gamma)$. This holds because $|\mathbf{z}^\star\mathbf{z}|$ and $|z[i-1]| : 1 \leq i \leq n$ are bounded due to the fact that $||\mathbf{x}||_1 \leq \beta$ and $|x[0]| > \gamma$ respectively. Also,

$\langle \mathbf{W}, \mathbf{H} \rangle = \langle \mathbf{W}, \mathbf{H}_T \rangle + \langle \mathbf{W}, \mathbf{H}_{T^\perp} \rangle \geq ||\mathbf{H}_{T^\perp}||_2$

where we use the following facts: $\langle \mathbf{W}, \mathbf{H}_T \rangle = 0$ (due to $\mathbf{W}\mathbf{x} = 0$), $\mathbf{H}_{T^\perp} \succcurlyeq 0$, $\mathbf{W}$ has a minimum eigenvalue $1$ in $T^\perp$. Hence,

$||\mathbf{H}_{T^\perp}||_2 \leq c_1(\beta,\gamma)\epsilon$.

Next, we bound $||\mathbf{H}_T||_2$. Since $|H[0,i]| \leq 4\epsilon : 0 \leq i \leq n-1$

$|H_T[0,i] + H_{T^\perp}[0,i]| \leq 4\epsilon \Rightarrow |H_T[0,i]| \leq (c_1(\beta,\gamma) + 4)\epsilon$

Since we can write $\mathbf{H}_T = \mathbf{x}\mathbf{w}^\star + \mathbf{w}\mathbf{x}^\star$ for some $\mathbf{w} = (w[0], w[1], ..., w[n-1])^T$, we have

$|w[0]| \leq (c_1(\beta,\gamma) + 4)\epsilon/(2|x[0]|)$

$|w[i]| \leq (c_1(\beta,\gamma) + 4)\epsilon/(2|x[0]|)(2 + |x[i]/x[0]|) : 1 \leq i \leq n-1$

Explicitly writing out $||\mathbf{H}_T||_2$ in terms of $\mathbf{x}$ and $\mathbf{w}$, and using the above bounds on $\mathbf{w}$, we get $||\mathbf{H}_T||_2 \leq c_2(\beta,\gamma)\epsilon$ for some constant $c_2(\beta,\gamma)$.

Hence, $||\mathbf{H}||_2 = ||\mathbf{H}_T||_2 + ||\mathbf{H}_T^\perp||_2 \leq C_0(\beta,\gamma)\epsilon$ for some constant $C_0(\beta,\gamma)$. This completes the proof. We wish to point out that the analysis provided here is not tight, the constants we observed in numerical simulations are *significantly* lesser than the ones derived in this section.

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