

ON CONDITIONS FOR UNIQUENESS IN SPARSE PHASE RETRIEVAL

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ABSTRACT

The phase retrieval problem has a long history and is an important problem in many areas of optics. Theoretical understanding of phase retrieval is still limited and fundamental questions such as uniqueness and stability of the recovered solution are not yet fully understood. This paper provides several additions to the theoretical understanding of sparse phase retrieval. In particular we show that if the measurement ensemble can be chosen freely, as few as $4k - 1$ phaseless measurements suffice to guarantee uniqueness of a k -sparse M -dimensional real solution. We also prove that $2(k^2 - k + 1)$ Fourier magnitude measurements are sufficient under rather general conditions.

Index Terms— Phase retrieval, complement property, compressive phase retrieval.

1. INTRODUCTION

In many areas in optics, physical limitations make it impossible to measure the phase. If the signal is real, then the sign is lost and if the signal is complex, the phase. Even though the phase is not measured, it often contains valuable information. For example, in X-ray crystallography [1, 2], only the magnitude of the Fourier transform is observed. If the phase would be observable, then the inverse Fourier transform would directly give the atomic structure of the crystal considered. Therefore the phase has to be retrieved before structural information can be explored.

The problem of retrieving the phase from intensity measurements is often referred to as the *phase retrieval problem*. The problem is by nature often ill-posed and early methods relied on additional information about the sought signal, such as band limitation, nonzero support, and nonnegativity to successfully recover the signal. The Gerchberg-Saxton algorithm is one of the popular methods for recovery. It utilizes a prior

on the support and alternates between the Fourier and inverse Fourier transforms to obtain a phase estimate from a set of Fourier magnitude measurements [3, 4]. A more recent development [5, 6, 7] has shown that *e.g.*, random collections of measurement vectors are rich enough to provide a well posed phase retrieval problem.

There has also been recent interest in sparse phase retrieval. In contrast to the literature on compressive sensing, which assumes a linear relation between measurements and the sparse unknown and is quite mature, the literature on sparse phase retrieval is still developing. Recent work has demonstrated that as in the case of linear measurements, the number of intensity measurements required to recover the true solution can be reduced by taking into account that the sought signal is sparse [8, 9, 10, 11, 12, 7, 13].

Even though [5, 6, 7] showed that there exist collections of measurement vectors that provide accurate phase estimates, it is still not fully understood what properties these sets need to satisfy for the phase retrieval map to be injective. The first attempt to try to characterize these properties was given in [14] (later refined in [15]). In particular the authors derived necessary and sufficient conditions for injectivity for a real signal and real collection of measurement vectors. Injectivity in the real case was also discussed in [7]. For the complex case (complex signal and complex collection of measurement vectors), [15] gave necessary conditions for injectivity.

As for sparse phase retrieval, it was shown in [7] that $\mathcal{O}(k \log(M/k))$ real measurement vectors are sufficient for stable recovery of a k -sparse M -dimensional real signal. This means that the number of measurements needed for recovery from quadratic measurements is the same, up to a multiplicative scalar, as for linear measurements. The work in [16] extended results presented in [14] and derived bounds on the number of measurements needed for unique recovery in the sparse real case (real measurement vectors and real sparse signal) and for the complex sparse case (complex measurement vectors and complex sparse signal). For a k -sparse signal, $4k - 1$ measurements were reported sufficient in the real case and $8k - 2$ in the complex case. However, no concrete characterization of the properties that lead to a unique recovery was given in [16]. In [17] the authors discuss sparse recovery from Fourier magnitude measurements and show that, under

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general conditions, the sought signal is uniquely defined by the magnitude of the full Fourier transform.

The contribution of the current paper is twofold. We first give properties leading to unique recovery for sparse signals. In particular we show that only $4k - 1$ phaseless measurements suffice to guarantee uniqueness of a k -sparse M -dimensional real solution while $2M - 1$ measurements are required for a general M -dimensional real solution. Note that [16] also showed that $4k - 1$ phaseless measurements suffice. However, the authors did not provide any condition for when this is sufficient. Secondly we consider the important case of sparse recovery from Fourier magnitude measurements. We show that under rather mild conditions, $2(k^2 - k + 1)$ Fourier magnitude measurements guarantee uniqueness. This improves on [17] which only considered recovery from a full Fourier ensemble, namely, M measurements.

2. THE PHASE RETRIEVAL PROBLEM

Define Φ as a collection of measurement vectors $\Phi = \{\varphi_n\}_{n=1}^N \in \mathbb{R}^M$ (or \mathbb{C}^M) and consider the problem of retrieving a vector \mathbf{x} from N intensity measurements

$$y_n = |\langle \varphi_n, \mathbf{x} \rangle|^2, \quad n = 1, \dots, N. \quad (1)$$

This problem is referred to as the phase retrieval problem. Introduce the operator \mathcal{A} as $(\mathcal{A}(\cdot))(n) = |\langle \varphi_n, \cdot \rangle|^2$. Note that if $\mathcal{A} : \mathbb{C}^M \rightarrow \mathbb{R}^N$ then $\mathcal{A}(c\mathbf{x}) = \mathcal{A}(\mathbf{x})$, $c \in \mathbb{C}$, $|c| = 1$, and if $\mathcal{A} : \mathbb{R}^M \rightarrow \mathbb{R}^N$ then $\mathcal{A}(\mathbf{x}) = \mathcal{A}(-\mathbf{x})$. The map $\mathcal{A}(\cdot)$ is hence not injective and \mathbf{x} can never be uniquely defined more than up to a global unit complex scalar if \mathbf{x} is complex and a global sign change if \mathbf{x} is real. Therefore, when referring to a unique solution and injectivity, it is always understood that it is either up to a unit complex scalar or a global sign change. We henceforth consider the map $\mathcal{A} : \mathbb{C}^M/\mathbb{T} \rightarrow \mathbb{R}^N$ (where \mathbb{T} is the complex unit circle) if \mathbf{x} is complex and $\mathcal{A} : \mathbb{R}^M/\{\pm 1\} \rightarrow \mathbb{R}^N$ if \mathbf{x} is known to be real.

As shown in [14, 15], the *complement property* is particularly useful when considering the theory of phase retrieval.

Definition 1 (Complement property [14, 15]). *We say that $\Phi = \{\varphi_n\}_{n=1}^N \in \mathbb{R}^M(\mathbb{C}^M)$ satisfies the complement property if for every $S \subseteq \{1, \dots, N\}$, either $\{\varphi_n\}_{n \in S}$ or $\{\varphi_n\}_{n \in S^c}$ span $\mathbb{R}^M(\mathbb{C}^M)$. Here $S^c = \{n : n \in \{1, \dots, N\}, n \notin S\}$.*

2.1. Real Measurement Vectors and a Real Signal

Using the complement property, the following theorem on the injectivity of intensity measurements using a real collection of measurement vectors was shown in [15]:

Theorem 1 (Injectivity in the real case (Thm. 3 of [15])). *Let $\mathcal{A} : \mathbb{R}^M/\{\pm 1\} \rightarrow \mathbb{R}^N$ be defined by*

$$(\mathcal{A}(\mathbf{x}))(n) = |\langle \varphi_n, \mathbf{x} \rangle|^2, \quad \varphi_n \in \mathbb{R}^M, \quad n = 1, \dots, N. \quad (2)$$

Then \mathcal{A} is injective iff $\Phi = \{\varphi_n\}_{n=1}^N \in \mathbb{R}^M$ satisfies the complement property.

It is now easy to show that $2M - 1$ intensity measurements are necessary for \mathcal{A} to be injective. This bound was also given (without a proof) in [15].

Corollary 2. *To satisfy the complement property we must have $N \geq 2M - 1$ intensity measurements. Any $N < 2M - 1$ intensity measurements do not provide an injective map \mathcal{A} .*

Proof. From Theorem 1 it is sufficient to show that $N < 2M - 1$ vectors can never satisfy the complement property. By definition, Φ satisfies the complement property if either $\{\varphi_n\}_{n \in S}$ or $\{\varphi_n\}_{n \in S^c}$ span \mathbb{R}^M for any $S \subseteq \{1, \dots, N\}$. Take $S^* \subseteq \{1, \dots, N\}$ to be any arbitrary set such that $|S^*| = M - 1$. In this case $|S^{*c}| = N - M + 1 < 2M - 1 - M + 1 = M$ if $N < 2M - 1$. Since both $|S^*| < M$ and $|S^{*c}| < M$, neither $\{\varphi_n\}_{n \in S^*}$ or $\{\varphi_n\}_{n \in S^{*c}}$ span \mathbb{R}^M . \square

It can easily be verified that $2M - 1$ measurement vectors independently drawn from *e.g.*, an M -dimensional standard Gaussian distribution (zero mean, unit variance) satisfy the complement property with probability 1. According to Theorem 1 it is hence possible to uniquely recover an M -dimensional real signal from $2M - 1$ intensity measurements.

2.2. Complex Measurement Vectors and a Complex Signal

Let us now consider the complex case, when the measurement vectors are complex and $\mathbf{x} \in \mathbb{C}^M$. It was recently shown in [15] that the complement property is a necessary condition for injectivity in this case.

Theorem 3 (Injectivity in the complex case (Thm. 7 of [15])). *Let $\mathcal{A} : \mathbb{C}^M/\mathbb{T} \rightarrow \mathbb{R}^N$ be defined by*

$$(\mathcal{A}(\mathbf{x}))(n) = |\langle \varphi_n, \mathbf{x} \rangle|^2, \quad \varphi_n \in \mathbb{C}^M, \quad n = 1, \dots, N. \quad (3)$$

If \mathcal{A} is injective then $\Phi = \{\varphi_n\}_{n=1}^N \in \mathbb{C}^M$ satisfies the complement property.

It is easy to verify that the complement property is only necessary and not sufficient for injectivity. An example of a set of measurement vectors that satisfies the complement property but does not provide an injective map is given in [15]. It was conjectured (but not proven) in [15] that $4M - 4$ generic (see [15] for definition) measurements are both necessary and sufficient for unique recovery.

3. UNIQUENESS IN SPARSE PHASE RETRIEVAL

We now build on previous results and generalize them to the analysis of sparse phase retrieval. We start by studying a collection of real measurement vectors and then extend the results to an important class of complex measurement vectors, a partial Fourier basis, in Section 3.2.

3.1. Real Measurement Vectors and a Sparse Real Signal

To handle sparse signals, it is convenient to introduce the following less restrictive version of the complement property:

Definition 2 (*k*-complement property). We say that $\Phi = \{\varphi_n\}_{n=1}^N$ satisfies the *k*-complement property if for every $S \subseteq \{1, \dots, N\}$ and subset $K \subseteq \{1, \dots, M\}$, $|K| = k$, either $\{\varphi_{n,K}\}_{n \in S}$ or $\{\varphi_{n,K}\}_{n \in S^c}$ span \mathbb{R}^k . The notation $\varphi_{n,K}$ denotes the elements indexed by K of the n th measurement vector φ_n .

The *k*-complement property reduces to the complement property of Definition 1 when $k = M$. If $k < M$ then the *k*-complement property is less restrictive. Furthermore, if Φ satisfies the *k*-complement property then it also satisfies the $(k-1)$ -complement property.

We are now ready to state the following theorem on unique recovery of a *k*-sparse real signal:

Theorem 4 (Unique recovery in the sparse real case). Let $\mathcal{A} : \mathbb{R}^M / \{\pm 1\} \rightarrow \mathbb{R}^N$ be defined by

$$(\mathcal{A}(\mathbf{x}))(n) = |\langle \varphi_n, \mathbf{x} \rangle|^2, \quad \varphi_n \in \mathbb{R}^M, n = 1, \dots, N, \quad (4)$$

and assume that we are given $\mathbf{y} = \mathcal{A}(\mathbf{x}_0) \in \mathbb{R}^N$. If \mathcal{A} satisfies the $2\|\mathbf{x}_0\|_0$ -complement property, then \mathbf{x}_0 is the unique real vector satisfying the given measurements with $\|\mathbf{x}_0\|_0$ or fewer nonzero elements. Thus, \mathbf{x}_0 can be found as the solution to

$$\mathbf{x}_0 = \arg \min_{\mathbf{x} \in \mathbb{R}^M} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{y} = \mathcal{A}(\mathbf{x}). \quad (5)$$

Proof. We prove the theorem by contradiction. Assume that $\tilde{\mathbf{x}} \neq \pm \mathbf{x}_0$, $\|\tilde{\mathbf{x}}\|_0 \leq \|\mathbf{x}_0\|_0$, $\mathbf{y} = \mathcal{A}(\tilde{\mathbf{x}}) = \mathcal{A}(\mathbf{x}_0)$, $\tilde{\mathbf{x}} \in \mathbb{R}^M$. Theorem 1 gives that if Φ associated with \mathcal{A} satisfies the $2\|\mathbf{x}_0\|_0$ -complement property, then $\{|\langle \varphi_{n,K}, \cdot \rangle|^2\}_{n=1}^N$ is injective for all subsets $K \subseteq \{1, \dots, M\}$, $|K| = 2\|\mathbf{x}_0\|_0$. Let $K^* \subseteq \{1, \dots, M\}$, $|K^*| = 2\|\mathbf{x}_0\|_0$, be an index set that includes the support of \mathbf{x}_0 and $\tilde{\mathbf{x}}$. Note that $\|\tilde{\mathbf{x}}\|_0 + \|\mathbf{x}_0\|_0 \leq 2\|\mathbf{x}_0\|_0 = |K^*|$. Then $\{|\langle \varphi_{n,K^*}, \mathbf{x}_{0,K^*} \rangle|^2\}_{n=1}^N = \{|\langle \varphi_{n,K^*}, \tilde{\mathbf{x}}_{K^*} \rangle|^2\}_{n=1}^N = \mathbf{y}$. Since $\{|\langle \varphi_{n,K}, \cdot \rangle|^2\}_{n=1}^N$ is injective for all subsets $K \subseteq \{1, \dots, M\}$ of size $|K| = 2\|\mathbf{x}_0\|_0$, it must also be injective for K^* . We therefore conclude that $\tilde{\mathbf{x}}_{K^*} = \mathbf{x}_{0,K^*}$ which implies that $\tilde{\mathbf{x}} = \pm \mathbf{x}_0$ since K^* includes the support of both vectors. \square

For a sufficiently sparse \mathbf{x} , unique recovery can hence be guaranteed from fewer measurements than in the dense case. We give this result as a corollary:

Corollary 5. A collection of $\min(4k-1, 2M-1)$ measurement vectors suffice to uniquely recover any *k*-sparse \mathbf{x} .

Before proving the corollary, we state the following lemma:

Lemma 6. A set of $4k-1$ independent samples from an M -dimensional standard Gaussian distribution satisfies the $2k$ -complement property with probability 1.

Proof of Lemma 6. Generate the collection of measurement vectors by independently drawing $4k-1$ samples from a M -dimensional standard Gaussian distribution. Introduce Φ as the $M \times (4k-1)$ -matrix obtained by arranging the $4k-1$ vectors of Φ into a matrix. Let $\Phi_{K,S}$ be the $|K| \times |S|$ -matrix obtained by picking out the rows indexed in K and columns indexed by S .

Consider the probability that Φ does not satisfy the $2k$ -complement property:

$$P(E) = P(\exists S, K : S \subset \{1, \dots, 4k-1\}, |K| = 2k, \lambda_{\min}(\Phi_{K,S} \Phi_{K,S}^*) = \lambda_{\min}(\Phi_{K,S^c} \Phi_{K,S^c}^*) = 0),$$

where λ_{\min} denotes the smallest eigenvalue. We now use Boole's inequality for unions of events

$$\begin{aligned} P(E) &\leq \sum_{s=1}^{4k-1} \binom{4k-1}{s} \binom{M}{2k} P(\text{a } 2k \times s\text{-submatrix of } \Phi \text{ has row-rank} < 2k) \\ &\quad \cdot P(\text{a } 2k \times (4k-1-s)\text{-submatrix of } \Phi \text{ has row-rank} < 2k) \\ &= 0, \end{aligned}$$

where we used that $P(\text{a } 2k \times s\text{-submatrix of } \Phi \text{ has row-rank} < 2k) = 0$ when $s \geq 2k$ and $P(\text{a } 2k \times (4k-1-s)\text{-submatrix of } \Phi \text{ has row-rank} < 2k) = 0$ when $s < 2k$, which follow from the Gaussianity of the entries of the submatrices. \square

Proof of Corollary 5. First, since $2M-1$ measurements are enough in the dense case, this provides an upper bound on the number of measurements. Second, Theorem 4 gives that $\mathbf{y} = \mathcal{A}(\mathbf{x})$ has a unique *k*-sparse solution for $\min(4k-1, 2M-1)$ measurements if the collection satisfies the $2k$ -complement property. Finally we have from Lemma 6 that such a collection exists since a set of $4k-1$ samples from an M -dimensional unit Gaussian distribution satisfies the $2k$ -complement property with probability 1. \square

3.2. Complex Measurement Vectors and Real Signal: Fourier Magnitude Measurements

A particularly interesting set of complex measurement vectors is the incomplete Fourier basis. This special case is of great importance since Fourier magnitude measurements (FMMs) are inherent in applications such as X-ray crystallography [1, 2], speckle imaging and blind channel estimation [17].

A complication in dealing with FMMs is that some properties are entirely embedded in the phase of the Fourier transform and therefore lost in the measuring process. In addition to the global sign shift previously discussed, we therefore include mirroring (reverse the ordering of the elements in \mathbf{x}) and shifts (circularly shift the elements in \mathbf{x}) in the set of invariances \mathbb{T} from here on.

Before discussing the results, note that even if a Fourier basis may satisfy some complex equivalent of the *k*-complement property, this is not enough to provide uniqueness up to the

invariances of \mathbb{T} . This was shown in [18] by giving an example of two signals, not equivalent with respect to \mathbb{T} , with the same autocorrelation. Such signals can thereby never be uniquely specified by the magnitude of their Fourier transforms. The k -complement property is therefore not enough to characterize when a signal is uniquely defined by its FMMs.

In deriving guarantees for FMMs, we need the concept of a *collision free vector* introduced in [17, Def. 1].

Definition 3 (Collision free vector). *Let $\mathbf{x}(i)$ denote the i th element of the vector \mathbf{x} . We say that \mathbf{x} is collision free if $\mathbf{x}(i) - \mathbf{x}(j) \neq \mathbf{x}(k) - \mathbf{x}(l)$, for all distinct $i, j, k, l \in \{i : i \in \{1, \dots, M\}, \mathbf{x}(i) \neq 0\}$.*

We are now ready to state the following theorem on the uniqueness of a sparse real solution given its FMMs.

Theorem 7. *Let $\{k_1, k_2, \dots, k_N\} \subseteq \{0, \dots, 2M - 1\}$,*

$$\varphi_n = [1 e^{-i2\pi k_n/2M} e^{-i4\pi k_n/2M} \dots e^{-i2\pi(2M-1)k_n/2M}]^T, \quad (6)$$

with $i = \sqrt{-1}$, and let $\mathcal{A} : \mathbb{R}^M / \mathbb{T} \rightarrow \mathbb{R}^N$ be defined by

$$(\mathcal{A}(\mathbf{x}))(n) = |\langle \varphi_n, [\mathbf{x}^T \quad \mathbf{0}_{1 \times M}]^T \rangle|^2, \quad n = 1, \dots, N. \quad (7)$$

Assume that we are given $\mathbf{y} = \mathcal{A}(\mathbf{x}_0) \in \mathbb{R}^N$ with N a prime integer larger than $2(\|\mathbf{x}_0\|_0^2 - \|\mathbf{x}_0\|_0 + 1)$. Then a collision free $\mathbf{x}_0 \in \mathbb{R}^M$ is uniquely defined by \mathbf{y} whenever

- $\|\mathbf{x}_0\|_0 \neq 6$, or
- $\|\mathbf{x}_0\|_0 = 6$ and $\mathbf{x}_0(i) \neq \mathbf{x}_0(j)$, for some $i, j \in \{i : i \in \{1, \dots, M\}, \mathbf{x}_0(i) \neq 0\}$.

The implication of the theorem is that we can guarantee a unique solution from FMMs as long as enough measurements are taken, the signal is sparse enough, collision free and the support constrained.

Proof. If there are no collisions and $\mathbf{x}_0 \in \mathbb{R}^M$ is k -sparse, then the autocorrelation $\mathbf{a} \in \mathbb{R}^{2M-1}$, defined as

$$\mathbf{a}(l) = \sum_{s=\max\{1, 1-l\}}^{\min\{M, M-l\}} \mathbf{x}_0(s)\mathbf{x}_0(s+l), \quad l = 1 - M, \dots, M - 1, \quad (8)$$

is $k^2 - k + 1$ -sparse (see for instance [17]). We further have that the autocorrelation is centro-symmetric, $\mathbf{a}(l) = \mathbf{a}(-l)$, $l = 0, \dots, M - 1$, and via Wiener-Khinchin's theorem that $\mathbf{a}(l)$, $l = 0, \dots, M - 1$, is related to $\mathbf{y}(n)$, $n = 1, \dots, N$, via $\mathbf{y}(n) = \langle \varphi_n, [\mathbf{a}(0) \dots \mathbf{a}(M - 1) \mathbf{0} \mathbf{a}(M - 1) \mathbf{a}(M - 2) \dots \mathbf{a}(1)]^T \rangle$.

Ignoring the symmetry, the problem of recovering the sparse autocorrelation from the partial FMMs \mathbf{y} can therefore be posed as

$$\begin{aligned} & \min_{\mathbf{q} \in \mathbb{R}^{2M}} \|\mathbf{q}\|_0 \\ \text{s. t. } & \mathbf{y}(n) = \langle \varphi_n, \mathbf{q} \rangle, \quad n = 1, \dots, N, \\ & \mathbf{0} = \mathbf{q}(M + 1). \end{aligned} \quad (9)$$

This is a well studied problem in compressive sensing (see for instance [19, 20]) and using the result of [21, Thm. 1.1] it can be shown that if N is prime and satisfies

$$2(\|\mathbf{x}_0\|_0^2 - \|\mathbf{x}_0\|_0 + 1) \leq N, \quad (10)$$

then (9) has a unique solution. This because $\mathbf{a}(1), \dots, \mathbf{a}(M - 1)$ contain $(\|\mathbf{x}_0\|_0^2 - \|\mathbf{x}_0\|_0)/2$ nonzero elements at most.

Finally, it was recently shown in [17] that whenever there are no collisions in \mathbf{x}_0 and the following conditions are satisfied, then the autocorrelation uniquely defines \mathbf{x}_0 :

- $\|\mathbf{x}_0\|_0 \neq 6$, or
- $\|\mathbf{x}_0\|_0 = 6$ and $\mathbf{x}_0(i) \neq \mathbf{x}_0(j)$, for some $i, j \in \{i : i \in \{1, \dots, M\}, \mathbf{x}_0(i) \neq 0\}$, or
- $\|\mathbf{x}_0\|_0 = 6$ and $\mathbf{x}_0(i) = \mathbf{x}_0(j)$, for all $i, j \in \{i : i \in \{1, \dots, M\}, \mathbf{x}_0(i) \neq 0\}$. In this case, the autocorrelation uniquely defines \mathbf{x}_0 almost surely.

Hence, under the conditions of the theorem, the FMMs \mathbf{y} uniquely define \mathbf{a} , and \mathbf{a} uniquely defines \mathbf{x}_0 , from which the theorem follows. \square

Note that the theorem does not require the Fourier basis vectors to be selected deterministically or randomly and therefore holds for both.

4. CONCLUSION

Even though phase retrieval is a longstanding problem in optics it is still not well understood whether a collection of measurements provides an injective map or not. It was recently shown that the complement property gives necessary and sufficient conditions for the uniqueness of a real signal and a real collection of measurement vectors. Here we show that if the measurement vectors satisfy a weaker version of the complement property then a sought sparse signal can be guaranteed to be uniquely defined by associated intensity measurements. We also consider a complex collection of measurement vectors and Fourier magnitude measurements. We show that in general, $2(k^2 - k + 1)$ Fourier magnitude measurements suffice to uniquely determine a k -sparse signal.

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