Department of Electrical Engineering
Electronics
Computers
Communications

Sub-Nyquist Sampling of Analog Signals

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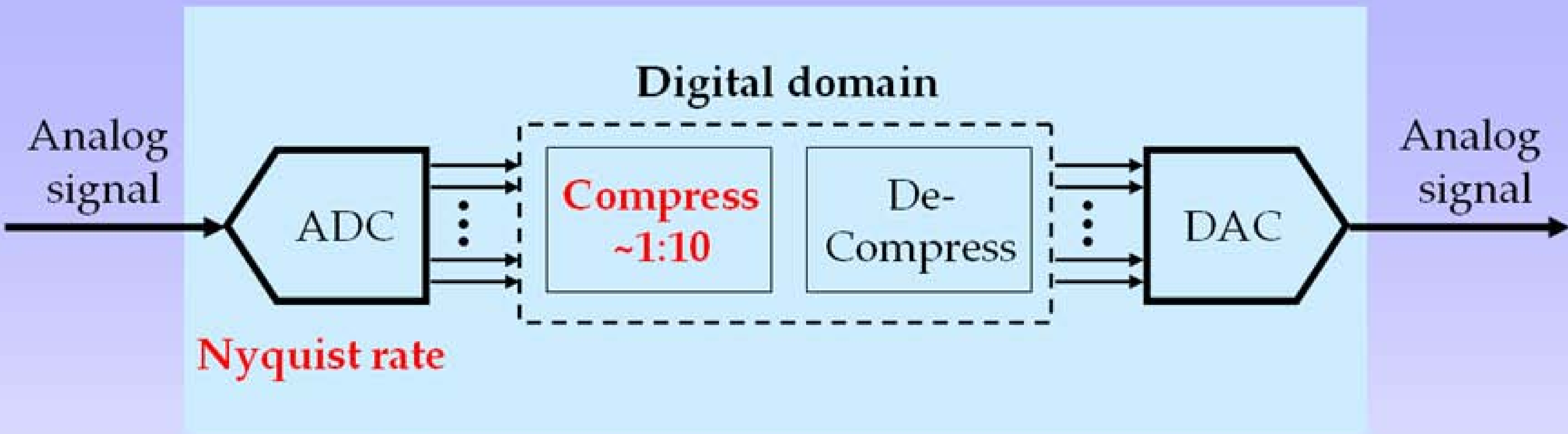
Compressed Sensing Workshop
Feb. 26th, 2009

"Analog Girl in a Digital World…"

Judy Gorman `99




Voice recorder

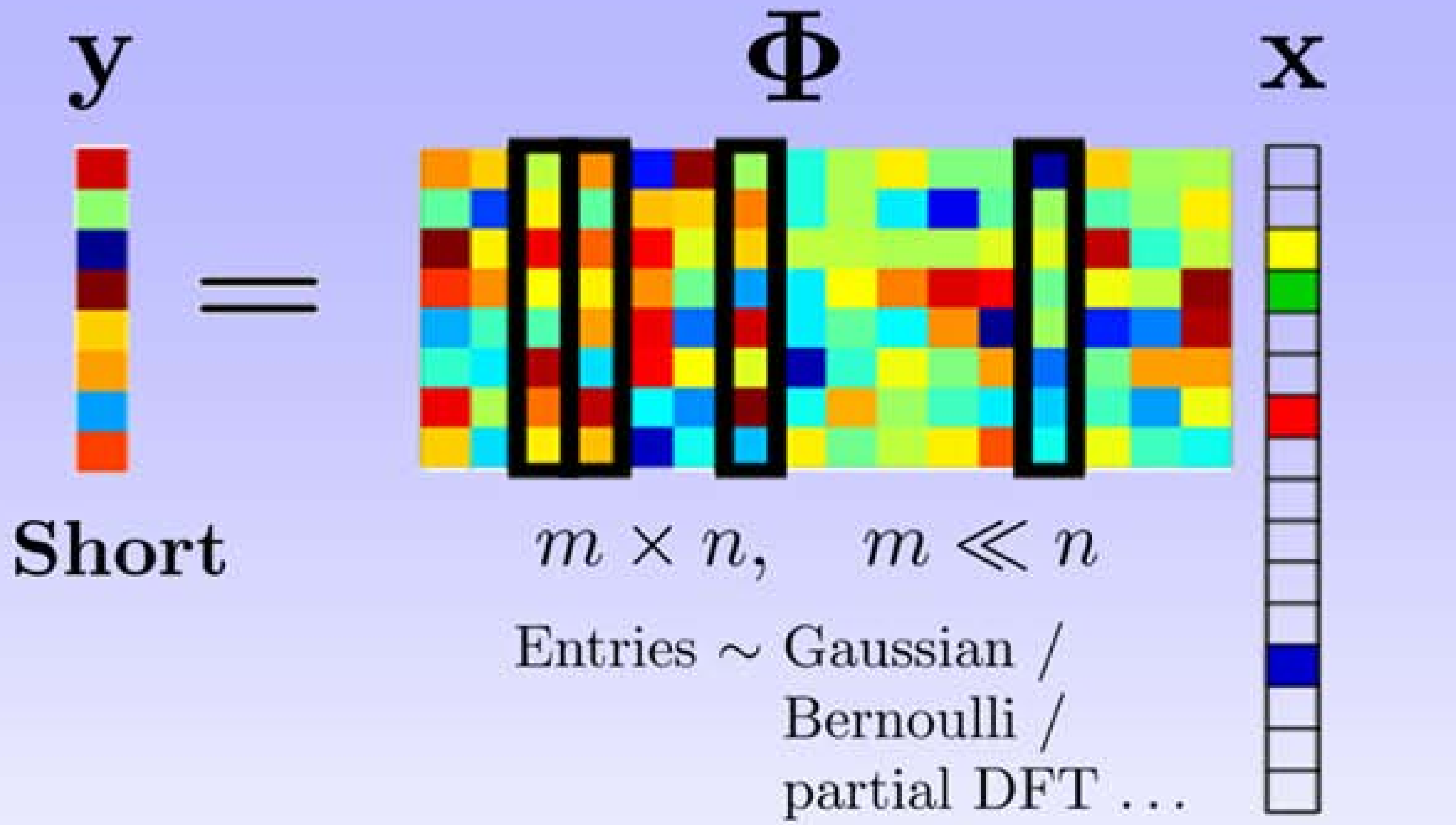

Camera


Medical imaging

"Can we not just **directly measure** the part that will not end up being thrown away ?"

(Donoho '06)

Compressed Sensing



Main ideas:

- Sensing = inner products $\mathbf{y}_i = \langle \mathbf{\Phi}_i, \mathbf{x} \rangle$
- Random projections
- Polynomial-time recovery algorithms

(Donoho '06)

(Candès, Romberg, Tao '06)

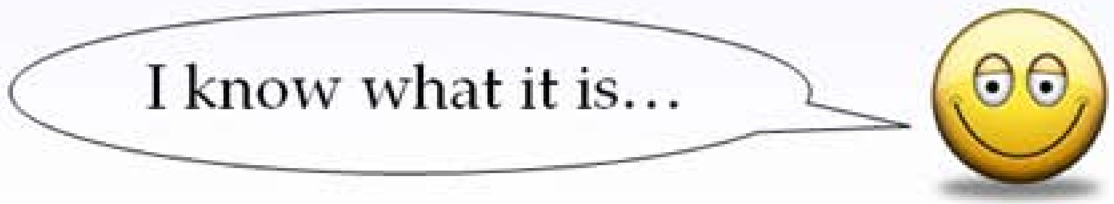
Naïve Extension to Analog Signals

Standard CS
Discrete Framework

$\|\mathbf{x}\|_0 \leq K$

Model

Analog Domain



Naïve Extension to Analog Signals

Standard CS
Discrete Framework

Analog Domain

Model

$\|\mathbf{x}\|_0 \leq K$

What is a sparse analog signal ?

What am I dealing with ?

Naïve Extension to Analog Signals

Standard CS
Discrete Framework

Analog Domain

Model

$\|\mathbf{x}\|_0 \leq K$

What is a sparse analog signal ?

Generalized sampling
$\mathbf{y}_i = \langle \mathbf{a}_i, \mathbf{x} \rangle$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Finite dimensional elements

$$\mathbf{y}[n] = \mathbf{A}\{\mathbf{x}(t)\}$$

Infinite sequence

Operator $L_2 \rightarrow l_2$

Continuous signal

I choose sampling, not the matrix

Naïve Extension to Analog Signals

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What is a sparse analog signal ?

Generalized sampling
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Infinite sequence

Operator $L_2 \to l_2$

Continuous signal

Sensing matrix

Random $\mathbf{A}$ is "good" w.h.p

Fully Random → Infinitely many $\mathbf{a}_i$

Must have some structure to implement

Naïve Extension to Analog Signals

Standard CS Discrete Framework		Analog Domain
$\|\mathbf{x}\|_0 \le K$	**Model**	What is a sparse analog signal ?
$\mathbf{y} = \mathbf{A}\mathbf{x}$ Finite dimensional elements	**Generalized sampling** $\mathbf{y}_i = \langle \mathbf{a}_i, \mathbf{x} \rangle$	$\mathbf{y}[n] = \mathbf{A}\{\mathbf{x}(t)\}$ Infinite sequence; Operator $L_2 \to l_2$; Continuous signal
Random $\mathbf{A}$ is "good" w.h.p	**Sensing matrix**	Fully Random → Infinitely many $\mathbf{a}_i$
$\min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_p}$ s.t. $\mathbf{y} = \mathbf{A}\mathbf{x}$ Finite program, well-studied	**Reconstruction**	$\min_{\mathbf{x}(t)} \|\mathbf{x}(t)\|_{\ell_p}$ s.t. $\mathbf{y}[n] = \mathbf{A}\{\mathbf{x}(t)\}$ Undefined program over a continuous signal

I can only do a reasonable amount of computations

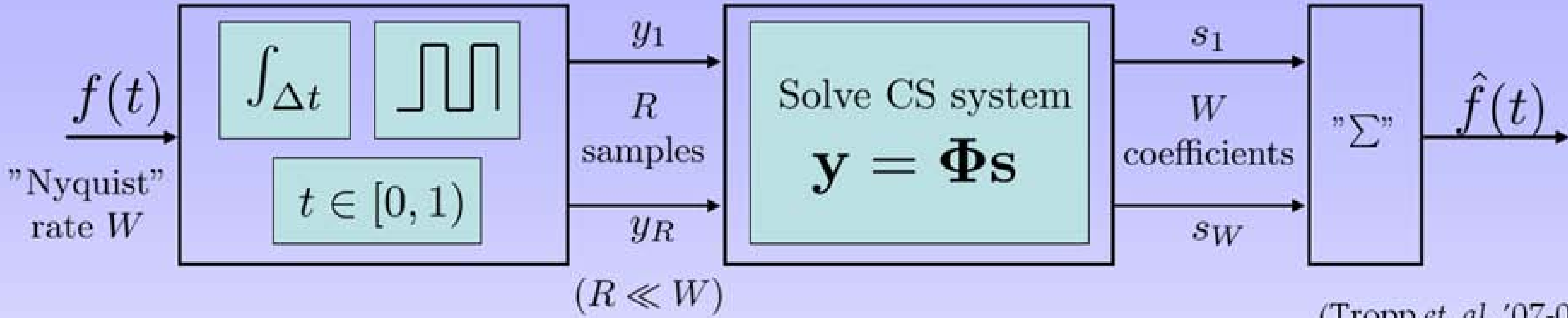
Goals for Analog CS

- Signal model
  - convenient enough to represent real-life situations
- Hardware implementation
- "Light" computational load
- Real-time processing

Approach: Combine CS with analog sampling ideas

Road Map

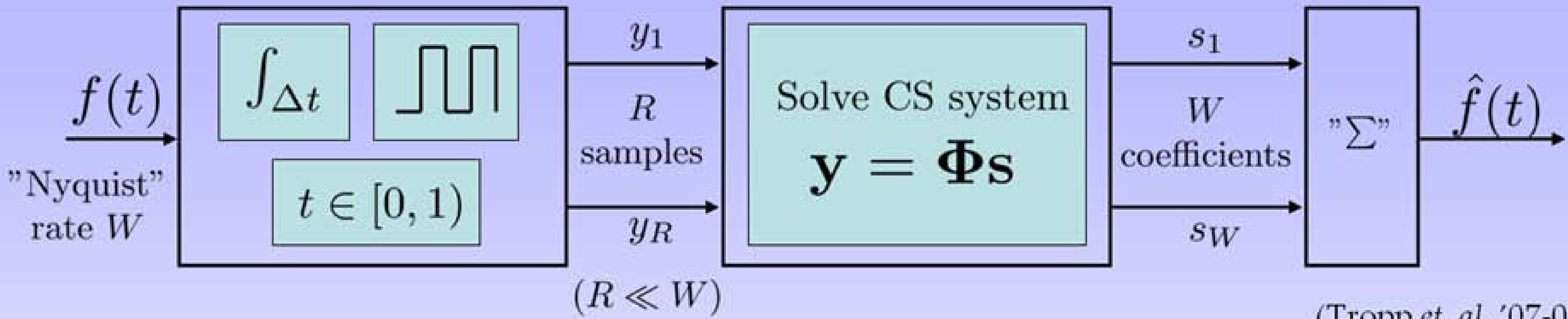
Time-domain approach:



(Tropp *et. al.* '07-09)

Road Map

Time-domain approach:



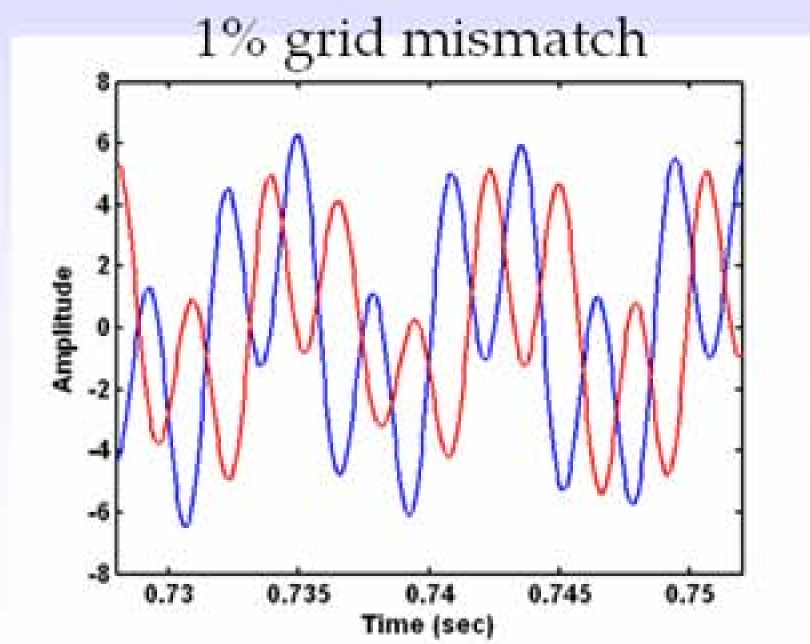
(Tropp *et. al.* '07-09)

Tones model (sensitive)

System "grid" must match signal tones grid

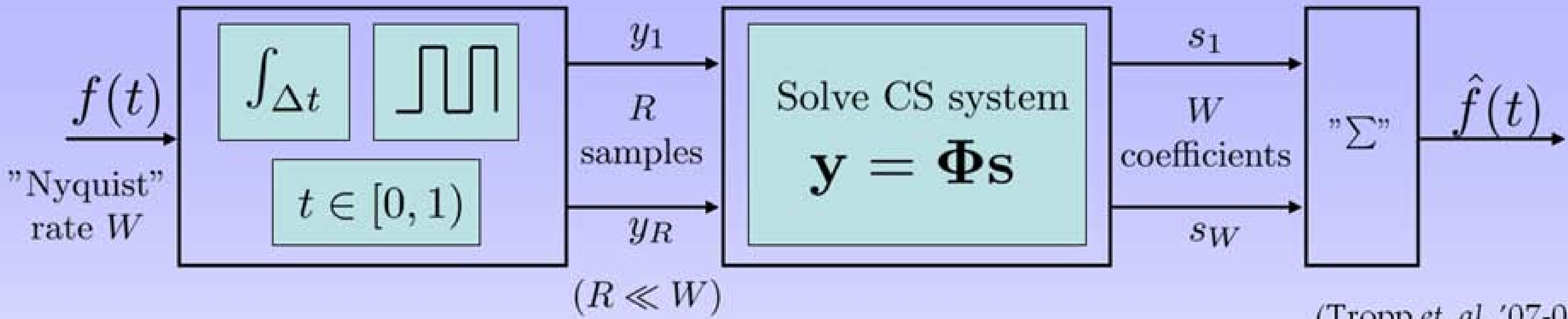
$$f(t) = \sum_{w \in \Omega} s_w \exp(j2\pi wt)$$

$$\frac{\|f(t) - \hat{f}(t)\|}{\|f(t)\|} = 1.7$$



Road Map

Time-domain approach:



(Tropp *et. al.* '07-09)

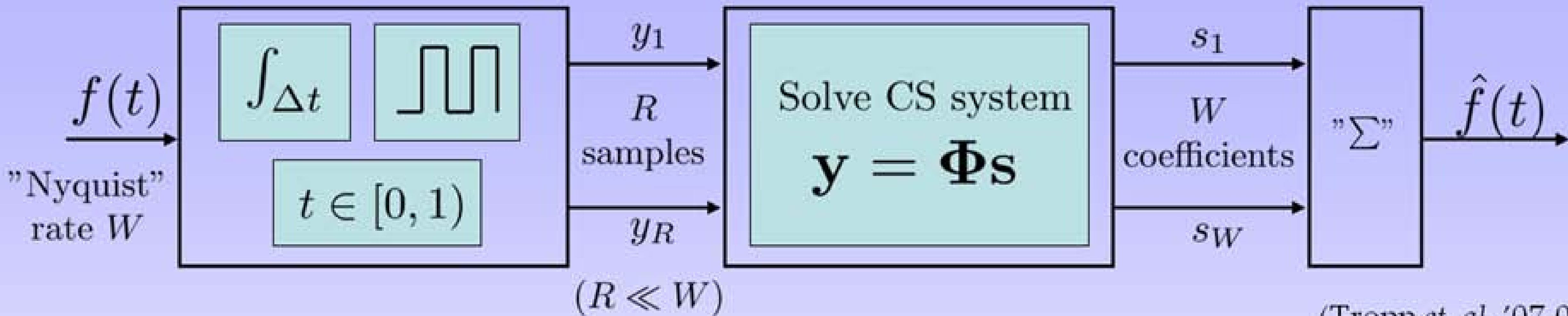
Tones model (sensitive)
System "grid" must match signal tones grid

High computational load

$$\mathbf{\Phi} = R \times W \xrightarrow[]{\substack{1 \text{ kHz bandwidth} \\ 10\% \text{ tones}}} \mathbf{\Phi} = 100 \times 1000$$

Road Map

Time-domain approach:



(Tropp *et. al.* '07-09)

Tones model (sensitive)
System "grid" must match signal tones grid

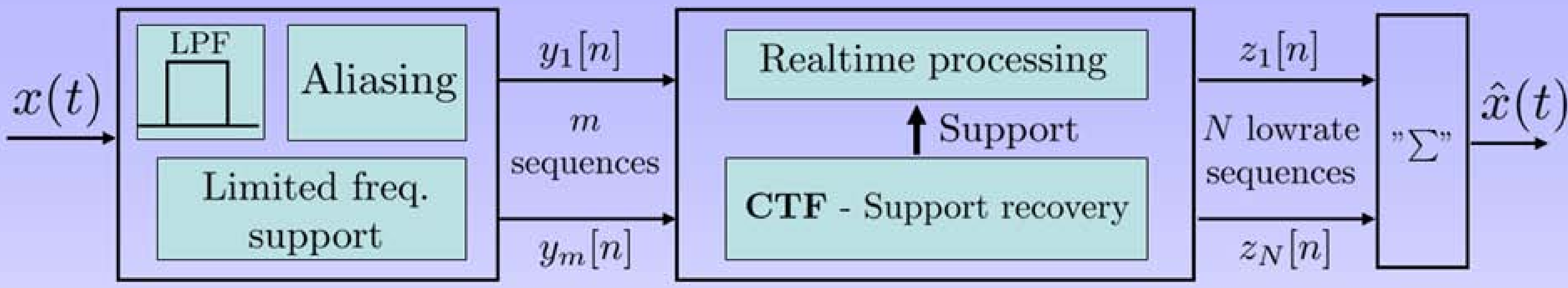
High computational load

Can't treat true analog signals

50 MHz information band with $W = 10$ GHz
Requires $\sim 100 \cdot 10^6$ tones $\rightarrow \mathbf{\Phi}$ is huge-scale $(10^7 \times 10^{10})$

Road Map

Frequency-domain approach:



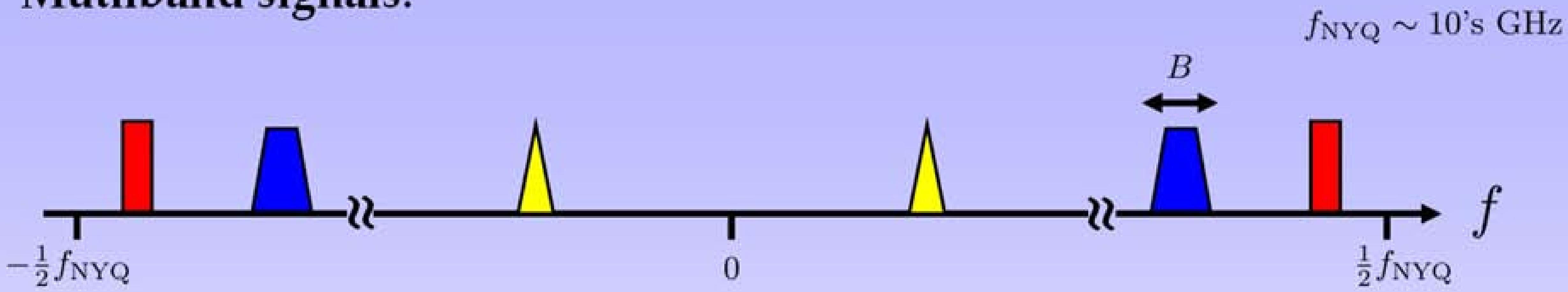
Outline

- Analog model – information bands in wide spectrum
- The "modulated wideband converter" – hardware implementation
- CTF – a "light" computational load on reconstruction
- Advantages – wideband regime, realtime, baseband

"**Think** in frequency-domain – **Implement** in time-domain"

Analog Signal Model

Mutliband signals:



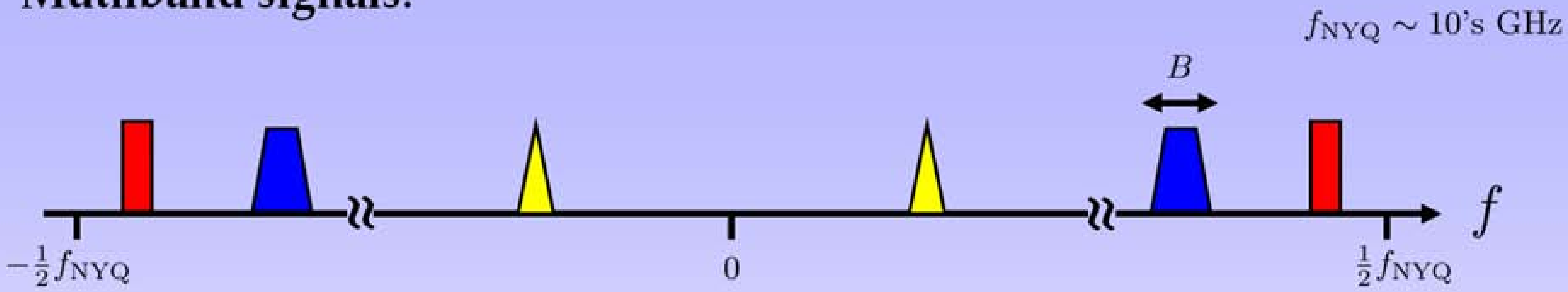
1. Each band has an uncountable number of non-zero elements
2. Band locations lie on the continuum
3. Band locations are unknown in advance

$\mathcal{M} = \{\, x(t) \mid$ no more than N bands, max width B, bandlimited to $[-\frac{1}{2}f_{\mathrm{NYQ}}, +\frac{1}{2}f_{\mathrm{NYQ}})\}$

(Mishali and Eldar 2007)

Analog Signal Model

Mutliband signals:



$\mathcal{M} = \{\, x(t) \mid$ no more than N bands, max width B, bandlimited to $[-\frac{1}{2}f_{\mathrm{NYQ}}, +\frac{1}{2}f_{\mathrm{NYQ}})\}$

(Mishali and Eldar 2007)

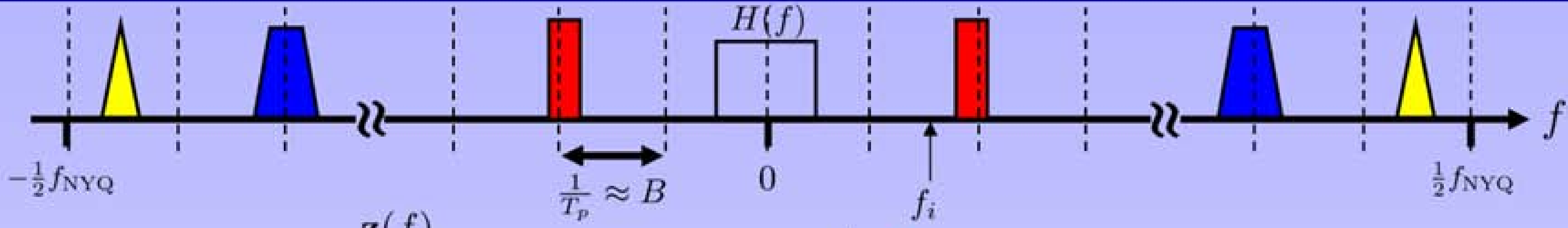
Sparse union of shift invariant subspaces:

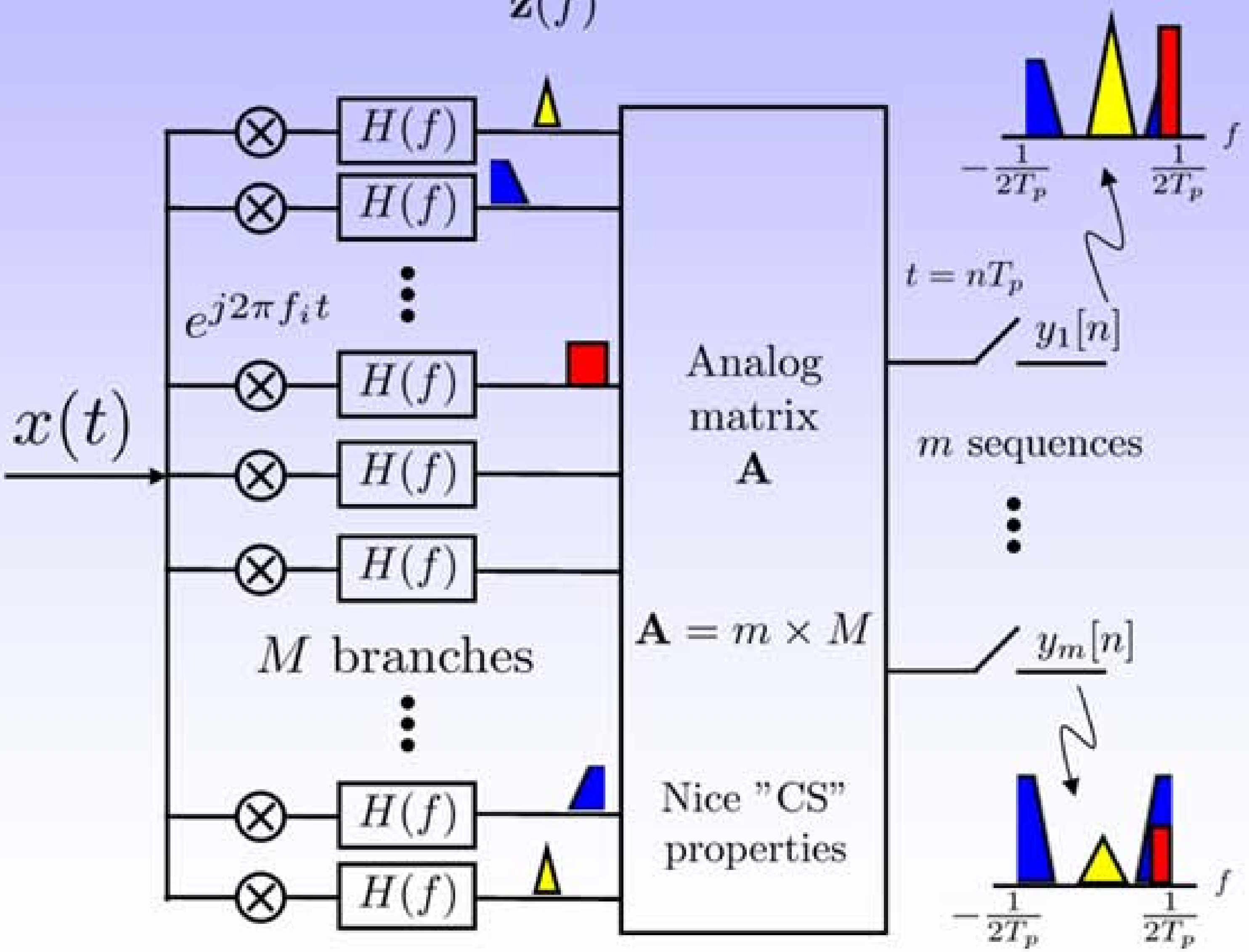
$$x(t) = \sum_{l=1}^{N} \sum_{n=-\infty}^{\infty} d_l[n] a_l(t-n)$$

only $K \ll N$ sequences $d_l[n]$ are non-zero

(Eldar 2008)

Conceptual System





Goals

$M \approx \frac{f_{\mathrm{NYQ}}}{B} = 10\text{'s} \div 200$

Sampling: only a few channels

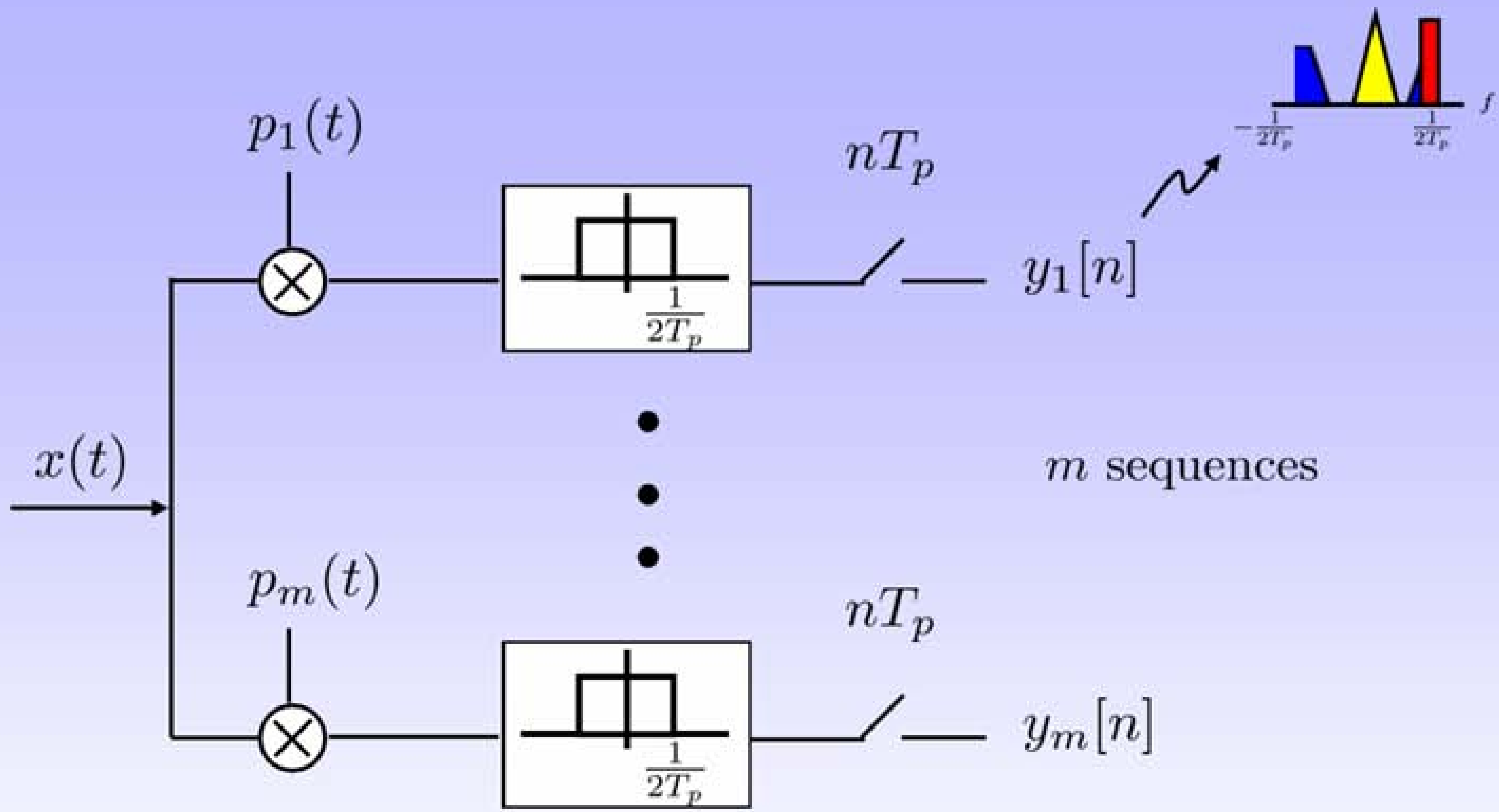
Aliasing (intentionally)

$$\mathbf{y}(f) = \mathbf{A}\mathbf{z}(f), \quad f \in [-\tfrac{1}{2T_p}, \tfrac{1}{2T_p})$$

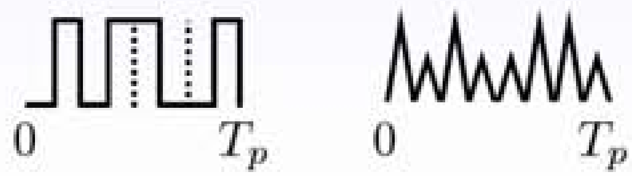
Recovery: solve inf. CS systems

No discretization ("implement in time…")

The Modulated Wideband Converter

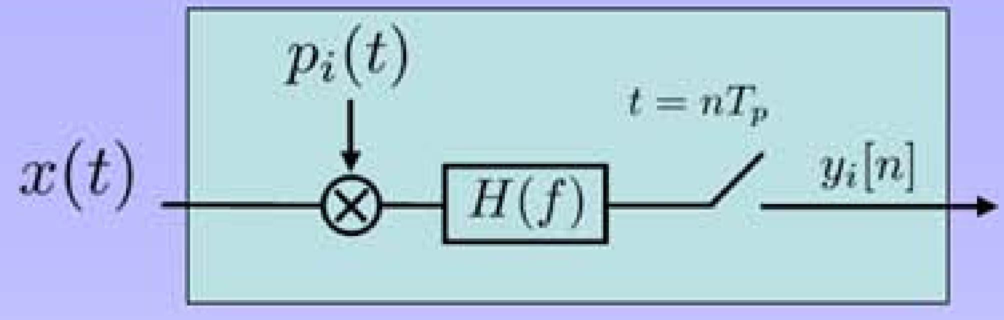


T_p−periodic $p_i(t)$ gives the desired aliasing effect



and many more...

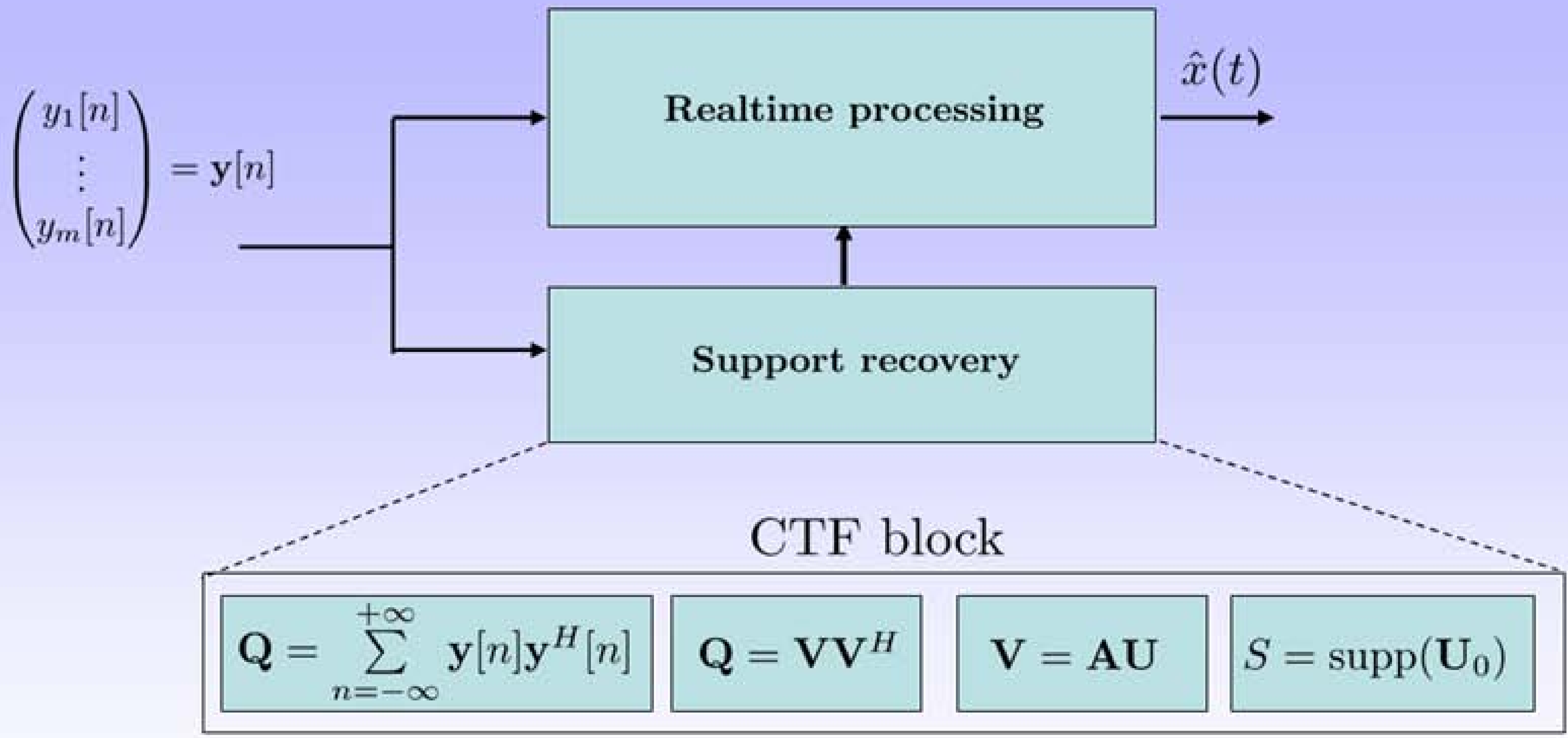
Advantages - Sampling



- Any periodic mixing function
- An accurate lowpass filter (any order)
- Flexible control of sampling rate at each channel
- Can implement the idea with a single channel
- One shift register can be used for all branches

- Parameter choice is **insensitive** to exact width B

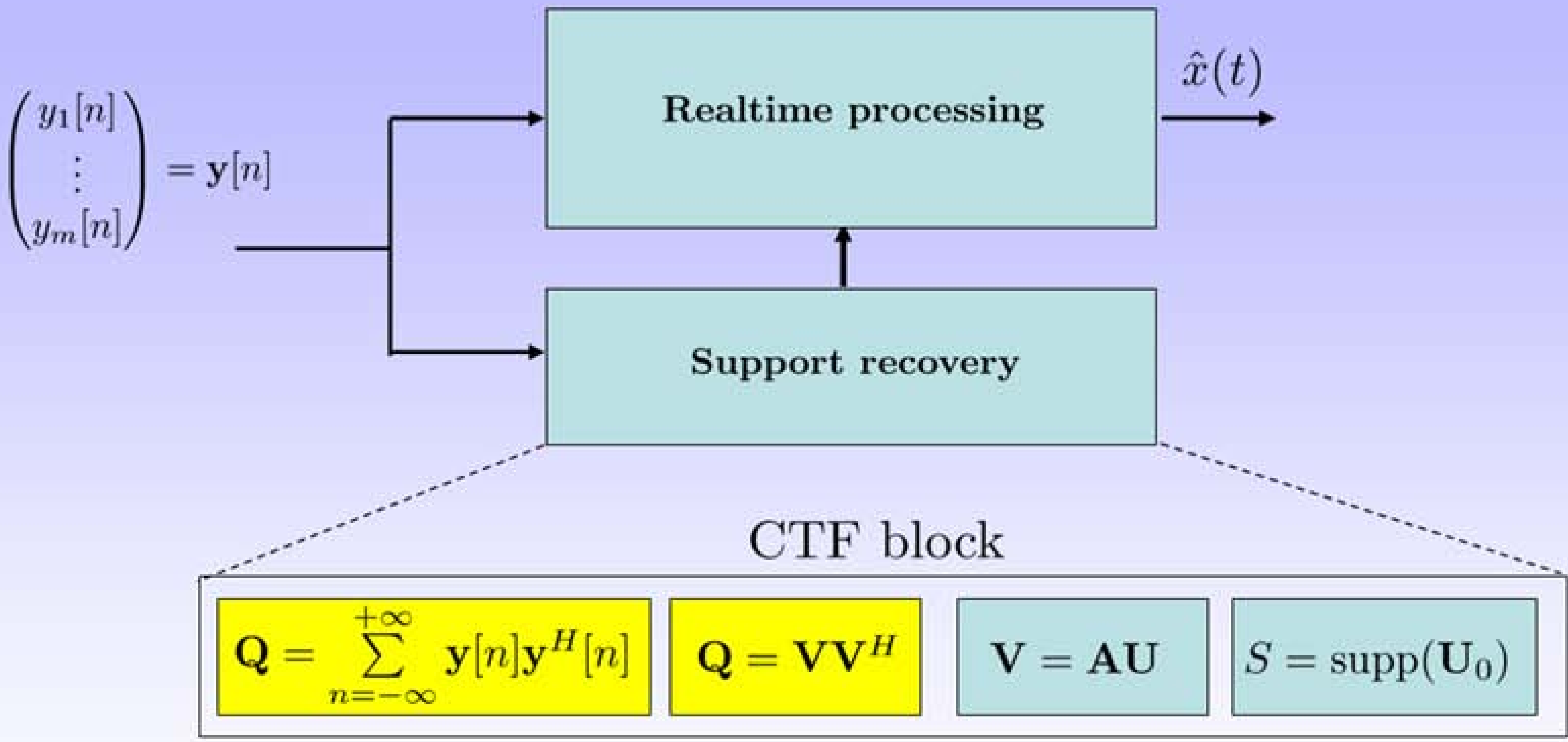
Reconstruction



Continuous → **Finite**

$$\mathbf{y}(f) = \mathbf{A}\mathbf{z}(f), \quad -\frac{1}{T_p} \leq f \leq \frac{1}{T_p} \longrightarrow \mathbf{V} = \mathbf{A}\mathbf{U}$$

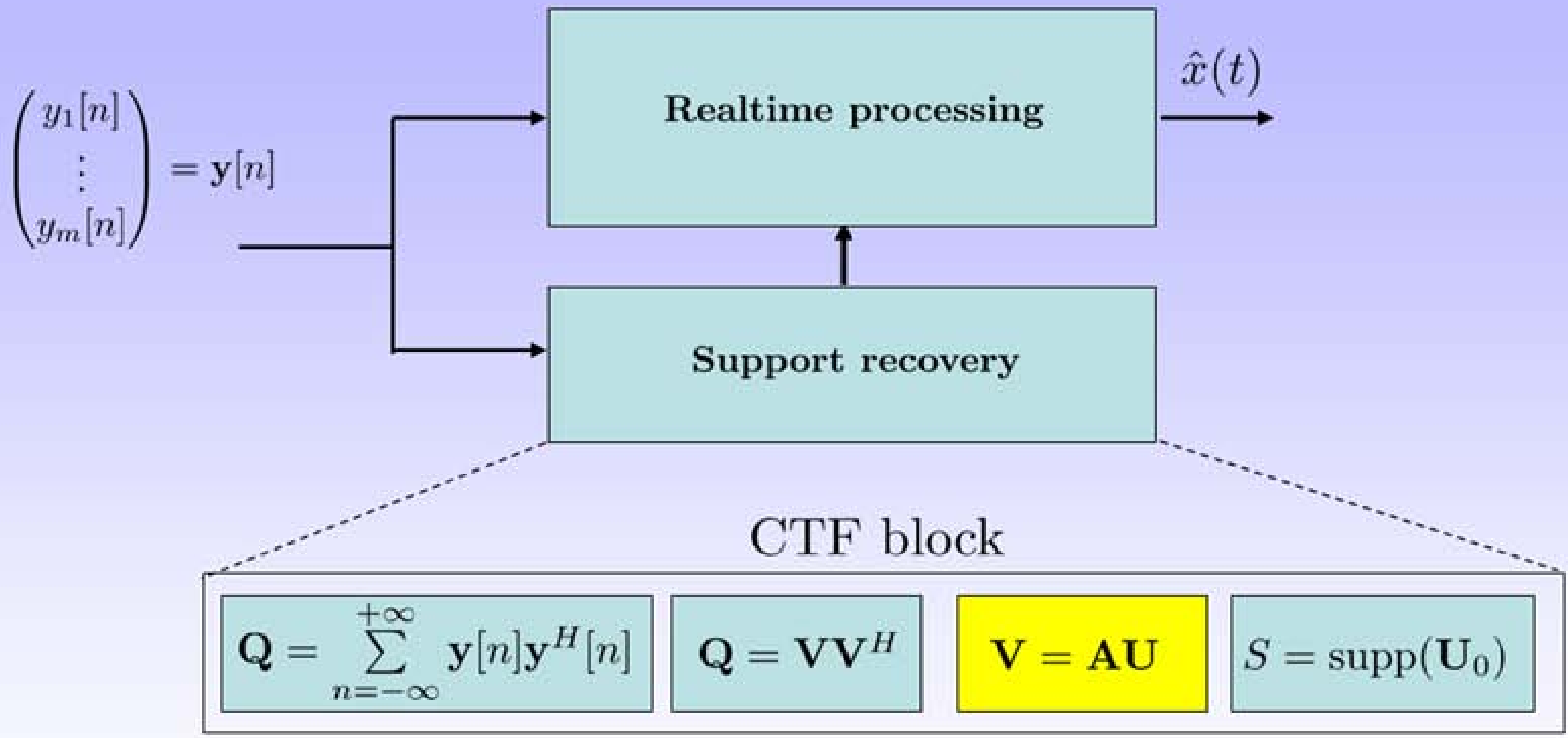
Reconstruction



Frame construction

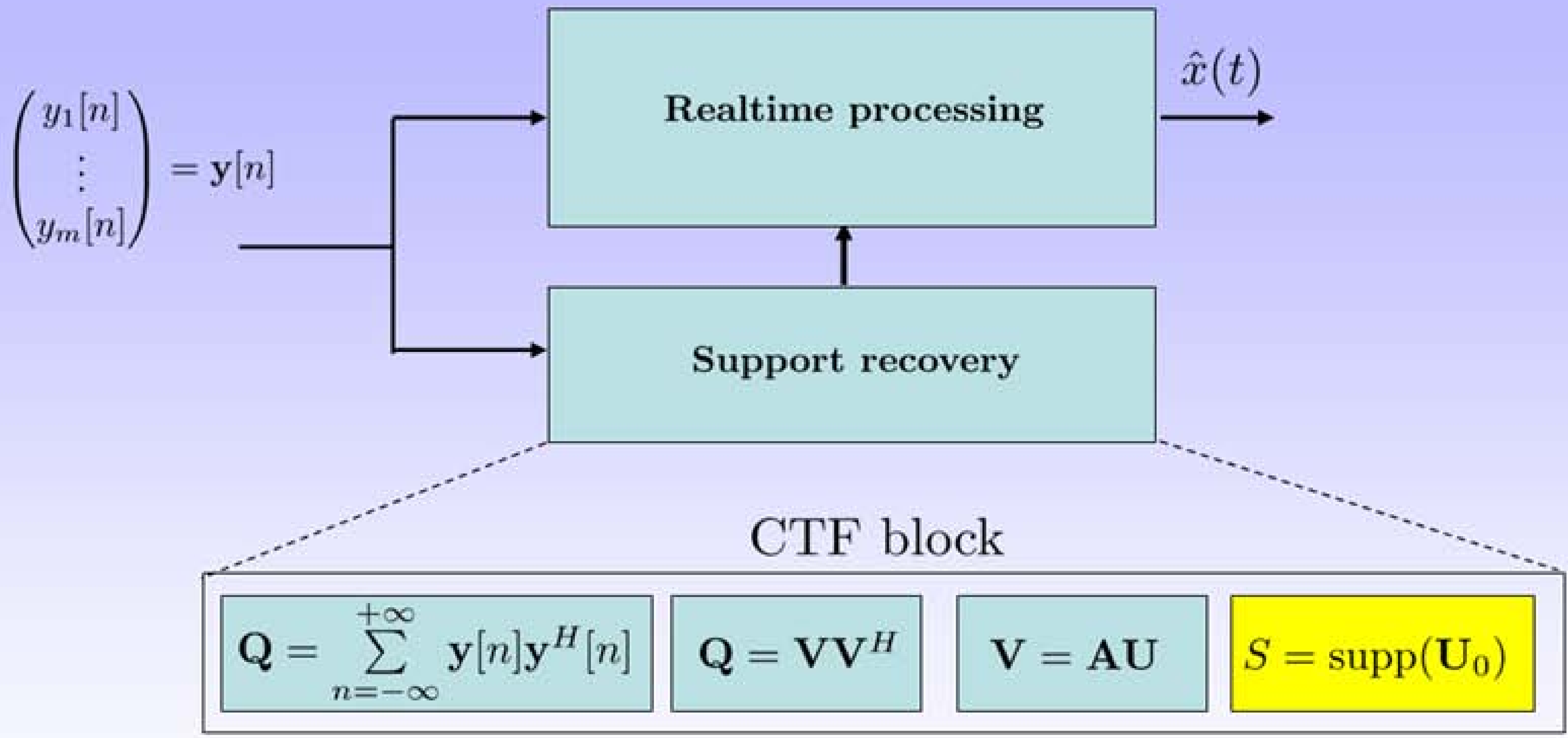
In practice, only a few time instances suffice

Reconstruction

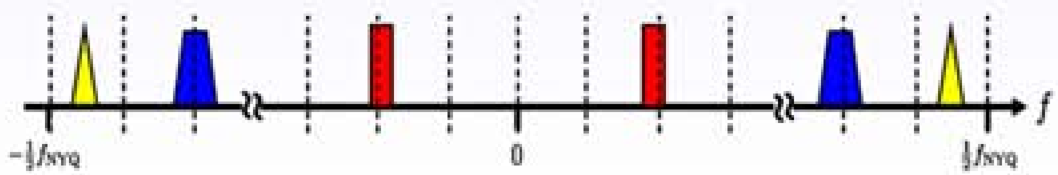


CS system (MMV)
Small size $m \times M$
$m \approx N \quad M \approx \frac{f_{\text{NYQ}}}{B}$
Using advanced techniques $m = 1$!

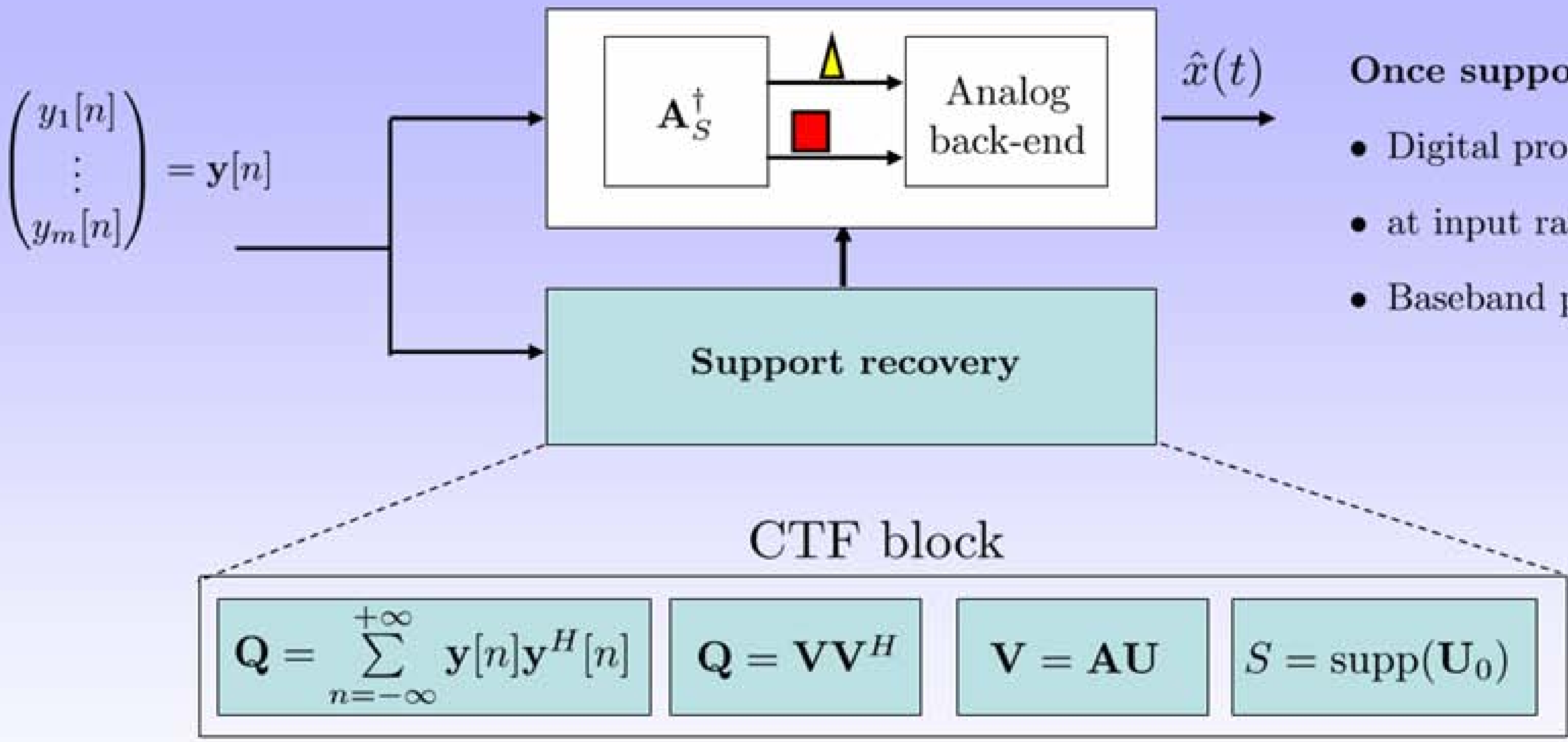
Reconstruction



Matrix support = Signal support



Reconstruction



Once support is known:

- Digital processing
- at input rate (realtime)
- Baseband processing

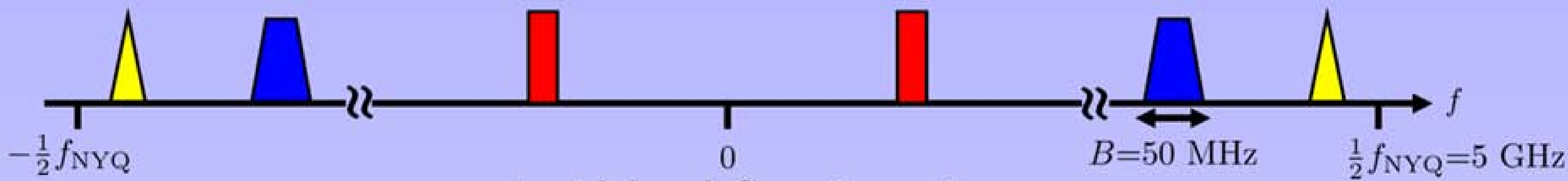
Advantages - Reconstruction

Realtime processing

CTF - Support recovery

- Perfect reconstruction for analog signals at lowest possible rate
- Decouples support recovery from signal reconstruction:
  - CTF works on small size CS system (fast, low memory req.)
  - Actual recovery works on signal dimension, but realtime (known support)
  - Baseband processing
- In practice, CTF requires only a small set of samples

Simulation



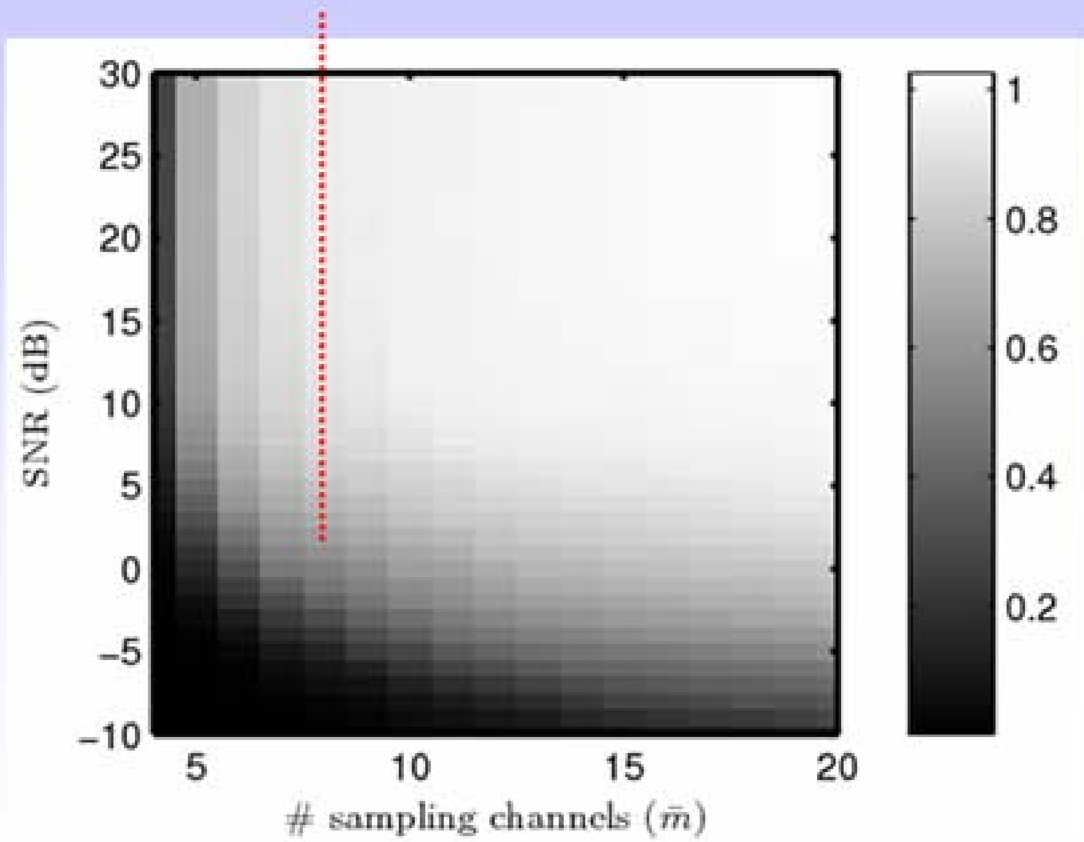
+ wideband Gaussian noise

Theory: P.R. requires 1.2 GHz

In practice: 99% recovery (out of 500 trials)
7 channels × 250 MHz each
=1.8 GHz (S-OMP algorithm)

CTF observes the input for 2 μsecs only !

Can further reduce the system to
4 channels × 450 MHz (CTF with 10 μsecs)
1 channel × 1.8 GHz (CTF with 40 μsecs)



The Underlying Theory

- Landau's theorem (for known support)

Theorem (non-blind recovery)

Let R be a sampling set for $\mathcal{B}_{\mathcal{F}} = \{x(t) \in L^2(\mathbb{R}) \mid \operatorname{supp} X(f) \subseteq \mathcal{F}\}$. Then,

$$D^-(R) \geq \lambda = |\mathcal{F}|$$

Landau (1967)

The Underlying Theory

- Landau's theorem (for known support)
- Minimal rate for multiband signals

Theorem (blind recovery)

Let R be a sampling set for $\mathcal{N}_\lambda = \bigcup_{|\mathcal{F}|\leq\lambda} \mathcal{B}_{\mathcal{F}}$.

Then, $$D^-(R) \geq \min\{2\lambda, f_{\mathrm{NYQ}}\}$$

Mishali and Eldar (2007)

$\Rightarrow$ Minimal rate for $\mathcal{M}$ is $2NB$

The Underlying Theory

- Landau's theorem (for known support)
- Minimal rate for multiband signals
- IMV model and support recovery (CTF)

Recovery with infinite measurement vectors

(IMV) $$\mathbf{y}(\lambda) = \mathbf{A}\mathbf{x}(\lambda), \quad \lambda \in \Gamma$$

Joint-sparsity assocciated with infinitely many CS systems

(CTF) If $\mathbf{x}(\Gamma)$ has $|S| \leq K$ and $\sigma(\mathbf{A}) \geq 2K$, then there exists a unique sparsest solution matrix $\mathbf{U}_0$ and $S = I(\mathbf{U}_0)$

Mishali and Eldar (2007)

The Underlying Theory

- Landau's theorem (for known support)
- Minimal rate for multiband signals
- IMV model and support recovery (CTF)
- Unique mapping conditions

Theorem (unique mapping analog $\Leftrightarrow$ digital)

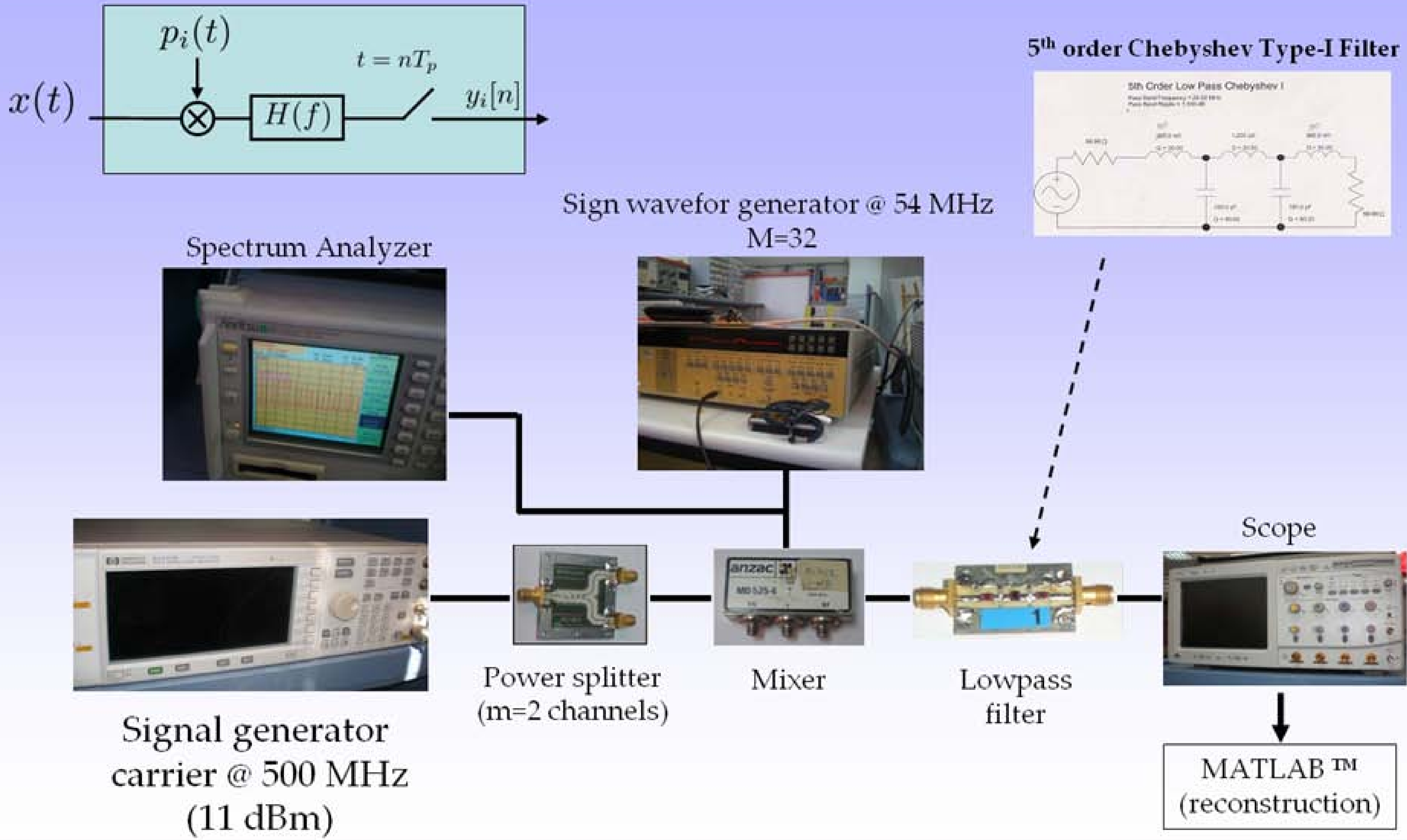
Let $x(t)$ be a multiband signal in $\mathcal{M}$. If,

- $mf_s \geq 2NB$
- $p_i(t)$ have enough "transients"

then, $x(t)$ is the unique analog input matching the samples.

Mishali and Eldar (2009)

Does the dream come true ?



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Thank you