SPECTRUM-BLIND RECONSTRUCTION OF MULTI-BAND SIGNALS

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ABSTRACT

We develop a perfect reconstruction scheme from point-wise sub-Nyquist rate samples for multi-band signals. Our approach is blind, namely the knowledge of the band locations is not used in the design of either the sampling or the reconstruction stage. This is in contrast to previous approaches to reconstruct this class of signals, which required information about the spectral support at least in the reconstruction stage. Our scheme guarantees exact recovery for a wide class of multi-band signals without the use of heuristics or discretization methods.

Index Terms— Landau-Nyquist rate, multiband sampling, nonuniform periodic sampling.

1. INTRODUCTION

We treat the problem of sampling and reconstructing a multi-band signal, namely a bandlimited signal whose Fourier transform is supported on several distinct intervals. The case of known band locations was studied in the literature and it was shown that this structure enables perfect reconstruction from sub-Nyquist rate samples. Landau [1] developed a minimal rate requirement for arbitrary sampling methods allowing perfect reconstruction. For multi-band signals the Landau rate is the sum of the band widths. Lin and Vaidynathan [2] introduced a perfect reconstruction method from periodic non-uniform sampling at the Landau rate, assuming knowledge of the band locations. Venkataramani and Bresler [3] suggested a half-blind system: the sampling is based on a blind multi-coset scheme whose design does not require the band locations. However, the reconstruction is performed using a non-blind filter-bank scheme requiring knowledge of the spectral support.

In this paper, we study the problem of blind perfect reconstruction. We begin by exploring the theoretical minimal rate requirement for blind reconstruction. It is shown that the minimal rate is higher than the lower bound for known-spectrum reconstruction, namely the Landau rate. This result is stated for arbitrary sampling and reconstruction and is based on a recent paper of Lu and Do [6].

The main contribution of this paper is the development of a fully-blind system. For sampling we use a blind multi-coset strategy satisfying the minimal rate requirement. This type of strategy was also used in [3], however the reconstruction of [3] is not blind. In order to reconstruct the signal, we develop theoretical results that enables transforming the continuous nature of the reconstruction problem into a finite dimensional problem without any discretization. In addition, we show that the solution of the finite dimensional problem can be obtained by finding the unique sparsest solution matrix from Multiple-Measurements-Vectors (MMV). Finding a sparsest solution matrix is discussed thoroughly in the literature of compressed sensing (CS) [4]. The proposed reconstruction scheme is formalized as a finite-step algorithm, and is proved to guarantee perfect reconstruction for a wide class of multi-band signals.

Our work differs from other main stream CS papers by the fact that the signals are treated in a continuous framework rather than a discrete one. The transformation into a finite dimensional problem is exact and does not involve any discretization technique. In addition, we use a deterministic sampling procedure as opposed to the use of random sampling operators in the CS literature. Thus, our theoretical results guarantee perfect reconstruction and do not condition this behavior by any probability.

The advantage of this work is that the design of a fully spectrum-blind sampling and reconstruction system can be done once and in advance. The reconstruction can then be applied to a wide range of multi-band signals differing in their band locations without the need to adjust either part of the system. The low sampling rate reduces the overall cost of such a system. Our method is therefore attractive for a variety of applications, including cellular networks, magnetic resonance imaging and military uses.

The paper is organized as follows. In Section 2 we detail the reconstruction objectives, which are the class of multi-band signals to be perfectly reconstructed, and the minimal rate requirement for blind reconstruction. An overview of the multi-coset sampling is given in Section 3. Conditions for blind prefect reconstruction using this sampling strategy are discussed in Section 4, in which we also present our reconstruction method. In Section 5 we describe numerical experiments demonstrating blind sampling and reconstruction using our approach.

2. RECONSTRUCTION OBJECTIVES

2.1. Multi-band signals class

We consider the class $\mathcal{M}$ of complex-valued multi-band signals bandlimited to $\mathcal{F} = [0, 1/T]$, so that $1/T$ is the Nyquist rate for signals in $\mathcal{M}$. In addition, the class $\mathcal{M}$ is assumed to contain multi-band signals with no more than N bands and each of the band widths is not greater than B. Note that the values of N and B are typically known in advance, whereas the exact band locations is not assumed.

Our goal is to perfectly reconstruct $x(t) \in \mathcal{M}$ when both sampling and reconstruction are required to be blind, which means knowledge of the band locations cannot be used in their design. In the next section, we study the minimal rate requirement for blind perfect reconstruction.

2.2. Minimal rate requirement

Let $R = \{r_n\}_{n=-\infty}^{\infty}$ be a countable set such that the sequence $x_R[n] = x(t = r_n)$ describes a point-wise sampling set for $x(t)$.

The corresponding average sampling rate is computed by the lower Beurling density function

$$D^-(R) = \lim_{r\to\infty} \inf_{y\in\mathbb{R}} \frac{|R\cap[y,y+r]|}{r}. \quad (1)$$

The numerator in the right hand side of (1) counts the number of points from R in every interval of width r.

In known-spectrum reconstruction the Fourier transform $X(f)$ is restricted to a known support $\mathcal{T} \subseteq \mathcal{F}$. A stable sampling set R requires the existence of constants $\alpha > 0$ and $\beta < \infty$ such that:

$$\alpha\|x(t)-y(t)\|^2 \le \|x_R[n]-y_R[n]\|^2 \le \beta\|x(t)-y(t)\|^2 \quad (2)$$

is valid for all signals $x(t), y(t)$ whose Fourier transform is supported on $\mathcal{T}$. Landau [1] proved that a stable sampling set must have a density $D^-(R)$ not less than the Landau rate, which is the Lebesgue measure of $\mathcal{T}$.

In blind-spectrum reconstruction we consider the set $\mathcal{N}_\Omega$ of signals bandlimited to $\mathcal{F}$ whose support measure is not greater than Ω/T for some $0 < \Omega < 1$. Note that $\mathcal{N}_\Omega$ contains signals with different support $\mathcal{T}$ and the exact spectral support of $x(t) \in \mathcal{N}_\Omega$ is not known. We say that R is a blind stable sampling set if there exists constants $\alpha > 0$ and $\beta < \infty$ such that (2) is valid for every $x(t), y(t) \in \mathcal{N}_\Omega$. The following theorem states the minimal density of a blind stable sampling set.

Theorem 1 (Minimal sampling rate) *Let R be a blind sampling set for $\mathcal{N}_\Omega$. Then,*

$$D^-(R) \ge \min\left\{\frac{2\Omega}{T}, \frac{1}{T}\right\}. \quad (3)$$

We point out that the theorem does not provide a method to achieve the lower bound. Nevertheless, for $\Omega > 0.5$ the solution is straightforward since sampling at the Nyquist rate achieves the lower bound of Theorem 1. An ideal low-pass filter guarantees perfect reconstruction for this sampling set. Thus, the resulting system is fully blind and perfectly reconstruct every $x(t) \in \mathcal{M}$. However, for $\Omega < 0.5$ Theorem 1 suggests that perfect reconstruction is allowed from samples at a sub-Nyquist rate. Throughout the paper, we therefore focus on this case of $\Omega < 0.5$.

The Landau rate for the set $\mathcal{M}$ is NB while the minimal rate for blind reconstruction is $2NB$ according to Theorem 1. Our goal is to develop a blind sampling and reconstruction scheme that does not assume knowledge of the band locations, and results in a rate close to the minimal rate.

3. SPECTRUM-BLIND SAMPLING

To sample the signal, we use the multi-coset blind technique proposed in [3] with a certain choice of parameters. Multi-coset sampling is a class of periodic non-uniform sampling in which the sampling set is taken from the uniform grid $x(t = nT)$. The grid is divided into blocks of L samples. A set $C = \{c_i\}_{i=1}^p$ describes the locations of p out of L samples that are kept in each block while the rest are ignored. The parameters L, p and the sampling pattern C determine the multi-coset sampling set.

Define the ith sampling sequence for $1 \le i \le p$ as

$$x_{c_i}[n] = \begin{cases} x(t=nT) & n = mL + c_i, \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Note that $x_{c_i}[n]$ is zero-padded by $L-1$ zeros between the non-zero samples, thus the implementation of this sampling strategy is carried out by p uniform sequences with period $1/(LT)$. Evidently, the average sampling rate is $p/(LT)$, which is lower than the Nyquist rate for $p < L$.

A linear relation between the sampling sequences and the signal exists in the frequency domain [3]

$$X_{c_i}(e^{j2\pi fT}) = \frac{1}{LT}\sum_{r=0}^{L-1} \exp\left(j\frac{2\pi}{L}c_i r\right) X\left(f + \frac{r}{LT}\right), \quad (5)$$
$$\forall f \in \mathcal{F}_0 = \left[0, \frac{1}{LT}\right), \ 1 \le i \le p,$$

where $X_{c_i}(e^{j2\pi fT})$ is the discrete-time Fourier transform of $x_{c_i}[n]$. Expressing (5) in a matrix form we have

$$\mathbf{y}(f) = \mathbf{A}\mathbf{x}(f), \ \ \forall f \in \mathcal{F}_0, \quad (6)$$

where $X_{c_i}(e^{j2\pi fT})$ is the ith entry of the vector $\mathbf{y}(f)$, and $\mathbf{x}(f)$ contains L unknowns for each f

$$\mathbf{x}_i(f) = X\left(f + \frac{i}{LT}\right), \ \ 0 \le i \le L-1, \ \ f \in \mathcal{F}_0. \quad (7)$$

The matrix $\mathbf{A}$ depends on the parameters L, p and the sampling pattern C but not on $x(t)$ and is defined by

$$\mathbf{A}_{ik} = \frac{1}{LT}\exp\left(j\frac{2\pi}{L}c_i k\right). \quad (8)$$

A sampling pattern C is called universal if every set of p columns of $\mathbf{A}$ is linearly independent [3]. Universal patterns are an essential element of both known and blind reconstruction since they result in the largest class of signals that can be perfectly reconstructed [3],[7]. Finding a universal pattern involves a combinatorial process which can be avoided for the following specific selections: The bunched pattern $C = \{0, 1, ..., p-1\}$ is proved to be universal [3]. Alternatively, choosing L to be prime renders every pattern universal [5].

Using the multi-coset strategy, our goal is to select the parameter L, p and a universal pattern C such that $\mathbf{x}(f)$ can be recovered from (6) for every $f \in \mathcal{F}_0$.

4. SPECTRUM-BLIND RECONSTRUCTION

The linear system of (6) is undetermined since for every $f \in \mathcal{F}_0$ there are less equations than unknowns, so that the matrix $\mathbf{A}$ is non-invertible. In addition, the solution $\mathbf{x}(f)$ typically does not belong to the null-orthogonal subspace of $\mathbf{A}$, which means it cannot be recovered by multiplying both sides of (6) by the pseudo-inverse $\mathbf{A}^\dagger$. Therefore, determining $\mathbf{x}(f)$ from (6) requires some prior knowledge. In our case, a multi-band signal has a special structure in the frequency domain: Its Fourier transform is zero outside the bands support. We use this characterization as a prior for solving (6). The following theorem expresses this prior in terms of the vector $\mathbf{x}(f)$. Specifically, it provides a parameters selection that ensures $\mathbf{x}(f)$ is N sparse, namely it contains no more than N non-zero values for every $f \in \mathcal{F}_0$.

Theorem 2 *Let $x(t) \in \mathcal{M}$ be a multi-band signal. If*

1. *$L \le 1/(BT)$,*
2. *$p \ge 2N$,*
3. *C is a universal pattern,*

then, for every $f \in \mathcal{F}_0$, the vector $\mathbf{x}(f)$ is N-sparse and $\mathbf{x}(f)$ is the unique sparsest solution of (6).

The sparsity of $\mathbf{x}(f)$, as noted in Theorem 2, results from the sparse structure in the frequency domain and the specific parameters selection. Evidently, perfect reconstruction is possible under the conditions of Theorem 2. Note that the number of non-zero entries in $\mathbf{x}(f)$ is no more than N while the length of $\mathbf{y}(f)$ is $p \geq 2N$. The origin of this factor of two comes from the following well known CS result for an arbitrary vector $\mathbf{y}$ and an arbitrary matrix $\mathbf{A}$.

Theorem 3 *Suppose $\bar{\mathbf{x}}$ is a solution of $\mathbf{y} = \mathbf{A}\mathbf{x}$. If $\bar{\mathbf{x}}$ is K-sparse and every set of $2K$ columns of $\mathbf{A}$ is linearly independent, then $\bar{\mathbf{x}}$ is the unique sparsest solution of the system.*

The requirement for a universal pattern in Theorem 2 provides the linear independency of every set of p columns in $\mathbf{A}$. According to Theorem 2, $\mathbf{x}(f)$ is N-sparse, which explains the limitation $p \geq 2N$. As explained in Section 3, the average sampling rate of the multi-coset strategy is $p/(LT)$. Thus, the limitation of $L \leq 1/(BT)$ and $p \geq 2N$ implies an average sampling rate of at least $2NB$, which satisfies the minimal rate requirement of Theorem 1.

The literature of CS suggests several techniques for finding the sparsest solution of a linear system. Theorem 2 guarantees that $\mathbf{x}(f)$ is the unique sparsest solution, thus a straightforward approach is to apply any of these methods on a dense grid of $f \in \mathcal{F}_0$. However, a discretization of this kind cannot guarantee perfect reconstruction of $\mathbf{x}(f)$ for *every* $f \in \mathcal{F}_0$. Instead, we propose an exact recovery of $\mathbf{x}(f)$ by first finding the finite set

$$S = \{k \mid \mathbf{x}_k(f) \neq 0, \text{ for some } f \in \mathcal{F}_0\}. \tag{9}$$

It is easy to see that (6) can be written as

$$\mathbf{y}(f) = \mathbf{A}_S\, \mathbf{x}_S(f), \tag{10}$$

where $\mathbf{A}_S$ is a sub-matrix of $\mathbf{A}$ containing only the columns whose indices belong to S. Similarly, $\mathbf{x}_S(f)$ is a vector of length $|S|$ that consists of the entries of $\mathbf{x}(f)$ in the locations described by S. It can be proved that under the conditions of Theorem 2, the pseudo-inverse of $\mathbf{A}_S$ satisfies

$$(\mathbf{A}_S)^{\dagger}\mathbf{A}_S = I, \tag{11}$$

which results in

$$\mathbf{x}_S(f) = (\mathbf{A}_S)^{\dagger}\mathbf{y}(f), \quad \forall f \in \mathcal{F}_0. \tag{12}$$

In addition, from (9)

$$\mathbf{x}_i(f) = 0, \quad \forall f \in \mathcal{F}_0,\, i \notin S. \tag{13}$$

Thus, once S is known perfect reconstruction is obtained by (12)-(13).

In order to find the set S the following operations are performed. First, we compute the $p \times p$ matrix

$$\mathbf{Q} = \int_0^{\frac{1}{LT}} \mathbf{y}(f)\mathbf{y}^H(f)df, \tag{14}$$

where $\mathbf{y}^H(f)$ denotes the conjugate transpose of $\mathbf{y}(f)$. Since $\mathbf{Q}$ is positive semi-definite, it can be decomposed as

$$\mathbf{Q} = \mathbf{V}\mathbf{V}^H \tag{15}$$

Algorithm 1 SBR

Input: $\mathbf{y}(f)$, **Assume:** The conditions of Theorem 4
Output: the set S, flag
1: Compute the matrix $\mathbf{Q}$ by (14) or by (17)
2: Decompose $\mathbf{Q} = \mathbf{V}\mathbf{V}^H$
3: Solve the MMV system $\mathbf{V} = \mathbf{A}\mathbf{U}$ for the sparsest solution $\mathbf{U}_0$
4: $S = \{$ the locations of non-identically zero rows of $\mathbf{U}_0$ $\}$
5: flag = $\{|S| \leq \frac{p}{2}\}$
6: **return** S, flag

with $\mathbf{V}$ having r orthogonal columns, where $r = \operatorname{rank}(\mathbf{Q})$. Note that this decomposition is not unique. We then define the finite dimensional linear system

$$\mathbf{V} = \mathbf{A}\mathbf{U}. \tag{16}$$

Finally, we seek the sparsest $L \times r$ solution matrix of (16), namely a matrix $\mathbf{U}_0$ solving (16) that has a minimal number of non-identical zero rows. The following theorem states conditions that ensures the desired set S is equal to the locations of the non-zero rows of $\mathbf{U}_0$.

Theorem 4 *Let $x(t) \in \mathcal{M}$ be a multi-band signal. If:*

1. *$L \leq 1/(BT)$,*
2. *$p \geq 4N$,*
3. *C is a universal pattern,*

then, for every choice of decomposition in (15), the system (16) has a unique sparsest solution $\mathbf{U}_0$. Moreover, the set S defined in (9) is the set of non-identical zero rows of $\mathbf{U}_0$ regardless of the specific selection of $\mathbf{V}$ in (15).

Theorem 4 is very powerful since it allows us to replace the continuous reconstruction problem with a finite dimensional one. In the CS literature, the linear system of (16) is referred to as Mutli-Measurement vectors (MMV). Finding the sparsest solution $\mathbf{U}_0$ of an MMV system is known to be NP-hard. Theorem 4 guarantees perfect reconstruction for every $x(t) \in \mathcal{M}$ as long as $\mathbf{U}_0$ is recovered correctly. To implement our method in practice, we suggest using sub-optimal efficient techniques from the CS literature to solve the MMV system (16). Several of these methods are given in [4].

The formal reconstruction steps are summarized in Algorithm 1, named SBR. The SBR algorithm also contains a flag variable reporting the success recovery of S in case a sub-optimal MMV technique is used. A failure indication means that the specific MMV approach used has failed and another one can be employed.

We note that $\mathbf{Q}$ can be computed directly from the sample sequences $\hat{x}_{c_i}[n] = x(nLT + c_iT)$:

$$\mathbf{Q}_{ik} = \sum_{m \in \mathbb{Z}} \hat{x}_{c_i}[m](\hat{x}_{c_k} * g_{ik})[m], \tag{17}$$

where g_{ik} are the set of digital filters

$$g_{ik}[m] = \int_0^{\frac{1}{LT}} \exp\left(j2\pi f(mL + (c_k - c_i))T\right) df. \tag{18}$$

The requirement $p \geq 4N$ of Theorem 4 is more constrained than $p \geq 2N$ of Theorem 2. Consequently, using the extreme values allowed for L, p by Theorem 4 yields an average sampling rate of at least $4NB$, which is twice the minimal required by Theorem 1. This increased rate is a drawback of the SBR algorithm, as according to Theorem 2 perfect reconstruction using multi-coset samples is also possible at the minimal rate. Our work in [7] provides a more complex version of the SBR algorithm that accomplishes perfect reconstruction and requires the minimal sampling rate.

5. NUMERICAL EXPERIMENTS

In this section we quantify the behavior of algorithm SBR when a sub-optimal technique is used to solve the MMV system. We consider an example of the class $\mathcal{M}$ with $\mathcal{F} = [0, 20\text{ GHz}]$, $N = 4$ and $B = 100$ MHz. In ordet to test the algorithm 1000 multi-band signals from this class were constructed such that each signal has exactly N bands and each of the band width is exactly B. Thus, the support measure of each signal is $NB = 400$ MHz. The bands were located uniformly at random on $\mathcal{F}$ and the complex values of $X(f)$ inside the bands were randomly generated from a normal distribution.

We constructed 29 different multi-coset stages with the following parameters. A prime value $L = 199$ is common in all the systems. The value of p varied from 4 to 32, and for each value of p the sampling pattern C was selected randomly among $\binom{L}{p}$ choices. Since L is prime, the pattern C is guaranteed to be universal. We conducted the experiments by sampling the signals by each one of these multi-coset systems. Then, we reconstructed the signals by the SBR algorithm, where the multi-orthogonal matching pursuit technique [4] was used to solve the MMV. The empirical success rate is calculated as the ratio of signals in which the correct solution was found. Note that if the flag variable indicates a failure, then we could have used another MMV technique to reconstruct the signal. However, since solving MMV is not our main focus here, we preferred to use only one technique. The results are depicted in Fig. 1.

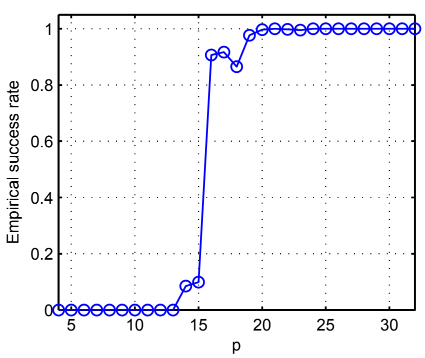

Fig. 1. Performance of the SBR Algorithm for signals $x(t) \in \mathcal{M}$.

The sampling rate for $p = 4N = 16$ is $p/(LT) = 1608$ MHz, which is slightly more than twice the minimal rate. Indeed, Fig. 1 shows that a high recovery is accomplished for $p \geq 16$ as ensured by Theorem 4. As expected, for $p < 8$ the signals could not be recovered since the average sampling rate does not satisfy the minimal requirement of Theorem 1.

To emphasize the exact recovery of the SBR algorithm, one of the tested signals is compared with its reconstructed version using $p = 16$. Fig. 2 plots the spectral support of both the original and the reconstructed signal. Fig. 3 presents both signals on a segment of the time domain. As seen, perfect reconstruction is accomplished. In fact, there is nothing special in the random distributions we used to construct the set of signals for this experiment. Perfect reconstruction occurs for other choices as well since our method assumes neither the exact signal support nor it the exact values of the Fourier transform.

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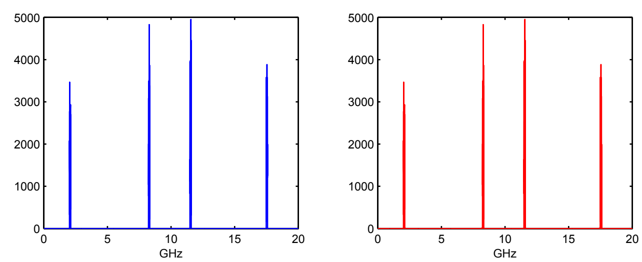
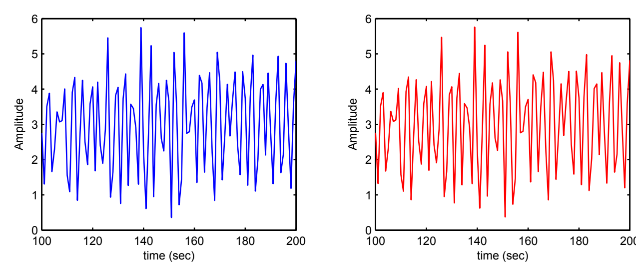


(a) (b)

Fig. 2. (a) Original and (b) reconstructed signal spectrum.



(a) (b)

Fig. 3. (a) Original and (b) reconstructed signal in time domain.