Multipath Medium Identification Using Efficient Sampling Schemes

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***Abstract*—Time delay estimation arises in many applications in which a multipath channel has to be identified using pulses transmitted through the medium. Various approaches have been proposed in the literature to identify the time delays of the multipath components. However, these methods require high sampling rates. In this paper, we develop a unified approach to time delay estimation from low rate samples of the output of a multipath medium. Our approach results in a sampling theorem for analog signals defined over an infinite union of subspaces. The proposed method leads to perfect recovery of the multipath delays from samples of the channel output at the lowest possible rate, which depends only on the number of multipath components and the transmission rate, and not on the bandwidth of the probing signal. By properly manipulating the low-rate samples, we show that the time delays can be recovered using the well-known ESPRIT algorithm. Combining results from sampling theory with those obtained in the context of direction of arrival estimation methods, we develop necessary and sufficient conditions on the transmitted pulse and the sampling functions in order to ensure perfect recovery of the channel parameters at the minimal possible rate.**

I. Introduction

Time delay estimation is an important signal processing problem with various applications such as radar [1], underwater acoustics [2], wireless communication [3], and more. In such applications, the goal is to identify a multipath medium by transmitting pulses with a known shape through it. Identification is performed by estimating the time delays and gain coefficients of each multipath component.

A classical solution to the time delay estimation problem is based on correlation techniques [1] which are effective only when the multipath components are well separated in time. To overcome this limitation, a number of super-resolution methods were proposed [4], [5], [6], [7], [8]. However, in these approaches the sampling stage of the analog received signal has not received much attention, and usually involves pointwise sampling at a high sampling rate. In [4], [5], [6], [8] the required sampling rate is the Nyquist rate of the transmitted pulse. The time domain algorithms proposed in [7] and [8] can theoretically recover the time delays by sampling at low rates, but no concrete conditions on the transmitted pulse were given, in order to ensure unique recovery of the delays.

In addition, the methods mentioned require the assumption that the receiver has access to multiple non-overlapping experiments on the medium. This implies that all the reflections from the medium, as a result of a pulse transmission, vanish before the next pulse is transmitted. This assumption can be problematic in some applications such as wireless communication.

In this paper we consider recovery of the parameters defining a multipath medium from samples of the medium output. Specifically, we assume that pulses with known shape are transmitted at a constant rate through the channel, and our aim is to recover the time delays and time-varying gain coefficients of each multipath component, from samples of the received signal taken at the lowest possible rate. We first propose a signal model that can describe the received signal. An advantage of our model is that it does not require the assumption of non-overlapping experiments, and allows for general pulse shapes. We then formulate the identification problem as a sampling problem, in which the set of parameters defining the medium have to be recovered from samples of the received signal. To this end we develop a general sampling scheme, which consists of filtering the received signal with a bank of p sampling filters and uniformly sampling their outputs. Given K multipath components, we show that at least $2K$ sampling filters are required in order to perfectly recover the time delays. We then develop explicit sampling strategies that achieve this minimal rate.

In order to recover the medium parameters from the given samples we combine results from standard sampling theory, with those of direction of arrival (DOA) estimation [9]. Specifically, by appropriate manipulation of the sampling sequences, we formulate our problem within the framework of DOA [10] and rely on the estimation of signal parameters via rotational invariance technique (ESPRIT) [10], developed in that context. Once the time delays are identified, the gain coefficients are recovered using standard sampling tools.

This paper is organized as follows. In Section II, we formulate the medium identification problem and describe our proposed signal model. A general sampling scheme is proposed in Section III. Section IV describes the recovery of the unknown delays from the samples, and provides sufficient conditions ensuring a unique recovery. Numerical experiments are presented in Section V.

II. Problem Formulation

We consider the problem of identifying a time-variant multipath channel. The medium is probed with pulses which are transmitted at a constant rate, so that the transmitted signal is given by

$$x_T(t) = \sum_{n\in\mathbb{Z}} g(t-nT), \tag{1}$$

where T is the probing period and $g(t)$ is the transmitted pulse shape. The impulse response of the time varying channel is modeled as

$$h(\tau,t) = \sum_{k=1}^{K} a_k(t)\,\delta(\tau - t_k), \tag{2}$$

where $a_k(t)$ is the time varying complex gain coefficient of the kth multipath component, $t_k \in [0,T)$ is the corresponding propagation delay and K denotes the total number of paths. We assume that the medium is slowly varying relatively to the probing period T, so that the gain coefficients can be considered as constant over one period. The output of the medium can then be expressed as

$$x(t) = \sum_{k=1}^{K}\sum_{n\in\mathbb{Z}} a_k[n]\,g(t-t_k-nT). \tag{3}$$

This model for the received signal does not require the assumption of non-overlapping experiments assumed in previous works. If we consider each period of T as an experiment, then our model allows

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interference of pulses from adjacent periods. Consequently (3) supports general pulse shapes, and does not require that the transmitted pulse is time limited.

Our problem now is to determine the delays $\tau = \{t_k\}_{k=1}^{K}$ and the gains $a_k[n]$ from samples of the received signal $x(t)$, at the minimal possible rate. Since these parameters uniquely define $x(t)$, our medium identification problem is equivalent to developing efficient sampling schemes for signals of the form (3), allowing perfect reconstruction of the signal from its samples.

The signal model (3) can be considered as a special case of a more general framework of signals that lie in a union of subspaces [11], [12], [13]. When the set τ is fixed, $x(t)$ lies in a shift-invariant (SI) subspace spanned by K generators $\{g(t-t_k)\}_{k=1}^{K}$. Thus, the set of all signals of the form (3) constitutes an infinite union of subspaces, as the unknown delays can take any continuous value in the interval $[0,T)$. Therefore, our results can be viewed as a method for sampling and recovering a signal over an infinite union of subspaces. As far as we know, this is the first example that allows recovery from minimal rate samples over an infinite union. We discuss this connection in more detail in the conclusion.

III. Sampling Scheme

To sample the signal $x(t)$ we propose a sampling scheme comprised of p parallel channels. In each channel $x(t)$ is pre-filtered using the filter $s_\ell^*(-t)$ and sampled uniformly at times $t = nT$ to produce the sampling sequence $c_\ell[n]$, as depicted in the left-hand side of Fig. 1. The superscript $(\cdot)^*$ represents complex conjugation. We assume that $p \geq K$; exact conditions on the number of sampling channels p and the choice of sampling filters will be given in the next sections.

The discrete-time Fourier transform (DTFT) of the ℓth sampling sequence is given by

$$C_\ell\left(e^{j\omega T}\right) = \frac{1}{T}\sum_{m\in\mathbb{Z}} S_\ell^*\left(\omega - \frac{2\pi}{T}m\right) X\left(\omega - \frac{2\pi}{T}m\right), \tag{4}$$

where $S_\ell(\omega)$ and $X(\omega)$ denote the Fourier transform of $s_\ell(t)$ and $x(t)$ respectively. From the definition of $x(t)$, its Fourier transform can be written as

$$X(\omega) \;=\; \sum_{k=1}^{K} A_k\left(e^{j\omega T}\right) G(\omega)\, e^{-j\omega t_k}, \tag{5}$$

where $A_k\left(e^{j\omega T}\right)$ denotes the DTFT of the sequence $a_k[n]$, and $G(\omega)$ denotes the Fourier transform of $g(t)$. Substituting (5) into (4), we have

$$\begin{aligned} C_\ell\left(e^{j\omega T}\right) \;=\; & \sum_{k=1}^{K} A_k\left(e^{j\omega T}\right) e^{-j\omega t_k}\frac{1}{T}\sum_{m\in\mathbb{Z}} S_\ell^*\left(\omega - \frac{2\pi}{T}m\right) \\ & \cdot G\left(\omega - \frac{2\pi}{T}m\right) e^{j\frac{2\pi}{T}mt_k}. \end{aligned} \tag{6}$$

Denoting by $\mathbf{c}\left(e^{j\omega T}\right)$ the length-p column vector whose ℓth element is $C_\ell\left(e^{j\omega T}\right)$ and by $\mathbf{a}\left(e^{j\omega T}\right)$ the length-K column vector whose kth element is $A_k\left(e^{j\omega T}\right)$, we can write (6) in matrix form as

$$\mathbf{c}\left(e^{j\omega T}\right) = \mathbf{M}\left(e^{j\omega T},\tau\right)\mathbf{b}\left(e^{j\omega T}\right), \tag{7}$$

where $\mathbf{M}(\omega,\tau)$ is a $p\times K$ matrix with ℓkth element

$$\begin{aligned} \mathbf{M}_{\ell k}\left(e^{j\omega T},\tau\right) \;=\; & \frac{1}{T}\sum_{m\in\mathbb{Z}} S_\ell^*\left(\omega - \frac{2\pi}{T}m\right)\cdot \\ & G\left(\omega - \frac{2\pi}{T}m\right) e^{j\frac{2\pi}{T}mt_k}, \end{aligned} \tag{8}$$

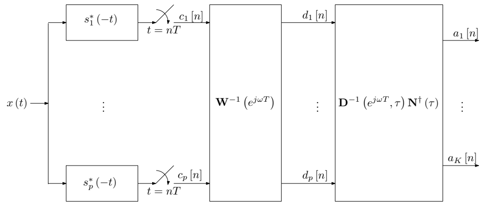


Figure 1. Sampling and reconstruction scheme

$\mathbf{b}\left(e^{j\omega T}\right)$ is a length-K column vector given by

$$\mathbf{b}\left(e^{j\omega T}\right) = \mathbf{D}\left(e^{j\omega T},\tau\right)\mathbf{a}\left(e^{j\omega T}\right), \tag{9}$$

and $\mathbf{D}\left(e^{j\omega T},\tau\right)$ is a diagonal matrix with kth diagonal element $e^{-j\omega t_k}$.

To proceed, we focus our attention on sampling filters $S_\ell(\omega)$ with finite support in the frequency domain, contained in the range

$$\mathcal{F} = \left[\frac{2\pi}{T}\gamma, \frac{2\pi}{T}(p+\gamma)\right], \tag{10}$$

where $\gamma\in\mathbb{Z}$ is an index which determines the working frequency band $\mathcal{F}$. This choice should be such that it matches the frequency occupation of $g(t)$. Under this choice of filters, the matrix $\mathbf{M}\left(e^{j\omega T},\tau\right)$ can be expressed as

$$\mathbf{M}\left(e^{j\omega T},\tau\right) = \mathbf{W}\left(e^{j\omega T}\right)\mathbf{N}(\tau) \tag{11}$$

where $\mathbf{W}\left(e^{j\omega T}\right)$ is a $p\times p$ matrix whose ℓmth element is given by

$$\begin{aligned} \mathbf{W}_{\ell m}\left(e^{j\omega T}\right) \;=\; & \frac{1}{T}S_\ell^*\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right)\cdot \\ & G\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right) \end{aligned} \tag{12}$$

and $\mathbf{N}(\tau)$ is a $p\times K$ Vandermonde matrix with mkth element

$$\mathbf{N}_{mk}(\tau) = e^{-j\frac{2\pi}{T}(m-1+\gamma)t_k}. \tag{13}$$

If $\mathbf{W}\left(e^{j\omega T}\right)$ is stably invertible, then we can define the modified measurement vector $\mathbf{d}\left(e^{j\omega T}\right)$ as

$$\mathbf{d}\left(e^{j\omega T}\right) = \mathbf{W}^{-1}\left(e^{j\omega T}\right)\mathbf{c}\left(e^{j\omega T}\right). \tag{14}$$

From (7) and (11), this vector satisfies

$$\mathbf{d}\left(e^{j\omega T}\right) = \mathbf{N}(\tau)\,\mathbf{b}\left(e^{j\omega T}\right). \tag{15}$$

Since $\mathbf{N}(\tau)$ is independent of ω, using the linearity of the DTFT, we can express (15) in the time domain as

$$\mathbf{d}[n] = \mathbf{N}(\tau)\,\mathbf{b}[n], \quad n\in\mathbb{Z}. \tag{16}$$

The elements of the vectors $\mathbf{d}[n]$ and $\mathbf{b}[n]$ are the discrete time sequences, obtained from the inverse DTFT of the elements of the vectors $\mathbf{b}\left(e^{j\omega T}\right)$ and $\mathbf{d}\left(e^{j\omega T}\right)$ respectively.

Equation (16) describes an infinite set of measurement vectors, each obtained by the same measurement matrix $\mathbf{N}(\tau)$, which depends on the unknown delays τ. This problem is reminiscent of the type of problems that arise in DOA estimation. Therefore, our approach is to rely on results developed in that context in order to first recover τ from the measurements. After τ is known, the vector $\mathbf{a}\left(e^{j\omega T}\right)$ can be found using the following linear filtering relation

$$\mathbf{a}\left(e^{j\omega T}\right) = \mathbf{D}^{-1}\left(e^{j\omega T},\tau\right)\mathbf{N}^{\dagger}(\tau)\,\mathbf{d}\left(e^{j\omega T}\right), \tag{17}$$

where $\mathbf{N}^{\dagger}(\tau)$ is the Moore-Penrose pseudo-inverse. The resulting sampling scheme is depicted in Fig. 1.

Our last step, therefore, is to derive conditions on the filters $s_\ell^*(-t)$ and the function $g(t)$ in order that the matrix $\mathbf{W}\left(e^{j\omega T}\right)$ is stably invertible. To this end, we can decompose $\mathbf{W}\left(e^{j\omega T}\right)$ as

$$\mathbf{W}\left(e^{j\omega T}\right) = \mathbf{S}\left(e^{j\omega T}\right)\mathbf{G}\left(e^{j\omega T}\right) \tag{18}$$

where $\mathbf{S}\left(e^{j\omega T}\right)$ is a $p \times p$ matrix with ℓmth element

$$\mathbf{S}_{\ell m}\left(e^{j\omega T}\right) = \frac{1}{T} S_\ell^*\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right) \tag{19}$$

and $\mathbf{G}\left(e^{j\omega T}\right)$ is a $p \times p$ diagonal matrix with mth diagonal element

$$\mathbf{G}_{mm}\left(e^{j\omega T}\right) = G\left(\omega + \frac{2\pi}{T}(m-1+\gamma)\right). \tag{20}$$

Each of these matrices needs to be stably invertible, leading to the following conditions:

Condition 1: the function $g(t)$ needs to satisfy

$$0 < a \leq |G(\omega)| \leq b < \infty \text{ a.e } \omega \in \mathcal{F}. \tag{21}$$

Condition 2: The filters $s_\ell^*(-t)$ should be chosen in such a way that they form a stably invertible matrix $\mathbf{S}\left(e^{j\omega T}\right)$.
Examples for choices of filters that satisfy condition (2) are given in [14]. These examples include a bank of complex bandpass filters and sampling channels with different time delays (interleaved sampling).

IV. Recovery of the Unknown Delays

A. Sufficient conditions for perfect recovery

We now derive sufficient conditions for a unique solution to the set of infinite equations (16).

We begin by introducing some notation. Let $\mathbf{d}[\Lambda]$ be the measurement set containing all measurement vectors $\mathbf{d}[\Lambda] = \{\mathbf{d}[n], n \in \mathbb{Z}\}$ and let $\mathbf{b}[\Lambda] = \{\mathbf{b}[n], n \in \mathbb{Z}\}$ be the unknown vector set. We can then rewrite (16) as

$$\mathbf{d}[\Lambda] = \mathbf{N}(\tau)\mathbf{b}[\Lambda]. \tag{22}$$

The following proposition provides sufficient conditions for a unique solution to (22).

Proposition 1: If $\left(\bar{\tau}, \bar{\mathbf{b}}[\Lambda]\right)$ is a solution to (22),

$$p > 2K - \dim\left(\text{span}\left(\bar{\mathbf{b}}[\Lambda]\right)\right) \tag{23}$$

and

$$\dim\left(\text{span}\left(\bar{\mathbf{b}}[\Lambda]\right)\right) \geq 1, \tag{24}$$

then $\left(\bar{\tau}, \bar{\mathbf{b}}[\Lambda]\right)$ is the unique solution of (22).
The notation span $\left(\bar{\mathbf{b}}[\Lambda]\right)$ is used for the subspace of minimal dimension containing $\bar{\mathbf{b}}[\Lambda]$. Condition (24) is needed to avoid the case where $\bar{\mathbf{b}}[\Lambda] = 0$.

Proposition 1 suggests that a unique solution to (16) is guaranteed, under proper selection of the number of sampling channels p. This parameter, in turn, determines the average sampling rate, which is given by p/T. Condition (23) depends on the value of $\dim(\text{span}(\mathbf{b}[\Lambda]))$, which is generally not known in advance. According to our assumption $\dim(\text{span}(\bar{\mathbf{b}}[\Lambda])) \geq 1$, therefore in order to satisfy the uniqueness condition (23) for every signal of the form (3), we must have $p > 2K - 1$ or a minimal sampling rate of $2K/T$.

It is shown in [14] that our signal model fits the framework proposed in [15], under the assumption that the unknown delays are taken from a discrete grid. The theoretical minimum sampling rate required for perfect recovery of the signal from its samples in this case is $2K/T$. Clearly this rate is necessary when the unknown delays can take any value in the continuous interval $[0, T)$. Therefore, according to of Proposition 1, our sampling scheme can achieve the minimal sampling rate required for signals of the form (3).

B. Recovering the unknown delays

We now describe an algorithm that recovers the unknown delays τ from the measurement set $\mathbf{d}[\Lambda]$, based on the ESPRIT [10] method. According to Proposition 1, in order be able to perfectly reconstruct every signal of the form (3), we must have $p \geq 2K$ sampling channels. We assume throughout that this condition holds.

One condition needed in order to apply ESPRIT is that

$$\mathbf{R}_{bb} = \sum_{n \in \mathbb{Z}} \mathbf{b}[n]\mathbf{b}^H[n], \tag{25}$$

is positive definite, which is equivalent to $\dim(\text{span}(\mathbf{b}[\Lambda])) = K$. The superscript $(\cdot)^H$ denotes conjugate transposition. In this case, which we refer to as the *uncorrelated case*, we can apply the ESPRIT method on the measurement set $\mathbf{d}[n]$, in order to recover τ. If $\dim(\text{span}(\mathbf{b}[\Lambda])) < K$, so that $\mathbf{R}_{bb} \succ 0$ does not hold, then an additional smoothing stage that was proposed in [16] will be used. This scenario is referred to as the *correlated case.* The decision whether we are in the uncorrelated or correlated case can be made directly using the measurements by forming the matrix

$$\mathbf{R}_{dd} \quad = \quad \sum_{n \in \mathbb{Z}} \mathbf{d}[n]\mathbf{d}^H[n] = \mathbf{N}(\tau)\mathbf{R}_{bb}\mathbf{N}^H(\tau).$$

Since $\mathbf{N}(\tau)$ has full column-rank, the ranks of $\mathbf{R}_{dd}$ and $\mathbf{R}_{bb}$ are equal. Therefore $\mathbf{R}_{bb} \succ 0$ only if the rank of $\mathbf{R}_{dd}$ is K.

Uncorrelated Case: In the uncorrelated case the ESPRIT method can by applied directly on $\mathbf{d}[\Lambda]$. It consists of the following stages:

1) Construct the correlation matrix $\mathbf{R}_{dd} = \sum_{n \in \mathbb{Z}} \boldsymbol{d}[n]\boldsymbol{d}^H[n]$.
2) Perform an SVD decomposition of $\mathbf{R}_{dd}$ and form the matrix $\mathbf{E}_s$ consisting of the K singular vectors associated with the non-zero singular values in its columns.
3) Compute the matrix $\mathbf{\Phi} = \mathbf{E}_{\mathbf{s}\downarrow}^\dagger \mathbf{E}_{\mathbf{s}\uparrow}$. The notations $\mathbf{E}_{s\downarrow}^\dagger$ and $\mathbf{E}_{\mathbf{s}\uparrow}$ denote the sub matrices extracted from $\mathbf{E}_s^\dagger$ and $\mathbf{E}_\mathbf{s}$ by deleting their last/first row respectively.
4) Compute the eigenvalues of $\mathbf{\Phi}$, $\lambda_i, i = 1, 2, \ldots, K$.
5) Retrieve the unknown delays by $t_i = -\frac{T}{2\pi}\arg(\lambda_i)$.

Correlated Case: For this case an additional stage based on the spatial smoothing technique proposed in [16] is used.

We define $M = p - K$ length-$(K+1)$ sub vectors

$$\boldsymbol{d}_i[n] = \begin{bmatrix} d_i[n] & d_{i+1}[n] & \ldots & d_{i+K}[n] \end{bmatrix}^T, \tag{26}$$

and define the smoothed correlation matrix $\overline{\mathbf{R}}_{dd}$ as

$$\overline{\mathbf{R}}_{dd} = \frac{1}{M}\sum_{i=1}^{M}\sum_{n \in \mathbb{Z}} \boldsymbol{d}_i[n]\boldsymbol{d}_i^H[n]. \tag{27}$$

It was shown in [14] that under the assumption $p \geq 2K$, the rank of $\overline{\mathbf{R}}_{dd}$ is K. Therefore, the unknown time delays can be recovered by applying the ESPRIT method on $\overline{\mathbf{R}}_{dd}$.

V. Numerical Experiments

In the setup of our simulations we choose

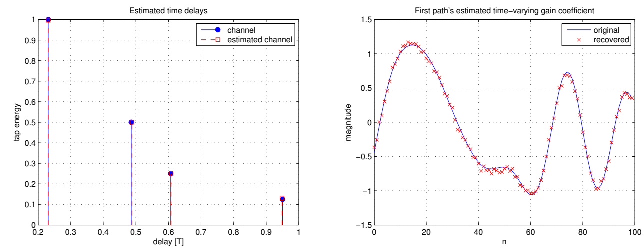
$$G(\omega) = 1, \quad \omega \in \mathcal{F}. \tag{28}$$

The sampling scheme is comprised of a bank of ideal complex bandpass filters:

$$S_\ell(\omega) = \begin{cases} T, & \omega \in \left[(\ell-1)\frac{2\pi}{T}, \ell\frac{2\pi}{T}\right] \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

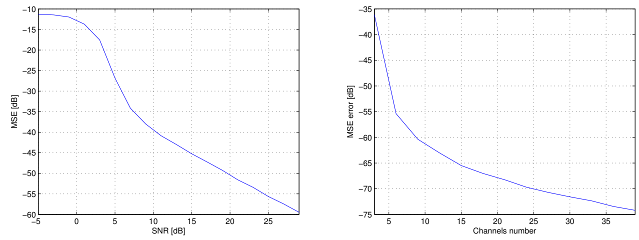
These selections satisfy Conditions 1 and 2.

In the first simulation we consider a time-varying medium, with $K = 4$ paths. The medium's time-varying gain coefficients $a_k[n]$



(a) (b)

Figure 2. Channel estimation with $p = 5$ sampling channels, and SNR=20dB. (a) Delays recovery. (b) Estimation of the time-varying gain coefficient of the first path.



(a)

Figure 3. (a) MSE of the delays estimation versus SNR, for $K = 2$ and $p = 4$. (b) MSE of the delays estimation versus the number of sampling channels p, for $K = 2$ and SNR=10dB.

are modeled according to Jakes' model [17] as a zero-mean complex-valued Gaussian stationary process with U-shape power spectral density, and energy of $(1/2)^{-k+1}$. To simulate a slow varying medium, the maximal Doppler shift, which is a parameter that controls the variation rate of the gain coefficients, was taken as $f_d = 0.05/T$. The samples at the output of each sampling channel were corrupted by complex-valued Gaussian white noise with an SNR of 20dB. The estimation of the delays is based on transmitting 100 consecutive pulses. To recover the medium's parameters $p = 5$ sampling channels were used. Although we have seen that $2K$ sampling channels are required for perfect recovery of every signal of the form (3), for some signals lowering the number of channels is possible. Indeed, according to Proposition 1, if $\dim(\text{span}(\mathbf{b}[\Lambda])) = K$, then the minimal number of channels is $K + 1$.

In Fig. 2(a) the original and estimated time delays and averaged energy of the gain coefficients are shown. In Fig. 2(b), we plot the magnitude of the original and estimated gains of the first path versus time. From Figs. 2(a) and (b) it is evident that our method can provide a good estimate of the channel's parameters, even when the samples are noisy, when sampling at the lowest possible rate.

In the next simulations we further examine the effect of noise. We choose $K = 2$ close multipath with delays, $t_1 = 0.4352T$ and $t_2 = 0.521T$. The sequences $a_k[n], k = 1, 2, n = 1, 2, \ldots 50$ are chosen as finite length sequences with unit power according to Jakes' model with $f_d = 0.05/T$. The results are based on averaging 1000 experiments. In Fig. 3(a), the mean-squared error (MSE) of the estimated time delays is shown versus the SNR, when using $p = 4$ sampling channels. In Fig. 3(b), the MSE of the estimated time delays is shown versus the number of sampling channels, for a constant SNR of 10dB. This simulation demonstrates that increasing the number of sampling channels can improve the performance of our method in the presence of noise.

VI. Conclusion

We considered the problem of estimating the time delays and time varying coefficients of a multipath medium, from low-rate samples of the received signal. We showed that if the medium has K multipath components a sampling rate of $2K/T$ is sufficient to guarantee perfect recovery of its parameters. This rate is independent of the probing pulse bandwidth, which can be very high. We developed necessary and sufficient conditions on the transmitted pulse and the sampling filters in order to guarantee perfect recovery at the minimal possible rate. To recover the unknown time delays we proposed an ESPRIT-type algorithm. Once the delays are properly identified, the time varying coefficients can be recovered using digital filtering.

Besides the application to medium identification, the problem we treated here can be seen as a first example of a systematic sampling theory for analog signals defined over an infinite union of subspaces. Recently, there has been growing interest in sampling theorems for signals over a union of subspaces [11], [12], [13], [15]. However, previous work addressing stability issues and concrete recovery algorithms have focused on finite unions. Here, we take a first step in the direction of extending these ideas to a broader setting that treats analog signals lying in an infinite union.

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