Sparse Source Separation from Orthogonal Mixtures

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Problem Formulation

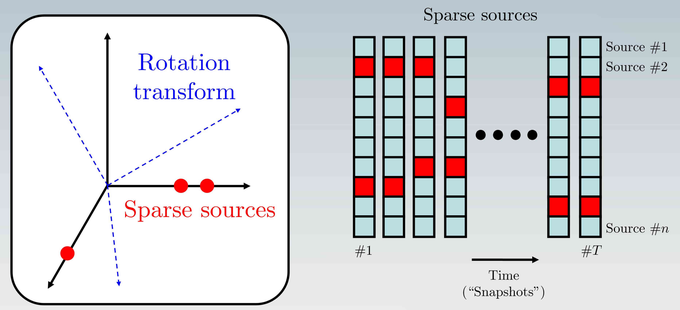
$$\mathbf{X} = \mathbf{\Psi}\mathbf{S}$$

Given $\mathbf{X}$ find an orthogonal $\mathbf{\Psi}$ and sparse $\mathbf{S}$

Assumptions:

- No noise*
- K active sources (exactly*) at each snapshot
- The nonzero values are drawn from some continuous distribution (or positive).

* See extensions for the general case



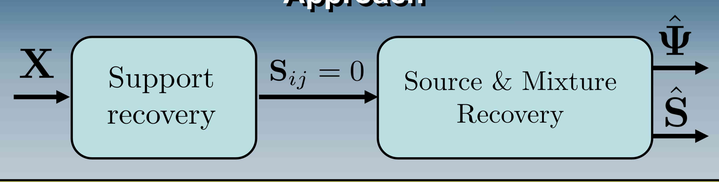
Motivation

$$(\hat{\mathbf{\Psi}}, \hat{\mathbf{S}}) = \arg\min \quad \|\mathbf{X} - \mathbf{\Psi}\mathbf{S}\|_F \quad \text{s.t.} \quad \mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}, \quad \|\mathbf{S}_i\|_0 = K, \quad 1 \le i \le T$$

Known methods:

- In [1]: $\min_{\mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}} \sum_{ik} |[\mathbf{\Psi}^T\mathbf{X}]_{ik}|$
  - Nonconvex
  - Many local minima in practice
- In [2-4]:
  - Combinatorial
  - Many snapshots
  - Nonorthogonal $\mathbf{\Psi}$

Approach



Idea: Exploit orth. and sparsity to simplify the problem !

Advantages:

- Few snapshots
- Non-combinatorial
- Significant improvement in learning (empirically).
- Orthogonal output is guaranteed.

Stage 1 – Support Recovery

Orthogonal mixture preserves inner products
$\mathbf{C} = \mathbf{X}^T\mathbf{X} = \mathbf{S}^T\mathbf{S}$

(**R1**) $\mathbf{C}_{kj} = 0$ implies that $\text{supp}(\mathbf{S}_k), \text{supp}(\mathbf{S}_j)$ are disjoint

(**R2**) $\mathbf{C}_{kj} \neq 0$ implies $\text{supp}(\mathbf{S}_k) \cap \text{supp}(\mathbf{S}_j)$ is not empty

(**R3**) Every column $\mathbf{S}_k$ contains exactly K non-zeros

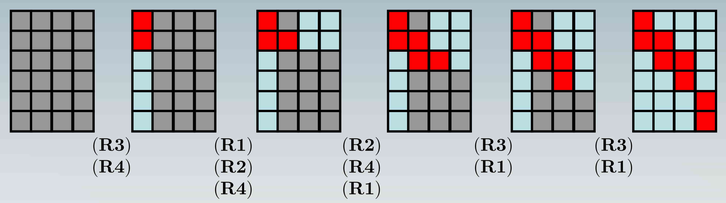
(**R4**) Row permutations of $\mathbf{S}$ are allowed

Construct the support by iteratively applying (**R1**) – (**R4**)

Toy-Example

$$\mathbf{C} = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot \end{pmatrix}$$

where "·" is a nonzero value



"*Rule*": In $\mathbf{S}_2$, at least one entry from positions $\mathcal{I} = \{1, 2\}$ is nonzero

Algorithm

Sketch (full pseudo-code in paper)

- `Maintain a dynamic list of rules`
- `Sweep through` $\mathbf{S}$ `columns and update estimated support and rules list`
- `Resolve singleton rules`
  `Resolve rules when symmetry allows`
- `Repeat till no further changes`

Rules list

In $\mathbf{S}_{20}$, $\mathcal{I} = \{19, 24, 14\}$

In $\mathbf{S}_{41}$, $\mathcal{I} = \{\cancel{5}, 10\}$

In $\mathbf{S}_{47}$, $\mathcal{I} = \{20, 24, 12\}$

In $\mathbf{S}_{80}$, $\mathcal{I} = \{\cancel{5}, 33, 42\}$

Iterations is bounded by nKT
(Typically much smaller)

Partial support recovery is usually "good enough"

Stage 2- Source and Mixture Recovery

$$(\hat{\mathbf{\Psi}}, \hat{\mathbf{S}}) = \arg\min \quad \|\mathbf{X} - \mathbf{\Psi}\mathbf{S}\|_F \quad \text{s.t.} \quad \mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}, \quad \mathbf{S}_{ik} = 0, \quad (i,k) \in \Omega_0$$

$$\Omega_0 = \{(i,k) \,|\, \mathbf{S}_{ik} = 0\}$$

Optimal $\mathbf{S}$ given $\mathbf{\Psi}$

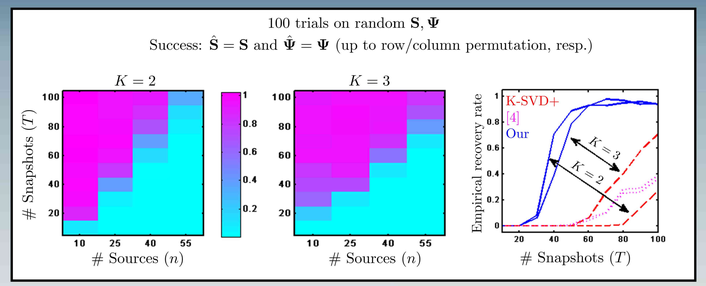
$$\mathbf{S} = \begin{cases} \mathbf{\Psi}^T\mathbf{X} & \text{on } \Omega_0^c \\ 0 & \text{on } \Omega_0 \end{cases}$$

Optimal $\mathbf{\Psi}$ given $\mathbf{S}$ [4]

$$\mathbf{X}\mathbf{S}^T = \mathbf{U}\Sigma\mathbf{V}^T \text{ (SVD)}, \quad \mathbf{\Psi} = \mathbf{U}\mathbf{V}^T$$

Alternate minimzation based on $\nabla\mathcal{L}(\mathbf{\Psi}, \mathbf{S}) = \mathbf{0}$

Numerical Experiments



K-SVD [3]: Learn an $n \times n$ dictionary $\mathbf{\Psi}$
+ orthogonalization step every few iterations

[4] : Sparse coding with dynamic threshold

Extensions

- Support recovery in the presence of noise
- Preprocessing to determine the sparsity levels $\|\mathbf{S}_i\|_0$
- Additional update rules based on support blockness

References (short list)

[1] R. Gribonval and K. Schnass, *EUSIPCO'08.*

[2] P. Georgiev, F. Theis, and A. Cichocki, *ICASSP'04.*

[3] M. Aharon, M. Elad, and A. M. Bruckstein, *Linear Algebra and Its Applications* 2006.

[4] S. Lesage, R. Gribonval, F.Bimot, and L. Benaroya, Tech. Rep. 2004.

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