Multichannel Sampling of Pulse Streams at the Rate of Innovation

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***Abstract*—We consider minimal-rate sampling schemes for streams of delayed and weighted versions of a known pulse shape. Such signals belong to the class of finite rate of innovation (FRI) models. The minimal sampling rate for these parametric signals, is the number of degrees of freedom per unit of time, referred to as the rate of innovation. Although sampling of pulse streams was treated in previous works, either the rate of innovation was not achieved, or the pulse shape was limited to diracs and the method was instable for high rates of innovation. In this work we propose a multichannel framework for pulse streams with arbitrary shape, operating at the rate of innovation. Our approach is based on modulating the input signal with a set of properly chosen waveforms, followed by a bank of integrators. We show that the pulse stream can be recovered from the proposed minimal-rate samples using standard tools taken from spectral estimation in a stable way even at high rates of innovation. In addition, we address practical implementation issues, such as reduction of hardware complexity and immunity to failure in the sampling channels. The resulting scheme is flexible and exhibits better noise robustness than previous approaches.**

I. Introduction

Digital processing has become ubiquitous, and is the most common way to process analog signals. Processing of analog signals by digital processing must be preceded by a sampling stage, carefully designed to retain the important features of the analog signal relevant for the signal processing task at hand. The well known Shannon-Nyquist theorem states that in order to perfectly reconstruct an analog signal from its samples, it must be sampled at the Nyquist rate, i.e., twice its highest frequency. This assumption is required when the only prior on the signal is that it is bandlimited. Other priors on signal structure [1], [2], which include subspace [3], sparsity [4], [5], or smoothness priors [1], can lead to more efficient sampling schemes.

An interesting class of priors was suggested by Vetterli et al. [6], [7], who considered signals with a finite number of degrees of freedom per unit time, termed by the authors as signals with *finite rate of innovation* (FRI). For such models, the goal is to design a sampling scheme operating at the innovation rate, which is the minimal possible rate from which perfect recovery is possible. A special case that was treated in detail are signals consisting of streams of short pulses. Pulse streams are prevalent in applications such as bio-imaging [8], neuronal activity and ultra-wideband communications. Since the pulses are highly compact in time, standard sampling methods require very high sampling rates. The main idea is to exploit the fact that the pulse shape is known, in order to characterize such signals by the time-delays and amplitudes of the various pulses. Targeting these parameters allows to reduce the sampling rate way beyond that dictated by the Shannon-Nyquist theorem.

Following this parametric point of view, low rate single-channel sampling schemes were developed for periodic streams of pulses [6], [7]. These methods rely on the observation that the time delays and amplitudes can be recovered in the frequency domain, once a set of Fourier series coefficients of the signal are known. This follows from the fact that in the frequency domain recovery translates into estimating the frequencies and amplitudes of a sum of complex sinusoids (cisoids), a problem which has been treated extensively in the context of spectral estimation [9].

In practical applications finite and infinite streams are usually encountered, rather than periodic streams. For the finite case Gaussian [6] and polynomial reproducing sampling kernels [10] were introduced. Both methods exhibit instability for large numbers of pulses per unit time [8], [10], [11]. An alternative sampling scheme was presented in [8], which allowed for a general sampling filter. Specific choices of this filter recover previous approaches such as [6]. However, the general formulation developed allows the design of more practical time-limited filters, that exhibit superior noise robustness over former techniques even at very high rates of innovation. Exploiting the compact support of the sampling kernels in [8], [10], both methods were extended to the infinite case. Unfortunately, neither approaches achieves the minimal sampling rate, which is the rate of innovation.

All previous methods were composed of a single sampling channel. Multichannel sampling schemes offer additional degrees of freedom which enable achieving the minimal sampling rate for the infinite setting. A simple multichannel system was proposed in [12] for the case of a single pulse per sampling period. However, our interest here is in higher rates of innovation. An alternative multichannel scheme comprised of a chain of integrators was proposed in [13]. As we show in simulations, this method is very instable in the presence of noise and for high rates of innovation. Moreover, both methods [12], [13] consider Dirac impulses only. To the best of our knowledge, a stable minimal rate sampling scheme for infinite pulse streams is still lacking.

Our first contribution treats finite pulse streams. We design a multichannel sampling system, based on oscillators, mixers and integrators. In each channel the signal is modulated by an appropriate waveform, followed by integration over a

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compact time interval. We derive conditions which guarantee that the output of each channel is a mixture of the Fourier coefficients of the signal. By properly choosing the mixing parameters, we show that the Fourier coefficients can be obtained from the samples. Once the set of Fourier coefficients is known, we use standard spectral estimation tools in order to recover the unknown times and amplitudes. As we show, the mixing scheme enables simple and practical generation of modulating waveforms. Furthermore, mixing the coefficients allows recovering the signal even when one or more sampling channels fails.

The integration over a finite interval enables a simple extension to the infinite setting. Our infinite sampling approach leads to perfect reconstruction of the signal, while sampling at the rate of innovation. As we show in simulations, our approach exhibits superior noise robustness compared to [13], and allows sampling at very high rates of innovation. In addition, we can accommodate general pulse shapes. We also discuss a special case of infinite streams having a shift-invariant structure, presented in [14], and point out some advantages of our method in certain scenarios. Finally, we describe how to practically generate the modulating waveforms, based on concepts presented in [15], [16] and derive conditions on these waveforms which guarantee perfect reconstruction of the signal.

Simulations show the advantage of our approach comparing to the integrator scheme [13], in terms of estimation error in the presence of noise. In addition, we demonstrate our practical sampling scheme, and show that shaping the waveforms with analog Chebyshev filters [17], closely approaches ideal performance. These results lead to practical sampling schemes which achieve the rate of innovation in a robust way, even in the presence of noise and high innovation rates.

The remainder of this paper is organized as follows. In Section II we derive a multichannel scheme for finite pulse streams. Section III extends our results to the infinite case. We discuss the generation of the modulating waveforms in Section IV, and present a practical sampling scheme which can be implemented in hardware. In Section V we discuss the relations of our results to previous work. Finally, in Section VI we present simulations demonstrating the performance of our method, and near-ideal performance of our scheme using practical analog filters.

II. Finite Streams of Pulses

A. Notations and Definitions

Matrices and vectors are denoted by bold font, with lowercase letters corresponding to vectors and uppercase letters to matrices. The nth element of a vector $\mathbf{a}$ is written as $\mathbf{a}_n$, and $\mathbf{A}_{ij}$ denotes the ijth element of a matrix $\mathbf{A}$. Superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ represent complex conjugation, transposition and conjugate transposition, respectively. The Moore-Penrose pseudo-inverse of a matrix $\mathbf{A}$ is written as $\mathbf{A}^\dagger$. We denote by $\text{diag}(\mathbf{a})$ a diagonal matrix having the elements of the vector $\mathbf{a}$ on its diagonal. The continuous-time Fourier transform (CTFT) of a continuous-time signal $x(t) \in L_2$ is defined by $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{d}t$.

B. Problem Formulation

Consider the finite stream of pulses

$$x(t) = \sum_{l=1}^{L} a_l h(t - t_l), \quad t_l \in I \subset [0, T), \, a_l \in \mathbb{C}, \tag{1}$$

where $h(t)$ is a known pulse shape, $\{t_l, a_l\}_{l=1}^{L}$ are the unknown delays and amplitudes, and I is a continuous time interval contained in $[0, T)$. The pulse can be arbitrary as long as

$$h(t - t_l) = 0, \, \forall t \notin [0, T) \quad l = 1 \dots L, \tag{2}$$

i.e., the signal $x(t)$ is confined to the time-window $[0, T)$. Since there are only $2L$ degrees of freedom in this model, at least $2L$ samples are required in order to represent the signal. Our goal is to design a sampling and reconstruction method which perfectly reconstructs the signal $x(t)$ from a minimal number of samples.

We now show that our sampling problem can be related to the well known problem of model-based complex sinusoids (cisoids) parameter estimation. This approach was originally taken by Hou and Wu [18], who were the first to show that time delay estimation can be converted into a problem of frequency estimation of a sum of cisoids [9]. This follows form noticing that the delays in the time domain are converted into modulations in the frequency domain. However, their method relied on Nyquist rate sampling of the signal, and their derivations were only approximate. Vetterli et al. [6], were the first to address this topic from an efficient sampling point of view, and derived a low-rate sampling and reconstruction scheme for periodic streams of Diracs. Their method was based on the same fundamental relation between the delays in the time domain, and the modulations in the frequency domain, but this time in a precise way, using the Fourier series coefficients of the signal. We follow a similar path, and derive low-rate sampling schemes from a parametric point of view. We first show that once a set of Fourier coefficients of the signal are known, the problem is indeed a sinusoidal parameter estimation problem. Following this derivation, we design low-rate sampling schemes for obtaining the set of Fourier coefficients.

Since $x(t)$ is confined to the interval $t \in [0, T)$, it can be expressed by its Fourier series

$$x(t) = \sum_{k \in \mathbb{Z}} X[k] e^{j \frac{2\pi}{T} k t}, \quad t \in [0, T) \tag{3}$$

where

$$X[k] = \frac{1}{T} \int_0^T x(t) e^{-j \frac{2\pi}{T} k t} \mathrm{d}t. \tag{4}$$

Substituting (1) into (4) we obtain

$$\begin{aligned} X[k] &= \frac{1}{T}\sum_{l=1}^{L} a_l \int_0^T h(t-t_l) e^{-j\frac{2\pi}{T}kt} \mathrm{d}t \\ &= \frac{1}{T}\sum_{l=1}^{L} a_l \int_{-\infty}^{\infty} h(t-t_l) e^{-j\frac{2\pi}{T}kt} \mathrm{d}t \\ &= \frac{1}{T}\sum_{l=1}^{L} a_l e^{j\frac{2\pi}{T}kt_l} \int_{-\infty}^{\infty} h(t) e^{-j\frac{2\pi}{T}kt} \mathrm{d}t \\ &= \frac{1}{T} H\left(\frac{2\pi}{T}k\right) \sum_{l=1}^{L} a_l e^{j\frac{2\pi}{T}kt_l}, \end{aligned} \tag{5}$$

where the second equality stems from the condition in (2), and $H(\omega)$ denotes the CTFT of $h(t)$.

Denote by $\mathcal{K}$ a set of K consecutive indices for which $H\left(\frac{2\pi}{T}k\right) \neq 0, \forall k \in \mathcal{K}$. We require that such a set exists, which is usually the case for short time-support pulses $h(t)$. Denote by $\mathbf{H}$ the $K \times K$ diagonal matrix with kth entry $\frac{1}{T}H\left(\frac{2\pi}{T}k\right)$, and by $\mathbf{V}(\mathbf{t})$ the $K \times L$ matrix with klth element $e^{-j\frac{2\pi}{T}kt_l}$, where $\mathbf{t} = \{t_1, \ldots, t_L\}$ is the vector of the unknown delays. In addition denote by $\mathbf{a}$ the length-L vector whose lth element is a_l, and by $\mathbf{x}$ the length-K vector whose kth element is $X[k]$. We may now write (5) in matrix form as

$$\mathbf{x} = \mathbf{H}\mathbf{V}(\mathbf{t})\mathbf{a}. \tag{6}$$

The matrix $\mathbf{H}$ is invertible by construction, and therefore we can define $\mathbf{y} = \mathbf{H}^{-1}\mathbf{x}$, which satisfies

$$\mathbf{y} = \mathbf{V}(\mathbf{t})\mathbf{a}. \tag{7}$$

Addressing the kth element of the vector $\mathbf{y}$ in (7) directly, we obtain

$$\mathbf{y}_k = \sum_{l=1}^{L} a_l e^{-j\frac{2\pi}{T}kt_l}. \tag{8}$$

It is now evident that given the vector $\mathbf{x}$, (7) conforms with the standard problem of finding the frequencies and amplitudes of a sum of L cisoids. The time-delays can be estimated using nonlinear techniques, e.g., the annihilating filter [6], MUSIC [19], [20] or ESPRIT [21] (see [9] for a review of this topic), as long as $K \geq 2L$ and the time-delays are distinct, i.e., $t_i \neq t_j$ for all $i \neq j$. Once the time-delays are known, the linear set of equations (7) may be solved via a least squares approach for the unknown amplitudes. Due to the Vandermonde form of $\mathbf{V}(\mathbf{t})$, it is left invertible as long as $K \geq L$, therefore the amplitudes may be obtained via $\mathbf{a} = \underline{(}V)^{\dagger}(\mathbf{t})\mathbf{y}$.

C. Direct Multichannel Sampling

As we have seen, given the vector of $K \geq 2L$ Fourier series coefficients $\mathbf{x}$, we may use standard tools from spectral analysis to determine the set $\{t_l, a_l\}_{l=1}^{L}$. In practice, the signal is sampled in the time-domain, and therefore we do not have direct access to samples of $\mathbf{x}$. Our goal now is to design a sampling scheme which will allow us to obtain the vector $\mathbf{x}$ from time-domain samples.

For simplicity, we set K to be an odd number, and choose the set $\mathcal{K} = \{-\lfloor K/2 \rfloor, \ldots, \lfloor K/2 \rfloor\}$. However, our results may be extended to any set $\mathcal{K}$ of consecutive indices, as long as $|\mathcal{K}| \geq 2L$. In order to obtain the Fourier coefficients, $X[k]$, we propose the multichannel sampling scheme depicted in Fig.1. Each channel consists of modulating the signal with a complex exponential, followed by an integrator operating over the window $[0, T)$. The sample taken by the kth channel is exactly $X[k]$, as in (4).

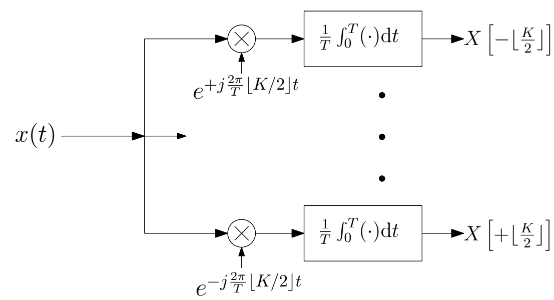

Fig. 1. Multichannel direct sampling of the Fourier series coefficients $X[k]$, $k \in \mathcal{K}$.

The direct sampling scheme is straightforward, and may be implemented using 3 basic building blocks: oscillators, mixers and integrators. However, from a practical point of view this approach has the disadvantage that it requires many oscillators, having frequencies which must be exact multiples of some common base frequency.

D. Mixing the Fourier Coefficients

We now generalize our framework, towards a more practical sampling scheme by mixing several Fourier coefficients together, rather than limiting ourselves to one coefficient per channel. The additional degrees of freedom offered by this extension will allow us to design waveforms that are easy to implement. In addition, in real-life scenarios one or more channels might fail, due to malfunction or noise corruption, and therefore we lose the information stored in that channel. When using the direct scheme we lose one Fourier coefficient, preventing us from recovering the unknown signal parameters. In contrast, when mixing the coefficients we distribute the information about each Fourier coefficient between several sampling channels. Consequently, when one or more channels fail, the required Fourier coefficients may still be recovered from the remaining operating channels. We discuss this feature more thoroughly in Section IV.

Consider a multichannel sampling scheme with p channels. In each channel, we modulate the signal using a weighted sum of cisoids given by

$$s_i(t) = \sum_{k \in \mathcal{K}} s_{ik} e^{-j\frac{2\pi}{T}kt}, \tag{9}$$

where the weights s_{ik} vary from channel to channel. The resulting sample of the ith channel is

$$\begin{aligned} c_i &= \int_0^T x(t) \sum_{k \in \mathcal{K}} s_{ik} e^{-j\frac{2\pi}{T}kt} \mathrm{d}t \\ &= \sum_{k \in \mathcal{K}} s_{ik} X[k], \end{aligned} \tag{10}$$

where we used our prior derivation (5). The resulting scheme is depicted in Fig. 2.

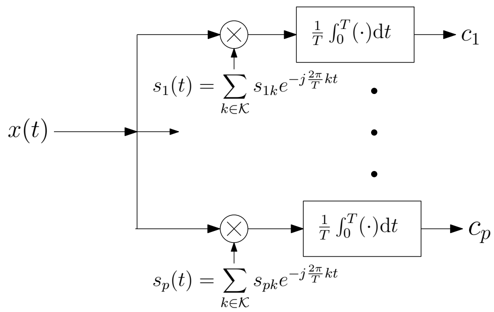

Fig. 2. Mixing the Fourier coefficients differently in each channel.

To relate the samples and the Fourier coefficients, we define the $p\times K$ matrix $\mathbf{S}$ with s_{ik} as its *ik*th element, and by $\mathbf{c}$ the length-p sample vector with *i*th element c_i. We may now write (10) in matrix form as

$$\mathbf{c}=\mathbf{S}\mathbf{x}. \tag{11}$$

As long as $\mathbf{S}$ has full column rank, where $p\geq K$ is a necessary condition, we can recover $\mathbf{x}$ from the samples by $\mathbf{x}=\mathbf{S}^{\dagger}\mathbf{c}$. The direct sampling scheme presented earlier is a special case of this more general approach, with $p=K$ and $\mathbf{S}=\mathbf{I}$. In Section IV we exploit the degrees of freedom this general scheme offers, and present sampling schemes which can simplify the hardware design, and are more robust to malfunctions in the sampling channels.

We summarize this result in the following theorem.

Theorem 1. *Consider a finite stream of pulses given by*

$$x(t)=\sum_{l=1}^{L}a_l h(t-t_l),\quad t_l\in I\subset[0,T),\, a_l\in\mathbb{C},\, l=1\dots L,$$

where $h(t)$ is a known pulse shape, and condition (2) *is satisfied. Choose a set $\mathcal{K}$ of consecutive indices for which $H(2\pi k/T)\neq 0,\,\forall k\in\mathcal{K}$. Consider the multichannel sampling scheme depicted in Fig. 2, for some choice of coefficients $\{s_{ik}\}_{k\in\mathcal{K}},\, i=1,\dots,p$. Then, the signal $x(t)$ can be perfectly reconstructed from the samples $\{c_i\}_{i=1}^{p}$ with*

$$c_i=\int_0^T x(t)\sum_{k\in\mathcal{K}} s_{ik}e^{-j\frac{2\pi}{T}kt}\mathrm{d}t, \tag{12}$$

as long as $p\geq|\mathcal{K}|\geq 2L$, and the coefficients matrix $\mathbf{S}$ in (11) *is left invertible.*

As we discuss in Section V-A, the method in [8] can be viewed as a special case of Fig. 2. Since our work is a generalization of [8], it benefits from the high noise robustness exhibited by [8], in contrast to previous work [6], [10]. It should be noted that Theorem 1 holds for a periodic pulse stream as well, since it can be similarly represented by a Fourier series, and all derivations remain intact.

We now demonstrate several useful modulating waveforms.

1) Cosine and Sine waveforms: First we set $p=K$. Then, we choose the first $\lfloor K/2\rfloor$ waveforms to be $\cos\left(\frac{2\pi}{T}kt\right)$, the next $\lfloor K/2\rfloor$ to be $\sin\left(\frac{2\pi}{T}kt\right)$, and the last to be the constant function 1. Clearly, these waveforms fit the form in (9), since cosine (sine) waves can be expressed as the sum (subtraction) of two complex exponentials. It is easily verified that this choice yields an invertible matrix $\mathbf{S}$. The practical advantage of the mixing scheme is already evident, since sine and cosine waves are real valued, whereas the direct multichannel scheme requires complex exponentials.

2) Periodic Waveforms: Every periodic waveform can be expanded into a Fourier series. Transferring such a waveform through some shaping filter, e.g., a low-pass filter, we can reject most of the coefficients, leaving only a finite set intact. Consequently, such a scheme meets the form of (9). In Section IV we elaborate on this concept, discuss design considerations, and show that properly chosen periodic waveforms yield a left invertible matrix $\mathbf{S}$.

One simple choice is periodic streams of rectangular pulses modulated by ± 1 [15]. The strength of the mixing scheme over the direct one will be emphasized in Section IV-B. We show that one periodic stream is sufficient for all channels, while each channel uses a delayed version of this common waveform. Therefore, the requirement for multiple oscillators and the need for accurate multiples of the basic frequency, are both removed. In addition, periodic streams are easily designed and implemented digitally, rather than somewhat complicated analog design of oscillators combined with analog circuits intended to create exact frequency multiples. Finally, if the period T changes, the analog circuit has to be modified substantially, whereas the flexibility of the digital design allows simple modifications.

III. Infinite Streams of Pulses

A. General Model

We now consider an infinite stream of pulses defined by

$$x(t)=\sum_{l\in\mathbb{Z}}a_l h(t-t_l),\quad t_l\in\mathbb{R},\, a_l\in\mathbb{C}. \tag{13}$$

We assume that there are no more than L pulses in any interval $I_m\triangleq[(m-1)T,mT],\ m\in\mathbb{Z}$. We further assume that within each interval condition (2) holds, and consequently, the intervals are independent of one another. The maximal number of degrees of freedom per unit time, also known as the rate of innovation [6], [7], is $2L/T$. We now present a multichannel sampling and reconstruction scheme which operates at the minimal rate possible, i.e., the rate of innovation.

Consider an extension of the sampling scheme presented in Section II-D, where we sample every T seconds. Upon each sample we reset the integrator, so that the *m*th sample results from the integral over the interval I_m. The resulting sampling scheme is depicted in Fig. 3. Since the *m*th sample is influenced by the interval I_m only, the infinite problem may be reduced into a sequence of finite streams of pulses. The resulting samples are given by

$$\mathbf{c}[m]=\mathbf{S}\mathbf{x}[m], \tag{14}$$

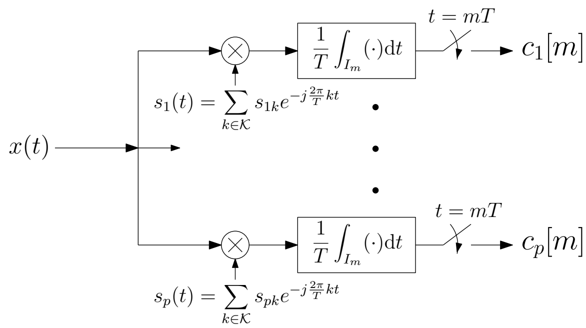

Fig. 3. Extended sampling scheme using modulating waveforms for an infinite pulse stream.

where the vector $\mathbf{x}[m]$ contains the Fourier series coefficients of the signal within the *m*th interval, I_m. As long as $\mathbf{S}$ is chosen so that it is left invertible, we can obtain the sequence of Fourier series coefficients by $\mathbf{x}[m] = \mathbf{S}^{\dagger}\mathbf{c}[m]$. Extending (7) to the infinite case we obtain:

$$\mathbf{y}[m] = \mathbf{H}^{-1}\mathbf{x}[m] = \mathbf{V}(\mathbf{t}[m])\mathbf{a}[m], \tag{15}$$

where $\mathbf{t}[m]$ and $\mathbf{a}[m]$ are the times and amplitudes of the pulses in the interval I_m, respectively, and the matrix $\mathbf{V}$ remains as in (6). For each m, equation (15) is a sum of cisoids problem, and thus may be solved as long as $K \geq 2L$. By choosing $p = K = 2L$ we present a sampling scheme which operates at the rate of innovation, and allows for perfect reconstruction of an infinite stream of pulses.

We state our result in a theorem.

Theorem 2. *Consider an infinite stream of pulses given by*

$$x(t) = \sum_{l\in\mathbb{Z}} a_l h(t - t_l), \quad t_l \in \mathbb{R}, \, a_l \in \mathbb{C}.$$

where $h(t)$ is a known pulse shape. Assume that there are no more than L pulses within any interval $I_m \triangleq [(m-1)T, mT]$, $m \in \mathbb{Z}$, and that condition (2) *holds for all intervals I_m, $m \in \mathbb{Z}$. Choose a set $\mathcal{K}$ of consecutive indices for which $H(2\pi k/\tau) \neq 0$, $\forall k \in \mathcal{K}$. Consider the multichannel sampling scheme depicted in Fig. 3, for some choice of coefficients $\{s_{ik}\}_{k\in\mathcal{K}}$, $i = 1, \ldots, p$. Then, the signal $x(t)$ can be perfectly reconstructed from the samples $\{c_i[m]\}_{i=1}^{p}$ with*

$$c_i[m] = \int_{I_m} x(t) \sum_{k\in\mathcal{K}} s_{ik} e^{-j\frac{2\pi}{T}kt}\mathrm{d}t, \quad m \in \mathbb{Z}, \tag{16}$$

as long as $p \geq |\mathcal{K}| \geq 2L$, and the coefficients matrix $\mathbf{S}$ in (11) *is left invertible.*

To the best of our knowledge Theorem 2 presents the first sampling scheme for pulse streams with arbitrary shape, operating at the rate of innovation. Furthermore, as we show in simulations, our method is more stable than previous approaches.

B. Stream of Pulses with Shift-Invariant Structure

We now focus on a special case of the infinite model (13), where the signal has an additional shift-invariant (SI) structure. This structure is expressed by the fact that in each period T, the delays are constant relative to the beginning of the period. Such signals can be described as

$$x(t) = \sum_{m\in\mathbb{Z}} \sum_{\ell=1}^{L} a_\ell[m]h(t - t_\ell - mT), \, t_\ell \in I \subset [0, T), \tag{17}$$

where $a_\ell[n] \in \ell_2$ denotes the ℓth pulse amplitude on the mth period. This model was first considered in [14]. We explore the relation to this work in Section V.

Assuming condition (2) holds here as well, (15) can be rewritten as

$$\mathbf{y}[m] = \mathbf{V}(\mathbf{t})\mathbf{a}[m], \tag{18}$$

since now the relative delays in each period are constant. Here, $\mathbf{a}[m]$ denotes the length-L vector whose ℓth element is given by $a_\ell[m]$.

The question arising here, is what is the minimal number of sampling channels which allows unique recovery of the delays $\mathbf{t}$ from the samples. Clearly, the condition for the general model $p \geq 2L$ is a sufficient condition here also, however as we show next, the additional prior we have on the signal's structure can relax this condition.

To answer this question, we rely on results obtained in [14], which treated the uniqueness conditions for a set of equations similar to (18). Following these results, a sufficient condition for unique recovery of the delays and vectors $\mathbf{a}[m]$ from (18) is given by:

$$K \geq 2L - \eta + 1, \tag{19}$$

where

$$\eta = \dim\left(\mathrm{span}\left(\{\mathbf{a}[m], m \in \mathbb{Z}\}\right)\right) \tag{20}$$

denotes the dimension of the minimal subspace containing the vector set $\{\mathbf{a}[m], m \in \mathbb{Z}\}$. This condition implies that in some cases K, and eventually the number of channels p (since $p \geq K$), can be reduced beyond the lower limit $2L$ for the general model, depending on the value of η.

In a similar way to [14], the recovery of the delays from (18) can be performed using the ESPRIT [21] or MUSIC [19] algorithms. These approaches, which are known as subspace methods, require that $\eta = L$. In this case they achieve the lower bound of condition (19), namely recover the delays using only $p \geq L + 1$ sampling channels. In cases where $\eta < L$, an additional smoothing [22] stage is required prior to the use of the subspace methods, and $p \geq 2L$ sampling channels are required.

To conclude this point, when the amplitudes of the pulses vary sufficiently from period to period, which is expressed by the condition $\eta = L$, the common information about the delays can be utilized in order to reduce the sampling rate to $(L+1)/T$. Moreover, the approach presented here can improve the delays estimation compared to the one used for the general model, since it uses the mutual information between periods, rather than recovering the delays for each period T separately.

This result is summarized in the following theorem.

Theorem 3. *Consider a stream of pulses given by*

$$x(t) = \sum_{m=0}^{N_p-1} \sum_{\ell=1}^{L} a_\ell[m] h(t - t_\ell - mT), \quad t_l \in I \subset [0, T)$$

where $h(t)$ *is a known pulse shape. Assume that condition* (2) *holds. Choose a set* $\mathcal{K}$ *of consecutive indices for which* $H(2\pi k/\tau) \neq 0$, $\forall k \in \mathcal{K}$. *Consider the multichannel sampling scheme depicted in Fig. 3, for which the samples in the* i*th channel are given by*

$$c_i[m] = \int_{I_m} x(t) \sum_{k\in\mathcal{K}} s_{ik} e^{-j\frac{2\pi}{T}kt} \mathrm{d}t, \quad m \in \mathbb{Z}, \tag{21}$$

for some choice of coefficients $\{s_{ik}\}_{k\in\mathcal{K}}$, $i = 1, \ldots, p$. *Then, the signal* $x(t)$ *can be perfectly reconstructed from the samples* $\{c_i[m]\}_{i=1}^{p}$, $m \in \mathbb{Z}$ *as long as the coefficients matrix* $\mathbf{S}$ *in* (11) *is left invertible and*

$$p \geq |\mathcal{K}| \begin{cases} \geq L+1 & \textit{when } \eta = L \\ \geq 2L & \textit{when } \eta < L, \end{cases}$$

where

$$\eta = \mathit{dim}\left(\mathit{span}\left(\{\mathbf{a}[m], m \in \mathbb{Z}\}\right)\right) \tag{22}$$

denotes the dimension of the minimal subspace containing the vector set $\{\mathbf{a}[m], m \in \mathbb{Z}\}$.

IV. Modulation Waveforms

In this section we treat thoroughly the example given in Section II-D of generating the modulation waveforms using periodic signals. We first address the case of general periodic waveforms, and then we focus on the special case of pulse sequences.

A. General Periodic Waveforms

Our aim is to show how to obtain the required modulating waveforms (9) using a set of p periodic functions, given by $p_i(t)$. Such waveforms can be expressed using their Fourier series expansion as

$$p_i(t) = \sum_{k\in\mathbb{Z}} d_i[k] e^{j\frac{2\pi}{T}kt}, \tag{23}$$

where the kth Fourier series coefficient of $p_i(t)$ is given by

$$d_i[k] = \frac{1}{T}\int_0^T p_i(t) e^{-j\frac{2\pi}{T}kt} dt. \tag{24}$$

The sum in (23) is generally infinite, in contrast to the finite sum in (9). Therefore, we propose filtering $p_i(t)$ with a filter $g(t)$ which rejects the unwanted elements in the sum (23). The filtered waveforms at the output of $g(t)$ are given by

$$\tilde{p}_i(t) = p_i(t) * g(t), \tag{25}$$

and are also periodic. Therefore they can be written as

$$\tilde{p}_i(t) = \sum_{k\in\mathbb{Z}} \tilde{d}_i[k] e^{j\frac{2\pi}{T}kt}, \tag{26}$$

where it can be easily verified that

$$\tilde{d}_i[k] = d_i[k] \cdot G\left(\frac{2\pi}{T}k\right). \tag{27}$$

Here $G(\omega)$ denotes the CTFT of $g(t)$. From (27), the shaping filter $g(t)$ has to satisfy

$$G(\omega) = \begin{cases} \text{nonzero} & \omega = \frac{2\pi}{T}k, \ k \in \mathcal{K} \\ 0 & \omega = \frac{2\pi}{T}k, \ k \notin \mathcal{K} \\ \text{arbitrary} & \text{elsewhere}, \end{cases} \tag{28}$$

so that $\tilde{d}_i[k] = 0$ for $k \notin \mathcal{K}$. This condition is similar to the one obtained in [8] for single channel sampling. Therefore, the class of filters developed there, can also be used here as a shaping filter.

Note that (28) implies that the frequency response of the filter $g(t)$ is specified only on the set of discrete points $\frac{2\pi}{T}k$, $k \in \mathbb{Z}$, offering large freedom when designing a practical analog filter. For instance, when implementing a lowpass filter (LPF) this requirement allows a smooth transition band between the passband and the stopband of the filter, with a width of $\frac{2\pi}{T}$.

The resulting scheme is depicted in Fig. 4. The elements of

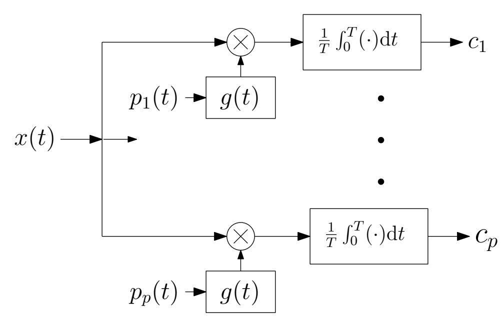

Fig. 4. Proposed sampling scheme, using modulating waveforms.

the mixing matrix $\mathbf{S}$, obtained from the proposed scheme are given by

$$\mathbf{S}_{ik} = \tilde{d}_i[-(k - \lfloor K/2 \rfloor)]. \tag{29}$$

The invertibility of the matrix $\mathbf{S}$ can be ensured, by proper selection of the periodic waveforms $p_i(t)$. In the next subsection we discuss one special case, which allows simple design of a left invertible mixing matrix $\mathbf{S}$.

B. Pulse Sequence Modulation

We follow practical modulation implementation ideas presented in [15], [16], and consider a set of waveforms given by

$$p_i(t) = \sum_{m\in\mathbb{Z}} \sum_{n=0}^{N-1} \alpha_i[n] p(t - nT/N - mT), \ i = 1, \ldots, p \tag{30}$$

where $p(t)$ is some pulse shape and $\alpha_i[n]$ is a length-N sequence. Our aim is to calculate the mixing matrix $\mathbf{S}$, when using the filtered version of (30) as modulating waveforms. To this end, we first calculate the Fourier series coefficients $d_i[k]$ of $p_i(t)$ as

$$\begin{aligned}
d_i[k] =&\frac{1}{T}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{N-1}\alpha_i[n]\int_0^T p(t-nT/N-mT)e^{-j\frac{2\pi}{T}kt}dt\\
=&\frac{1}{T}\sum_{n=0}^{N-1}\alpha_i[n]\sum_{m\in\mathbb{Z}}\int_{-mT}^{-(m-1)T} p(t-nT/N)e^{-j\frac{2\pi}{T}kt}dt\\
=&\frac{1}{T}\sum_{n=0}^{N-1}\alpha_i[n]\int_{-\infty}^{\infty} p(t-nT/N)e^{-j\frac{2\pi}{T}kt}dt\\
=&\frac{1}{T}\sum_{n=0}^{N-1}\alpha_i[n]P\left(\frac{2\pi}{T}k\right)e^{-j\frac{2\pi}{N}kn}, \qquad (31)
\end{aligned}$$

where $P(\omega)$ denotes the CTFT of $p(t)$.

Combining (31) with (29) and (27), the ikth element of $\mathbf{S}$ can be expressed as

$$\mathbf{S}_{ik} = \frac{1}{T}\sum_{n=0}^{N-1}\alpha_i[n]P\left(\frac{2\pi}{T}k'\right)G\left(\frac{2\pi}{T}k'\right)e^{-j\frac{2\pi}{N}k'n}, \quad (32)$$

where we defined $k' = -(k - \lfloor K/2 \rfloor)$. This matrix can be decomposed as

$$\mathbf{S} = \mathbf{A}\mathbf{W}\mathbf{\Phi}, \qquad (33)$$

where $\mathbf{A}$ is a $p \times N$ matrix with inth element equal to $\alpha_i[n]$, $\mathbf{W}$ is an $N \times K$ matrix with nkth element equal to $e^{-j\frac{2\pi}{N}k'n}$, and $\mathbf{\Phi}$ is a $K \times K$ diagonal matrix with kth diagonal element

$$\mathbf{\Phi}_{kk} = \frac{1}{T}P\left(\frac{2\pi}{T}k'\right)G\left(\frac{2\pi}{T}k'\right). \qquad (34)$$

From this decomposition it is clear that each one of these matrices has to be left invertible, in order to guarantee the invertibility of $\mathbf{S}$. We now examine each one of these matrices.

We start with the matrix $\mathbf{\Phi}$. From (28), $G\left(\frac{2\pi}{T}k\right) \neq 0$ for $k \in \mathcal{K}$. Therefore, we only need to require that $P\left(\frac{2\pi}{T}k\right) \neq 0$ for $k \in \mathcal{K}$ in order for $\mathbf{\Phi}$ to be invertible. The matrix $\mathbf{W}$ is a Vandermonde matrix, and therefore has full column rank as long as $N \geq K$ [23]. The last matrix $\mathbf{A}$, can be designed to be left invertible, by proper selection of the sequences $\alpha_i[n]$, where a necessary condition is that $p \geq N$.

We summarize our results in the following proposition:

Proposition 1. *Consider the system depicted in Fig. 4, where the modulation waveforms are given by* (30)*. If the following conditions hold*

1) $p \geq N \geq K$
2) *The frequency response of the shaping pulse $g(t)$ satisfies* (28),
3) *The frequency response of the pulse $p(t)$ satisfies $P\left(\frac{2\pi}{T}k\right) \neq 0$ for $k \in \mathcal{K}$,*
4) *The sequences $\alpha_i[n]$ are chosen such that the matrix $\mathbf{A}$, whose inth element is given by $\alpha_i[n]$, has full column rank,*

then the mixing matrix S in (11) *is left invertible.*

We now give two useful configurations of pulse sequence modulation schemes, that satisfy the conditions of Proposition 1.

1) Single Generator: We create the sequence on the ith channel, by taking a cyclic shift of one common sequence $\alpha[n]$ as

$$\alpha_i[n] = \alpha[n - i + 1 \bmod N]. \qquad (35)$$

Clearly, the corresponding waveforms can be created by using only one pulse generator, where the waveform at the ith channel is a delayed version of the generator output, delayed by $(i-1)T/N$ time units. This suggests, that in contrast to the direct scheme in Fig. 1, which requires multiple frequency sources, here only one pulse generator is required which simplifies the hardware design. It is easy to see that with this choice, $\mathbf{A}$ will be a circulant matrix. Such a matrix can be decomposed [24] as

$$\mathbf{A} = \mathbf{F}^H \mathrm{diag}\left(\mathbf{F}\boldsymbol{\alpha}\right)\mathbf{F}, \qquad (36)$$

where $\mathbf{F}$ is a $N \times N$ unitary discrete Fourier transform (DFT) matrix, and $\boldsymbol{\alpha}$ is a length-N vector containing the elements of the sequence $\alpha[n]$. Therefore, for $\mathbf{A}$ to be invertible the DFT of the sequence $\alpha[n]$ can not take on the value zero.

We now give an example for such a selection of the system's parameters. We set $p = N = K$, and choose

$$p(t) = \begin{cases} 1 & t \in \left[0, \frac{T}{N}\right] \\ 0 & t \notin \left[0, \frac{T}{N}\right]. \end{cases} \qquad (37)$$

The frequency response of this pulse satisfies

$$P(\omega) = \frac{T}{N}e^{-j\frac{T}{2N}\omega}\cdot \mathrm{sinc}\left(\frac{T}{2\pi N}\omega\right). \qquad (38)$$

Therefore,

$$\left|P\left(\frac{2\pi}{T}k\right)\right| = \frac{T}{N}\mathrm{sinc}\left(\frac{k}{N}\right), \qquad (39)$$

which is non-zero for $k \in \mathcal{K}$. In addition we choose the sequences $\alpha_i[n]$ as a sequences of ± 1s, created from cyclic shifts of one basic sequence, in a way that yields an invertible matrix $\mathbf{A}$. Such rectangular pulses with alternating signs can be easily implemented in hardware [16]. In Figs. 5 and 6, one modulating waveform is shown in the time and frequency domains, for $p = N = K = 7$. The original time-domain waveform is comprised of rectangular pulses, whereas lowpass filtering results in a smooth modulating waveform. Switching to the frequency domain, the Fourier series coefficients are shaped by $P(\omega)$, the CTFT of the pulse shape. The shaping filter frequency response, $G(\omega)$, is designed to transfer only the Fourier coefficients whose index is a member of the set $\mathcal{K} = \{-3, \ldots, 3\}$, suppressing all other coefficients.

2) Robustness to Sampling Channels Failure: Next we provide a setup which can overcome failures in a given number of the sampling channels. The identification of the malfunctioning channels is assumed to be performed by some external hardware.

We consider p sampling channels, and maximal number of malfunctioning channels which is denoted by p_e. Since disregarding of p_e channels is equivalent to the deletion of p_e rows in the matrix $\mathbf{A}$, it is required to have at least $p \geq N + p_e$ sampling channels. In addition the new submatrix $\tilde{\mathbf{A}}$, which is obtained from $\mathbf{A}$ by omitting of the corresponding rows

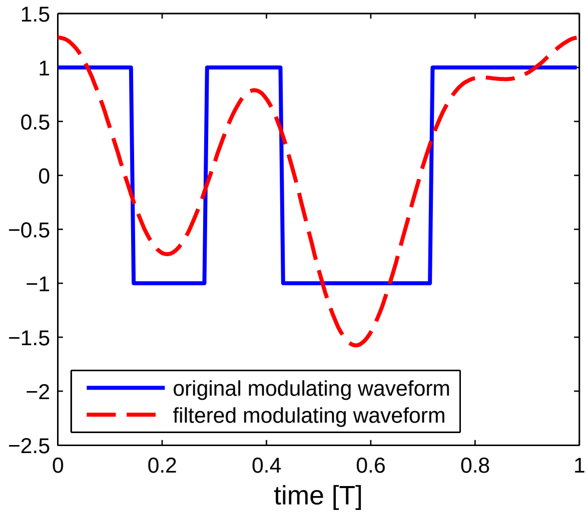

Fig. 5. Modulating waveform in the time domain, before and after filtering.

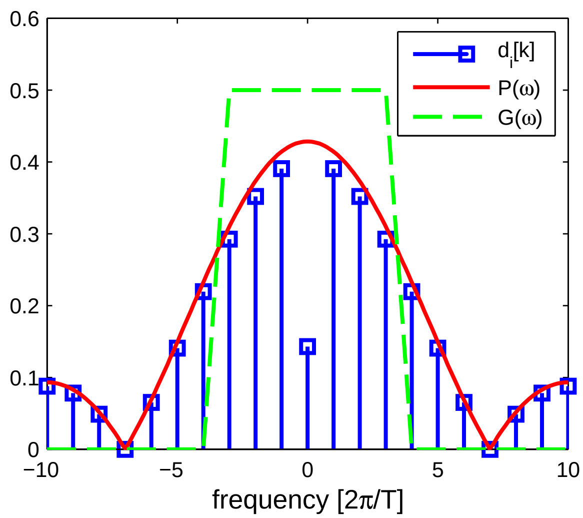

Fig. 6. Modulating waveform in the frequency domain.

should be left invertible. This has to be satisfied for every possible selection of p_e rows, therefore, we require that the matrix $\mathbf{A}$ should be designed, such that any $p - p_e$ rows will form a rank-N matrix. Following our ideas from the previous discussion, we demonstrate how to reduce the number of required generators, for the current setting. For simplicity, we assume $p \geq 2p_e$ and that two different generators are used. The first half of the sampling channels use delayed versions of the first generator output, and the second half uses the second generator. By proper selection of the two sequences, the condition mentioned above can be satisfied.

We now give a numerical example for a such choice. We set $N = K = 9$ and use $p = 18$ sampling channels, which are based on two generators only. Each generator produces a different sequence of ± 1, which are chosen randomly. In Fig. 7 we plot the log of the maximal condition number of $\tilde{\mathbf{A}}$, obtained when going over all possible options for omitting p_e rows from $\mathbf{A}$, i.e

$$\max_{\tilde{\mathbf{A}}} \log(\text{cond}(\tilde{\mathbf{A}})) \text{ s.t } \tilde{\mathbf{A}} \in \{p - p_e \text{ rows submatrices of } \mathbf{A}\}. \tag{40}$$

It can be seen that for $p_e \leq 6$ a relatively low condition number of the matrix $\tilde{\mathbf{A}}$ is achieved in the worst case, suggesting that its rank is N as required and $\tilde{\mathbf{A}}^{\dagger}$ is not ill-conditioned. Therefore, we can overcome failure in up to 6 sampling channels, using this system.

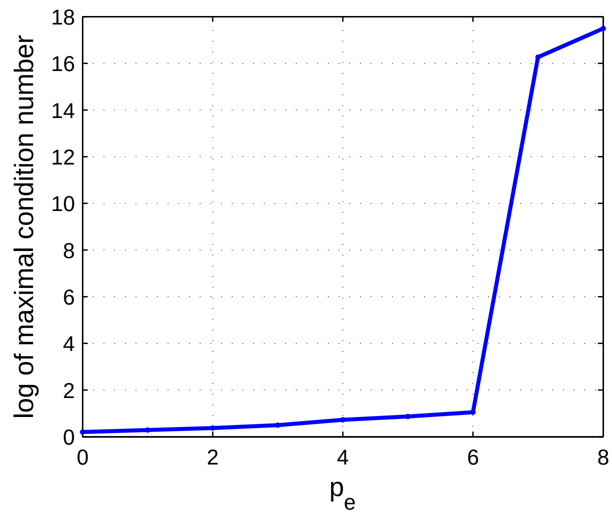

Fig. 7. Robustness to sampling channels failure example, $N = K = 9$, $p = 18$, two pulse generators.

V. Related Work

A. Single-channel Sampling with the SoS Filter

The work in [8] considered single-channel sampling schemes for pulse streams. The proposed sampling scheme is based on a filter which is comprised of a Sum of Sincs (SoS) in the frequency domain. This filter can be expressed in the time domain as

$$g(t) = \text{rect}\left(\frac{t}{T}\right) \sum_{k \in \mathcal{K}} b_k e^{j\frac{2\pi}{T}kt}, \tag{41}$$

where $\mathcal{K}$ is the chosen index set, and the coefficients $\{b_k\}_{k \in \mathcal{K}}$ can be chosen to be arbitrary nonzero values.

We first focus on periodic streams of pulses, with period T, for which the signal is filtered by the SoS filter prior to uniform sampling, as depicted in Fig. 8. The resulting samples are given by

$$c[n] = \sum_{k \in \mathcal{K}} b_k X[k] e^{j\frac{2\pi}{T}knT_s}, \quad n = 0, \ldots, p-1, \tag{42}$$

where $T_s = T/p$ is the sampling period. Using the matrix $\mathbf{V}$ defined in (6) only now with parameter $\mathbf{t}_s = \{0, T_s, \ldots, (p-1)T_s\}$, and defining the diagonal matrix $\mathbf{B}$ with kth diagonal element b_k, (42) can be written in matrix form as

$$\mathbf{c} = \mathbf{V}(-\mathbf{t}_s)\mathbf{B}\mathbf{x}. \tag{43}$$

Therefore, this is a special case of our mixing multichannel sampling scheme in (11) with mixing matrix $\mathbf{S} = \mathbf{V}(-\mathbf{t}_s)\mathbf{B}$.

The matrix $\mathbf{B}$ is invertible by construction, and $\mathbf{V}(-\mathbf{t}_s)$ is a Vandermonde matrix with distinct times, so that $\mathbf{S}$ is left-invertible as long as $p \geq K$. In that case, the p samples taken over one period in [8], are equal to the samples at the output of the p channels in our scheme.

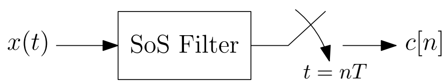

Fig. 8. Single-channel sampling scheme using the SoS filter.

Exploiting the compact support of the SoS filter, the method was extended to the finite and infinite settings as well [8]. The extension is based on using an r-fold periodic continuation of the SoS filter $g(t)$, where the parameter r depends on the support of the pulse-shape $h(t)$. However, the infinite scheme does not achieve the rate of innovation. In addition, implementing an r-fold periodic continuation of the filter $g(t)$ is harder to implement than the scheme we present in this paper.

We now examine which modulation waveforms are obtained when using the mixing matrix from [8]. By substituting the elements of this matrix into (9) we get

$$s_i(t) = \sum_{k \in \mathcal{K}} b_k e^{-j\frac{2\pi}{T}k(t-iT/p)}. \tag{44}$$

It is easily shown that these waveforms can be expressed as

$$s_i(t) = \tilde{g}(-(t - iT/p)), \tag{45}$$

where $\tilde{g}(t)$ is the periodic continuation of the SoS filter $g(t)$. Therefore, in each channel the signal is modulated by a delayed version of the periodic SoS filter. The equivalence of the schemes is easy to explain: sampling the convolution between the input signal and the SoS filter in [8], is equivalent to performing inner products (multiplication followed by integration) with delayed and reflected versions of this filter. This relation provides us another valid class of modulation waveforms.

B. Chain of Integrators

Sampling schemes for infinite streams of pulses have already been presented [8], [10], however, they do not achieve the minimal sampling rate, i.e., the rate of innovation. A recent work [13] presented a sampling scheme operating at the rate of innovation. The proposed method was based on multichannel sampling, similar to our approach. However, the signal model is limited to streams of Dirac impulses, whereas our scheme supports more general pulse shapes.

The method in [13] is based on the observation that the mth integral over a Dirac $\delta(t-t_l)$ on the interval $[0,T]$, is given by $(T-t_l)^{m-1}$. Following this observation, a sampling scheme based on successive integration of the signal is proposed, i.e., the sample at the mth channel is an m-fold integration of the signal. The resulting sample at the output of the *m*th channel is given by

$$y_m = \sum_{l=1}^{L} a_l (T-t_l)^{m-1} \quad m = 0, 1, \ldots, p-1. \tag{46}$$

Recovering the delays t_l from y_m is a problem commonly encountered in spectral analysis, which is closely related to the sum of cisoids topic. It may be solved by the same methods, e.g., the annihilating filter, as long as the number of integrations, p, satisfies $p \geq 2L$. Under the infinite model in (13) with $h(t) = \delta(t)$, this sampling scheme allows perfect reconstruction while operating at the rate of innovation.

However, as we will show in simulations, our method exhibits much higher noise robustness than the integrator approach. Moreover, for high rates of innovation, typically $L \geq 4$, the integrator scheme is unstable, whereas our approach remains stable even for very high values of L.

In addition, each integration enlarges the dynamic range of the output signal. As a result, the resolution of the analog to digital converters (ADC) has to grow rapidly when advancing through the channels. This imposes very strong requirements on the ADC converters in use. In contrast, our scheme is well-balanced, i.e., the dynamic range stays approximately the same throughout all channels, imposing no restricting requirements on the ADC converters used in the system.

C. Multichannel Schemes for Shift-Invariant Pulse Streams

Another related work is the one presented in [14] which treats the SI signal model (17) presented in Section III-B. The model considered in [14], is more general than the one in this work, since condition (2) is not required. However, as we see next, when (2) holds, the proposed method has some advantages over [14].

The sampling scheme proposed in [14] is depicted in Fig. 9. In each channel, the input signal is filtered by a band-limited sampling kernel $s_\ell^*(-t)$ followed by a uniform sampler operating at a rate of $1/T$. After the sampling process, a properly designed digital filter correction bank, whose frequency response in the DTFT domain is denoted here by $\mathbf{M}(e^{j\omega T})$, is applied on the sampling sequences. The exact form of this filter bank is detailed in [14]. It was shown in [14], that the ESPRIT algorithm can be applied on the new corrected samples, in order to recover the unknown delays.

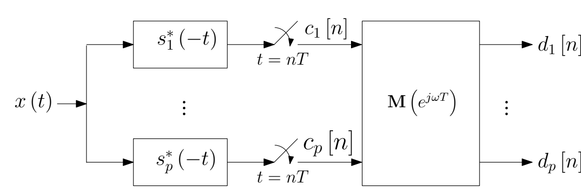

Fig. 9. Proposed sampling scheme in [14].

The sampling rate achieved by the method in [14] is generally $2L/T$, where for certain signals it can be reduced to $(L+1)/T$. Such signals satisfy dim(span($\{\mathbf{b}[m], m \in \mathbb{Z}\}$)) $= L$, where the vectors $\mathbf{b}[m]$ are related to the vectors $\mathbf{a}[m]$ through some filtering operations that are defined in [14]. This condition is different than the one obtained here, which directly depends on the vectors $\mathbf{a}[n]$ and not on a filtered version of them. Therefore the sampling rate, when using the scheme in [14], can be reduced to $(L+1)/T$ for different signals. However, this fact is not surprising, since each approach has a different analog sampling stage, and therefore coresponds to a different sampling operator. Each of these operators has a different invertibility condition. However, in any case, the worst-case minimal sampling rate for both methods is $2L/T$.

The proposed method has several advantages over the one presented in [14]. First the equivalent stage for the digital correction in [14], is replaced by the inversion of the matrices $\mathbf{H}$ and $\mathbf{S}$, since

$$\mathbf{y}[m] = (\mathbf{SH})^{-1}\mathbf{c}[m]. \quad (47)$$

This can be viewed as a one-tap digital correction filter bank, in contrast to the filter $\mathbf{M}(e^{\jmath\omega T})$, which generally has a larger number of taps. Therefore, the correction stage of the proposed method, is much simpler, and requires lower computational complexity, than the one in [14].

An additional advantage of our method, is that the approach of [14] requires collection of an infinite number of samples, even in the case where the input signal contains a finite number of periods. Moreover, if one is interested only in a finite time interval of the signal, the method in [14] does not allow processing it separately. This is in contrast to the proposed scheme, which integrates on finite time intervals, and can collect samples only from the relevant periods.

D. Modulated Wideband Converter

As mentioned before, the concept of using modulation waveforms, is based on ideas which were presented in [15] for a different signal model. In the sequel, we briefly present the sampling problem treated in [15] and its relation to our setup. As we show the practical hardware implementation of both systems is similar.

The model in [15] is of multiband signals: signals whose CTFT is concentrated on N_{bands} frequency bands, and the width of each band is no greater than B. The location of the bands is unknown in advance. An example of such a signal is depicted in Fig. 10. A low rate sampling scheme allowing recovery of such signals, at a rate of $4BN_{\text{bands}}$ was proposed in [5]. This scheme exploits the sparsity of the multiband signals in the frequency domain, to reduce the sampling rate well below the traditional Nyquist rate.

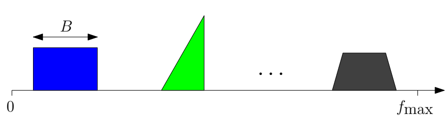

Fig. 10. Multiband signal model.

Later in [15], this sampling scheme was extended to a more practical one, which uses a modulation stage and referred to as the Modulated Wideband Converter (MWC). In each channel of the proposed multichannel sampling scheme, the input signal is modulated with some periodic waveform, and then sampled using a LPF followed by a low rate uniform sampler. The main idea here is that in each channel, the spectrum of the signal is scrambled, such that a portion of the energy of all bands appears at baseband. Therefore, the input to the sampler contains a mixture of all the bands. Mixing of the frequency bands in [15] is analogous to mixing the Fourier coefficients in our scheme.

Using the method in [15] on our signal model is highly inefficient, since our signal model is generally non-bandlimited, and does not have a sparse structure in the frequency domain. Therefore, we choose a different path presented in this paper, exploiting the sparse structure of the signal in the time domain. Nevertheless the modulation concepts introduced in [15] is used here as well, and eventually the sampling stages of both methods are similar.

We note here some differences between the schemes. First, following the mixing stage, we use an integrator in contrast to the LPF used in [15]. This difference, of course, is a result of the different signal quantities measured, i.e., Fourier coefficients in our work as opposed to the frequency bands content in [15]. The second difference is in the purpose of the mixing procedure. In [15] the mixing is done in order to reduce the sampling rate relatively to the Nyquist rate. In our setting, the mixing is used in order to simplify the hardware implementation and to improve the robustness to failure in one of the sampling channels.

Nonetheless, the hardware considerations in the mixing stage in both systems is similar. Recently, a prototype of the MWC has been implemented in hardware [16]. This design is composed of $p = 4$ sampling channels, where the repetition rate of the modulating waveforms is $1/T \approx 20$ MHz. In each period there are $N = 108$ rectangular pulses. This prototype, with certain modifications, can be used in order to implement our sampling scheme as well.

VI. Simulations

We now demonstrate the performance of our proposed method, in the presence of noise. We compare our results to those achieved by the integrators method [13]. We examine three modulation waveforms which are supported by our method: cosine and sine waveform (tones), filtered rectangular alternating pulses (rectangular) and waveforms obtained from delayed versions of the SoS filter (SoS). Note that, as discussed in Section V-A, the samples obtained by the third configuration (SoS) are exactly equal to the samples of the single channel method in [8], and therefore both setups provide the same performance. For the rectangular pulses scheme, the modulation waveforms are generated using a single generator, as discussed in Section IV-B. The shaping filter $g(t)$ is taken as an ideal LPF with transition band of width $2\pi/T$. White gaussian noise is added to the samples.

We first consider a finite stream of $L = 2$ Diracs with $\mathbf{t} = [0.256T, 0.38T]^T$, and amplitudes $\mathbf{a} = [1, 0.8]^T$. We set the system parameters as $p = K = N = 5$. The estimation error of the time-delays versus the SNR is depicted in Fig. 11, for the various approaches. Evidently, our approach outperforms the integrators method in terms of noise robustness, for all configurations. There is a slight advantage of 2dB for the schemes based on tones and SoS, over the scheme which uses alternating pulses, where the first two configurations have similar performance.

Turning to higher order problems, in Fig. 12 we show the results for $L = 10$ Diracs with times randomly distributed in

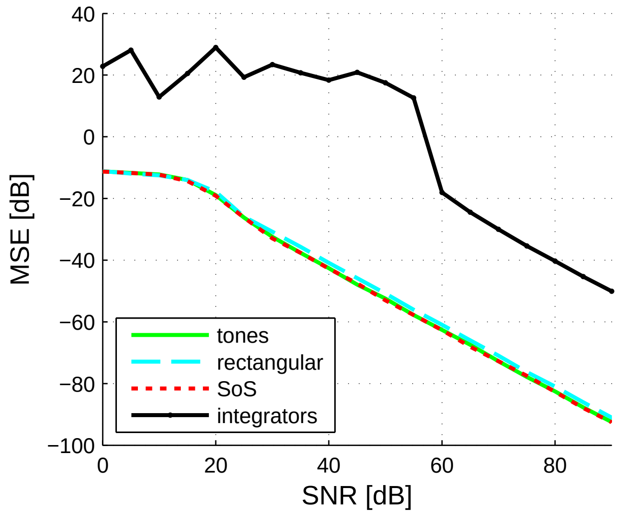

Fig. 11. Performance in the presence of noise: time estimation error of our method vs. integrators approach [13]. The signal consists of $L = 2$ pulses.

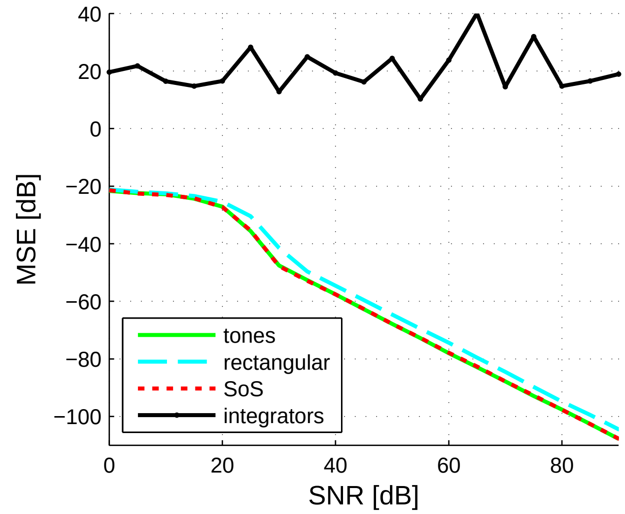

Fig. 12. Our method vs. integrators approach [13], for $L = 10$ pulses.

the interval $[0, T)$ and amplitudes equal one, where we set $N = p = K = 21$. The instability of the integrator method becomes apparent in this simulation on the one hand where, on the other hand, our approach achieves good estimation results. This demonstrates that our method is stable even for high model order.

The performance advantage of the tones and SoS based schemes is now around 3.5dB. We conclude that from a noise robustness point of view, using multiple frequency sources or SoS waveforms is preferable over the use of a single pulse generator. However, as discussed in Section II-D, pulse sequences based schemes have advantages from practical implementation considerations, and reduce the hardware complexity. In addition, the performance degradation is reasonable, and the estimation error is still significantly lower than that of the integrators approach [13]. Therefore, the flexibility of our approach, allows the system designer to decide between better performance in the presence of noise, or lower hardware complexity.

We now explore the use of practical shaping filters, for the rectangular pulses scheme, rather than the ideal ones used above. Once practical filters are used, the rejection of coefficients whose index is not in the set $\mathcal{K}$ is not perfect. We set the shaping filter $g(t)$ to be a Chebyshev (Type I) LPF [17] of various orders, with ripple 3 dB. The Chebyshev filter is a good choice for our requirements since it has a steeper roll-off than other filters, resulting in better rejection of the undesired coefficients. The rapid transition between the pass-band and stop-band of the Chebyshev filter comes at the expense of larger ripple in the pass-band, however, ripple is of minor concern for our method since it is digitally corrected when inverting the matrix $\mathbf{S}$. The cutoff frequency was set to $\frac{2\pi}{T}\lfloor K/2 \rfloor$. The frequency response of the various filters is shown in Fig. 13.

The estimation error of the time-delays versus the SNR is depicted in Fig. 14, for various filter orders. The simulation consists of $L = 2$ Diracs with $\mathbf{t} = [0.256T, 0.46T]^T$, and

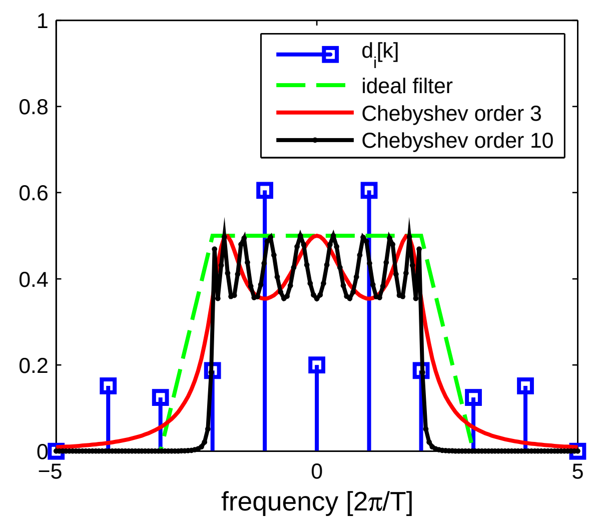

Fig. 13. Frequency responses of the shaping filters: ideal shaping filter vs. practical Chebyshev filters.

amplitudes $\mathbf{a} = [1, 0.8]^T$. Clearly, a Chebyshev filter of order 10 closely approaches the performance of an ideal LPF. In addition, for SNR levels below 50dB, using a Chebyshev filter of order 6 provides good approximation. Therefore, the modulation waveform generation stage of our proposed method can be implemented using practical analog filters.

VII. Conclusion

In this work, we proposed a new class of sampling schemes for pulse streams. The proposed method allows recovery of the delays and amplitudes defining such a signal, while operating at the rate of innovation. In contrast to previous works [12], [13] which achieved the rate of innovation, our approach supports general pulse shapes, rather than diracs only. In addition, as we demonstrate by simulations, our method exhibits better noise robustness than the one presented in [13], and can accommodate high rates of innovation.

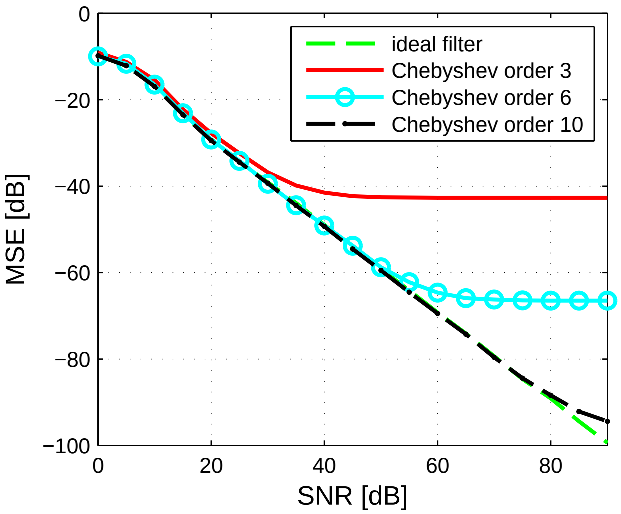

Fig. 14. Performance of practical shaping filters of various orders vs. ideal filtering.

The proposed scheme is based on multiple channels, each one comprised of mixing with a properly chosen waveform followed by an integrator. We exploit the degrees of freedom in the waveforms selection, and provide several useful configurations, which allow simplified hardware implementation and robustness to channel failure. Using simulations we further explored practical issues, and showed that standard analog filters could be used in the waveform generation stage. Moreover, we draw connections with the work in [15], [16] and show that the hardware prototype implemented there, can be used for our scheme, with certain modifications.

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