Uncertainty Relations for Shift-Invariant Analog Signals

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***Abstract*— The past several years have witnessed a surge of research investigating various aspects of sparse representations and compressed sensing. Most of this work has focused on the finite-dimensional setting in which the goal is to decompose a finite-length vector into a given finite dictionary. Underlying many of these results is the conceptual notion of an uncertainty principle: a signal cannot be sparsely represented in two different bases. Here, we extend these ideas and results to the analog, infinite-dimensional setting by considering signals that lie in a finitely-generated shift-invariant (SI) space. This class of signals is rich enough to include many interesting special cases such as multiband signals and splines. By adapting the notion of coherence defined for finite dictionaries to infinite SI representations, we develop an uncertainty principle similar in spirit to its finite counterpart. We demonstrate tightness of our bound by considering a bandlimited lowpass train that achieves the uncertainty principle. Building upon these results and similar work in the finite setting, we show how to find a sparse decomposition in an overcomplete dictionary by solving a convex optimization problem. The distinguishing feature of our approach is the fact that even though the problem is defined over an infinite domain with infinitely many variables and constraints, under certain conditions on the dictionary spectrum our algorithm can find the sparsest representation by solving a finite-dimensional problem.**

I. Introduction

Uncertainty relations date back to the work of Weyl and Heisenberg who showed that a signal cannot be localized simultaneously in both time and frequency. This basic principle was then extended by Landau, Pollack, Slepian and later Donoho and Stark to the case in which the signals are not restricted to be concentrated on a single interval [1], [2], [3], [4]. The uncertainty principle has deep philosophical interpretations. For example, in the context of quantum mechanics it implies that a particle's position and momentum cannot be simultaneously measured. In harmonic analysis it imposes limits on the time-frequency resolution [5].

Recently, there has been a surge of research into discrete uncertainty relations in more general finite-dimensional bases [6], [7], [8]. This work has been spurred in part by the relationship between sparse representations and the emerging field of compressed sensing [9], [10]. In particular, several works have shown that discrete uncertainty relations can be used to establish uniqueness of sparse decompositions in different bases representations. Furthermore, there is an intimate connection between uncertainty principles and the ability to recover sparse expansions using convex programming [6], [7], [11].

The vast interest in representations in redundant dictionaries stems from the fact that the flexibility offered by such systems can lead to decompositions that are extremely sparse, namely use only a few dictionary elements. However, finding a sparse expansion in practice is in general a difficult combinatorial optimization problem. Two fundamental questions at the heart of overcomplete representations are what is the smallest number of dictionary elements needed to represent a given signal, and how can one find the sparsest expansion in a computationally efficient manner. In recent years, several key papers have addressed both of these questions in a discrete setting, in which the signals to be represented are finite-length vectors [6], [7], [11], [12], [13], [14], [10], [8].

The discrete generalized uncertainty principle for pairs of orthonormal bases states that a vector in $\mathbb{R}^N$ cannot be simultaneously sparse in two orthonormal bases. The number of non-zero representation coefficients is bounded below by the inverse coherence [6], [7]. The coherence is defined as the largest absolute inner product between vectors in each basis [15], [6]. This principle has been used to establish conditions under which a convex ℓ_1 optimization program can recover the sparsest possible decomposition in a dictionary consisting of both bases [6], [7], [11]. These results where later generalized in [13], [12], [14] to representations in arbitrary dictionaries and to other efficient reconstruction algorithms [14].

The classical uncertainty principle is concerned with expanding a continuous-time analog signal in the time and frequency domains. However, the generalizations outlined above are mainly focused on the finite-dimensional setting. In this paper, our goal is to extend these recent ideas and results to the analog domain by first deriving uncertainty relations for more general classes of analog signals and arbitrary analog dictionaries, and then suggesting concrete algorithms to decompose a continuous-time signal into a sparse expansion in an infinite-dimensional dictionary.

In our development, we focus our attention on continuous-time signals that lie in shift-invariant (SI) subspaces of L_2 [16], [17], [18]. Such signals can be expressed in terms of

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linear combinations of shifts of a finite set of generators:

$$x(t)=\sum_{\ell=1}^{N}\sum_{n\in\mathbb{Z}}a_\ell[n]\phi_\ell(t-nT), \tag{1}$$

where $\phi_\ell(t), 1\leq \ell \leq N$ are the SI generators, and $a_\ell[n]$ are the expansions coefficients. Clearly, $x(t)$ is characterized by infinitely many coefficients $a_\ell[n]$. Therefore, the finite results which provide bounds on the number of non-zero expansion coefficients in pairs of bases decompositions are not immediately relevant here. Instead, we characterize analog sparsity as the number of active generators that comprise a given representation, where the ℓth generator is said to be active if $a_\ell[n], n\in\mathbb{Z}$ is not identically zero.

Starting with expansions in two orthonormal bases, we show that the number of active generators in each representation obeys an uncertainty principle similar in spirit to that of finite decompositions. The key to establishing this relation is in defining an analog coherence between the two bases. Our definition replaces the inner product in the finite setting by the largest spectral value of the sampled cross-correlation between basis elements, in the analog case. The similarity between the finite and infinite cases can also be seen by examining settings in which the uncertainty bound is tight. In the discrete scenario, the lower uncertainty limit is achieved by decomposing a spike train into the spike and Fourier bases, which are maximally incoherent [4]. To generalize this result to the analog domain we first develop an analog spike-Fourier pair and prove that it is maximally incoherent. The analog spike basis is obtained by modulations of the basic lowpass filter (LPF), which is maximally spread in frequency. In the time domain, these signals are given by shifts of the sinc function, whose samples generate shifted spikes. The discrete Fourier basis is replaced by an analog Fourier basis, in which the elements are frequency shifts of a narrow LPF in the continuous-time frequency domain. Tightness of the uncertainty relation is demonstrated by expanding a train of narrow LPFs in both bases.

We next address the problem of sparse decomposition in an overcomplete dictionary, corresponding to using more than N generators in (1). In the finite setting, it can be shown that under certain conditions on the dictionary, a sparse decomposition can be found using computationally efficient algorithms such as ℓ_1 optimization [19], [7], [11], [9]. However, directly generalizing this result to the analog setting is challenging. Although in principle we can define an ℓ_1 optimization program similar in spirit to its finite counterpart, it will involve infinitely many variables and constraints and therefore it is not clear how to solve it in practice. Instead, we develop an alternative approach by exploiting recent results on analog compressed sensing [20], [21], [22], [23], that leads to a finite-dimensional convex problem whose solution can be used to find the analog sparse decomposition. Our algorithm is based on a three-stage process: In the first step we sample the analog signal ignoring the sparsity, and formulate the decomposition problem in terms of sparse signal recovery from the given samples. In the second stage, we exploit results on infinite measurement models (IMV) and multiple measurement vectors (MMV) [24], [22], [25], [26] in order to determine the active generators, by solving a finite-dimensional convex optimization problem. Finally, we use this information to simultaneously solve the resulting infinite set of equations by inverting a finite matrix [27]. Our method works under certain technical conditions, which we elaborate on in the appropriate section. We also indicate how these results can be extended to more general classes of dictionaries.

The paper is organized as follows. In Section II we review the generalized discrete uncertainty principle and introduce the class of analog signals we will focus on. The analog uncertainty principle is formulated and proved in Section III. In Section IV we consider a detailed example illustrating the analog uncertainty relation and its tightness. In particular we introduce the analog version of the maximally incoherent spike-Fourier pair. Sparse decompositions in two orthonormal analog bases are discussed in Section V. These results are extended to arbitrary dictionaries in Section VI.

In the sequel, we denote signals in L_2 by lower case letters *e.g.,* $x(t)$, and SI subspaces of L_2 by $\mathcal{A}$. Vectors in $\mathbb{C}^N$ are written as boldface lowercase letters *e.g.,* $\mathbf{x}$, and matrices as boldface uppercase letters *e.g.,* $\mathbf{A}$. The ith element of a vector $\mathbf{x}$ is denoted x_i. The identity matrix of appropriate dimension is written as $\mathbf{I}$. For a given matrix $\mathbf{A}$, $\mathbf{A}^T$, $\mathbf{A}^H$ are its transpose and conjugate transpose respectively, $\mathbf{A}_\ell$ is its ℓth column, and $\mathbf{A}^\ell$ is the ℓth row. The standard Euclidean norm is denoted $\|\mathbf{x}\|_2=\sqrt{\mathbf{x}^H\mathbf{x}}$, $\|\mathbf{x}\|_1=\sum_i|x_i|$ is the ℓ_1 norm of $\mathbf{x}$, and $\|\mathbf{x}\|_0$ is the cardinality of $\mathbf{x}$ namely the number of non-zero elements. The complex conjugate of a complex number a is denoted $\overline{a}$. The Fourier transform of a signal $x(t)$ in L_2 is defined as $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$. We use the convention that upper case letters represent Fourier transforms. The discrete-time Fourier transform (DTFT) of a sequence $x[n]$ in ℓ_2 is defined by $X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$. To emphasize the fact that the DTFT is 2π-periodic we use the notation $X(e^{j\omega})$.

II. Problem Formulation

A. Discrete Uncertainty Principles

The generalized uncertainty principle is concerned with pairs of representations of a vector $\mathbf{x}\in\mathbb{R}^N$ in two different orthonormal bases [6], [7]. Suppose we have two orthonormal bases for $\mathbb{R}^N$: $\{\boldsymbol{\phi}_\ell, 1\leq\ell\leq N\}$ and $\{\boldsymbol{\psi}_\ell, 1\leq\ell\leq N\}$. Any vector $\mathbf{x}$ in $\mathbb{R}^N$ can then be decomposed uniquely in terms of each one of these vector sets:

$$\mathbf{x}=\sum_{\ell=1}^{N}a_\ell\boldsymbol{\phi}_\ell=\sum_{\ell=1}^{N}b_\ell\boldsymbol{\psi}_\ell. \tag{2}$$

Since the bases are orthonormal, the expansion coefficients are given by $a_\ell=\boldsymbol{\phi}_\ell^T\mathbf{x}$ and $b_\ell=\boldsymbol{\psi}_\ell^T\mathbf{x}$. Denoting by $\boldsymbol{\Phi},\boldsymbol{\Psi}$ the matrices with columns $\boldsymbol{\phi}_\ell,\boldsymbol{\psi}_\ell$ respectively, (2) can be written as $\mathbf{x}=\boldsymbol{\Phi}\mathbf{a}=\boldsymbol{\Psi}\mathbf{b}$, with $\mathbf{a}=\boldsymbol{\Phi}^T\mathbf{x}$ and $\mathbf{b}=\boldsymbol{\Psi}^T\mathbf{x}$.

The uncertainty relation sets limits on the sparsity of the decomposition for any vector $\mathbf{x} \in \mathbb{R}^N$. Specifically, let $A = \|\mathbf{a}\|_0$ and $B = \|\mathbf{b}\|_0$ denote the number of non-zero elements in each one of the expansions. The generalized uncertainty principle [7], [6] states that

$$\frac{1}{2}(A+B) \geq \sqrt{AB} \geq \frac{1}{\mu(\mathbf{\Phi}, \mathbf{\Psi})}, \tag{3}$$

where $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ is the coherence between the bases $\mathbf{\Phi}$ and $\mathbf{\Psi}$ and is defined by

$$\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \max_{\ell, r} |\boldsymbol{\phi}_\ell^T \boldsymbol{\psi}_r|. \tag{4}$$

The coherence measures the similarity between basis elements. This definition was introduced in [15] to heuristically characterize the performance of matching pursuit, and later used in [6], [7], [12], [14] in order to analyze the basis pursuit algorithm.

It can easily be shown that $1/\sqrt{N} \leq \mu(\mathbf{\Phi}, \mathbf{\Psi}) \leq 1$ [6]. The upper bound follows from the Cauchy-Schwarz inequality and the fact that the bases elements have norm 1. The lower bound is the result of the fact that the matrix $\mathbf{M} = \mathbf{\Phi}^T \mathbf{\Psi}$ is unitary and consequently $\mathbf{M}^T\mathbf{M} = \mathbf{I}_N$. This in turn implies that the sum of the squared elements of $\mathbf{M}$ is equal to N. Since there are N^2 variables, the value of the largest cannot be smaller than $1/\sqrt{N}$. The lower bound of $1/\sqrt{N}$ can be achieved, for example, by choosing the two orthonormal bases as the spike (identity) and Fourier bases [4]. With this choice, the uncertainty relation (3) becomes

$$A + B \geq 2\sqrt{AB} \geq 2\sqrt{N}. \tag{5}$$

Assuming $\sqrt{N}$ is an integer, the relations in (5) are all satisfied with equality when $\mathbf{x}$ is a spike train with spacing $\sqrt{N}$, resulting in $\sqrt{N}$ non-zero elements. This follows from the fact that the discrete Fourier transform of $\mathbf{x}$ is also a spike train with the same spacing. Therefore, $\mathbf{x}$ can be decomposed both in time and in frequency into $\sqrt{N}$ basis vectors.

As we discuss in Section V, the uncertainty relation provides insight into how sparse a signal $\mathbf{x}$ can be represented in an overcomplete dictionary consisting of $\mathbf{\Phi}$ and $\mathbf{\Psi}$. It also sheds light on the ability to compute such decompositions using computationally efficient algorithms. Most of the research to date on sparse expansions has focused on the discrete setting in which the goal is to represent a finite-length vector $\mathbf{x}$ in $\mathbb{R}^N$ in terms of a given dictionary using as few elements as possible. First general steps towards extending the notions and ideas underlying sparse representations and compressed sensing to the analog domain have been developed in [20], [22], [23], [28]. Here we would like to take a further step in this direction by extending the discrete uncertainty principle to the analog setting.

B. Shift-Invariant Signal Expansions

In order to develop a general framework for analog uncertainty principles we first need to describe the set of signals we consider. A popular model in signal and image processing are signals that lie in SI spaces. A finitely generated SI subspace in L_2 is defined as [16], [17], [18]:

$$\mathcal{A} = \left\{ x(t) = \sum_{\ell=1}^{N} \sum_{n \in \mathbb{Z}} a_\ell[n] \phi_\ell(t - nT) : a_\ell[n] \in \ell_2 \right\}. \tag{6}$$

The functions $\phi_\ell(t)$ are referred to as the generators of $\mathcal{A}$. Examples of SI spaces include multiband signals [20], [23] and spline functions [29], [27]. Expansions of the type (6) are also encountered in communication systems, when the analog signal is produced by pulse amplitude modulation. In the Fourier domain, we can represent any $x(t) \in \mathcal{A}$ as

$$X(\omega) = \sum_{\ell=1}^{N} A_\ell(e^{j\omega T}) \Phi_\ell(\omega), \tag{7}$$

where

$$A_\ell(e^{j\omega T}) = \sum_{n \in \mathbb{Z}} a_\ell[n] e^{-j\omega n T} \tag{8}$$

is the DTFT of $a_\ell[n]$ at frequency ωT, and is $2\pi/T$ periodic.

In order to guarantee a unique stable representation of any signal in $\mathcal{A}$ by a sequence of coefficients $a_\ell[n]$, the generators $\phi_\ell(t)$ are typically chosen such that the functions $\{\phi_\ell(t - nT), n \in \mathbb{Z}, 1 \leq \ell \leq N\}$ form a Riesz basis for L_2. This means that there exist constants $\alpha > 0$ and $\beta < \infty$ such that

$$\alpha \|\mathbf{a}\|^2 \leq \left\| \sum_{\ell=1}^{N} \sum_{n \in \mathbb{Z}} a_\ell[n] \phi_\ell(t - nT) \right\|^2 \leq \beta \|\mathbf{a}\|^2, \tag{9}$$

where $\|\mathbf{a}\|^2 = \sum_{\ell=1}^{N} \sum_{n \in \mathbb{Z}} |a_\ell[n]|^2$, and the norm in the middle term is the standard L_2 norm. Condition (9) implies that any $x(t) \in \mathcal{A}$ has a unique and stable representation in terms of the sequences $a_\ell[n]$. By taking Fourier transforms in (9) it follows that the shifts of the generators $\phi_\ell(t)$ form a Riesz basis if and only if [17]

$$\alpha \mathbf{I} \preceq \mathbf{M}_{\phi\phi}(e^{j\omega}) \preceq \beta \mathbf{I}, \quad \text{a.e. } \omega, \tag{10}$$

where

$$\mathbf{M}_{\phi\phi}(e^{j\omega}) = \begin{bmatrix} R_{\phi_1\phi_1}(e^{j\omega}) & \dots & R_{\phi_1\phi_N}(e^{j\omega}) \\ \vdots & \vdots & \vdots \\ R_{\phi_N\phi_1}(e^{j\omega}) & \dots & R_{\phi_N\phi_N}(e^{j\omega}) \end{bmatrix}, \tag{11}$$

and for any two functions $\phi(t), \psi(t)$ with Fourier transforms $\Phi(\omega), \Psi(\omega)$,

$$R_{\phi\psi}(e^{j\omega}) = \frac{1}{T} \sum_{k \in \mathbb{Z}} \overline{\Phi}\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right) \Psi\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right). \tag{12}$$

Note that $R_{\phi\psi}(e^{j\omega})$ is the DTFT of the cross correlation sequence $r_{\phi\psi}[n] = \langle \phi(t - nT), \psi(t) \rangle$, where the inner product on L_2 is defined as

$$\langle s(t), x(t) \rangle = \int_{-\infty}^{\infty} \overline{s}(t) x(t) dt. \tag{13}$$

In Section VI we consider overcomplete signal expansions in which more than N generators $\phi_\ell(t)$ are used to represent a signal $x(t)$ in $\mathcal{A}$. In this case (9) can be generalized to allow for stable overcomplete decompositions in terms of a frame for $\mathcal{A}$. The functions $\{\psi_\ell(t-nT), n\in\mathbb{Z}, 1\leq\ell\leq M\}$ form a frame for the SI space $\mathcal{A}$ if there exist constants $\alpha>0$ and $\beta<\infty$ such that

$$\alpha\|x(t)\|_2^2 \leq \sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}} |\langle \psi_\ell(t-nT), x(t)\rangle|^2 \leq \beta\|x(t)\|_2^2 \tag{14}$$

for all $x(t)\in\mathcal{A}$, where $\|x(t)\|_2^2=\langle x(t),x(t)\rangle$.

Our main interest is in expansions of a signal $x(t)$ in a SI subspace $\mathcal{A}$ of L_2 in terms of orthonormal bases for $\mathcal{A}$. The generators $\{\phi_\ell(t)\}$ of $\mathcal{A}$ form an orthonormal basis[1] if

$$\langle \phi_\ell(t-nT), \phi_r(t-mT)\rangle = \delta_{nm}\delta_{\ell r}, \tag{15}$$

for all ℓ, r, n, m, where $\delta_{nm}=1$ if $n=m$ and 0 otherwise. Since $\langle \phi_\ell(t-nT), \phi_r(t-mT)\rangle = \langle \phi_\ell(t-(n-m)T), \phi_r(t)\rangle$, (15) is equivalent to

$$\langle \phi_\ell(t-nT), \phi_r(t)\rangle = \delta_{n0}\delta_{\ell r}. \tag{16}$$

Taking the Fourier transform of (16), the orthonormality condition can be expressed in the Fourier domain as

$$R_{\phi_\ell\phi_r}(e^{j\omega}) = \delta_{\ell r}. \tag{17}$$

Given an orthonormal basis $\{\phi_\ell(t-nT)\}$ for $\mathcal{A}$, the unique representation coefficients $a_\ell[n]$ in (6) are given by $a_\ell[n]=\langle\phi_\ell(t-nT),x(t)\rangle$. This can be seen by taking the inner product of $x(t)$ in (6) with $\phi_r(t-mT)$ and using the orthogonality relation (15). Evidently, computing the expansion coefficients in an orthonormal decomposition is straightforward. There is also a simple relationship between the energy of $x(t)$ and the energy of the coefficient sequence in this case, as incorporated in the following proposition:

Proposition 1: Let $\{\phi_\ell(t), 1\leq\ell\leq N\}$ generate an orthonormal basis for a SI subspace $\mathcal{A}$, and let $x(t)=\sum_{\ell=1}^{N}\sum_{n\in\mathbb{Z}} a_\ell[n]\phi_\ell(t-nT)$. Then

$$\|x(t)\|^2 = \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{\ell=1}^{N}\left|A_\ell(e^{j\omega T})\right|^2 d\omega, \tag{18}$$

where $\|x(t)\|_2^2=\langle x(t),x(t)\rangle$ and $A_\ell(e^{j\omega})$ is the DTFT of $a_\ell[n]$.

Proof: See Appendix I. ∎

C. Analog Problem Formulation

In the finite-dimensional setting, sparsity is defined in terms of the number of non-zero expansion coefficients in a given basis. In an analog decomposition of the form (1), there are in general infinitely many coefficients so that it is not immediately clear how to define the notion of analog sparsity.

In our development, analog sparsity is measured by the number of generators needed to represent $x(t)$. In other words, some of the sequences $a_\ell[n]$ in (1) may be identically zero, in which case

$$x(t) = \sum_{|\ell|=A}\sum_{n\in\mathbb{Z}} a_\ell[n]\phi_\ell(t-nT), \tag{19}$$

where the notation $|\ell|=A$ means a sum over at most A elements. Evidently, in our definition, sparsity is determined by the energy of the entire sequence $a_\ell[n]$ and not by the values of the individual elements.

In general, the number of zero sequences depends on the choice of basis. Suppose we have an alternative representation

$$x(t) = \sum_{|\ell|=B}\sum_{n\in\mathbb{Z}} b_\ell[n]\psi_\ell(t-nT), \tag{20}$$

where $\{\psi_\ell(t)\}$ also generate an orthonormal basis for $\mathcal{A}$. An interesting question is whether there are limitations on A and B. In other words, can we have two representations that are simultaneously sparse so that both A and B are small? This question is addressed in the next section and leads to an analog uncertainty principle, similar to (3). In Section IV we prove that the relation we obtain is tight, by constructing an example in which the lower limits are satisfied.

As in the discrete setting we expect to be able to use fewer generators in a SI expansion by allowing for an overcomplete dictionary. In particular, if we expand $x(t)$ using both sets of orthonormal bases we may be able to reduce the number of sequences in the decomposition beyond what can be achieved using each basis separately. The problem is how to find a sparse representation in the joint dictionary in practice. Even in the discrete setting this problem is NP-complete. However, results of [7], [13], [12], [14] show that under certain conditions a sparse expansion can be determined by solving a convex optimization problem. Here we have an additional essential complication due to the fact that the problem is defined over an infinite domain so that it has infinitely many variables and infinitely many constraints. In Section V we show that despite the combinatorial complexity and infinite dimensions of the problem, under certain conditions on the bases functions, we can recover a sparse decomposition by solving a finite-dimensional convex optimization problem.

III. Uncertainty Relations in SI Spaces

We begin by developing an analog of the discrete uncertainty principle for signals $x(t)$ in SI subspaces. Specifically, we show that the minimal number of sequences required to express $x(t)$ in terms of any two orthonormal bases has to satisfy the same inequality (3) as in the discrete setting, with an appropriate modification of the coherence measure.

Theorem 1: Suppose we have a signal $x(t)\in\mathcal{A}$ where $\mathcal{A}$ is a SI subspace of L_2. Let $\{\phi_\ell(t), 1\leq\ell\leq N\}$ and

[1]Here and in the sequel, when we say that a set of signals $\{\phi_\ell(t)\}$ form (or generate) a basis, we mean that the basis functions are $\{\phi_\ell(t-nT), n\in\mathbb{Z}, 1\leq\ell\leq N\}$.

$\{\psi_\ell(t), 1 \le \ell \le N\}$ denote two orthonormal generators of $\mathcal{A}$, so that $x(t)$ can be expressed in both bases with coefficient sequences $a_\ell[n], b_\ell[n]$:

$$x(t) = \sum_{|\ell|=A} \sum_{n\in\mathbb{Z}} a_\ell[n]\phi_\ell(t-nT) = \sum_{|\ell|=B} \sum_{n\in\mathbb{Z}} b_\ell[n]\psi_\ell(t-nT). \quad (21)$$

Then,

$$\frac{1}{2}(A+B) \ge \sqrt{AB} \ge \frac{1}{\mu(\Phi,\Psi)}, \quad (22)$$

where

$$\mu(\Phi,\Psi) = \max_{\ell,r} \operatorname{ess\,sup}_{\omega} \left| R_{\phi_\ell\psi_r}(e^{j\omega}) \right|, \quad (23)$$

and $R_{\phi\psi}(e^{j\omega})$ is defined by (12).

The coherence $\mu(\Phi,\Psi)$ of (23) is a generalization of the notion of discrete coherence (4) defined for finite-dimensional bases. To see the analogy, note that $R_{\phi\psi}(e^{j\omega})$ is the DTFT of the correlation sequence $r_{\phi\psi}[n] = \langle \phi(t-nT), \psi(t) \rangle$. On the other hand, the finite-dimensional coherence can be written as $\mu(\mathbf{\Phi},\mathbf{\Psi}) = (1/N)\max_{\ell,r} |\hat{\boldsymbol{\phi}}_\ell^H \hat{\boldsymbol{\psi}}_r|$, where $\hat{\mathbf{x}}$ is the discrete Fourier transform of $\mathbf{x}$ and N is the length of $\mathbf{x}$.

Proof: Without loss of generality, we assume that $\|x(t)\|_2 = 1$. Since $\{\phi_\ell(t)\}$ and $\{\psi_\ell(t)\}$ both generate orthonormal bases, we have from Proposition 1 that

$$\begin{aligned} 1 &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{|\ell|=A} |A_\ell(e^{j\omega T})|^2 d\omega \\ &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{|\ell|=B} |B_\ell(e^{j\omega T})|^2 d\omega. \end{aligned} \quad (24)$$

Using the norm constraint and expressing $X(\omega)$ once in terms of $\Phi_\ell(\omega)$ and once in terms of $\Psi_\ell(\omega)$:

$$\begin{aligned} 1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{\substack{|\ell|=A \\ |r|=B}} \overline{A}_\ell(e^{j\omega T}) B_r(e^{j\omega T}) \overline{\Phi}_\ell(\omega)\Psi_r(\omega) d\omega \\ &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{\substack{|\ell|=A \\ |r|=B}} \overline{A}_\ell(e^{j\omega T}) B_r(e^{j\omega T}) R_{\phi_\ell\psi_r}(e^{j\omega T}) d\omega \\ &\le \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{\substack{|\ell|=A \\ |r|=B}} \left|A_\ell(e^{j\omega T})\right| \left|B_r(e^{j\omega T})\right| \left|R_{\phi_\ell\psi_r}(e^{j\omega T})\right| d\omega \\ &\le \frac{\mu(\Phi,\Psi)T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{|\ell|=A} \left|A_\ell(e^{j\omega T})\right| \sum_{|r|=B} \left|B_r(e^{j\omega T})\right| d\omega. \end{aligned} \quad (25)$$

The third equality follows from rewriting the integral over the entire real line as the sum of integrals over intervals of length $2\pi/T$ as in (109) in Appendix I, and the second inequality is a result of (23). Applying the Cauchy-Schwarz inequality to the integral in (25) we have

$$\begin{aligned} &\left[\int_0^{\frac{2\pi}{T}} \sum_{|\ell|=A} \left|A_\ell(e^{j\omega T})\right| \sum_{|r|=B} \left|B_r(e^{j\omega T})\right| d\omega\right]^2 \\ &\le \int_0^{\frac{2\pi}{T}} \left(\sum_{|\ell|=A} \left|A_\ell(e^{j\omega T})\right|\right)^2 d\omega \int_0^{\frac{2\pi}{T}} \left(\sum_{\ell=1}^{B} \left|B_\ell(e^{j\omega T})\right|\right)^2 \end{aligned} \quad (26)$$

Using the same inequality we can upper bound the sum in (26):

$$\left(\sum_{|\ell|=A} \left|A_\ell(e^{j\omega T})\right|\right)^2 \le A \sum_{|\ell|=A} \left|A_\ell(e^{j\omega T})\right|^2. \quad (27)$$

Combining with (26), (25) and (24) leads to

$$1 \le \mu(\Phi,\Psi)\sqrt{AB}. \quad (28)$$

Using the well-known relation between the arithmetic and geometric means completes the proof. ■

An interesting question is how small $\mu(\Phi,\Psi)$ can be made by appropriately choosing the bases. From Theorem 1 the smaller $\mu(\Phi,\Psi)$, the stronger the restriction on the sparsity in both decompositions. As we will see in Section V, such a limitation is helpful in recovering the true sparse coefficients. In the finite setting we have seen that $1/\sqrt{N} \le \mu(\mathbf{\Phi},\mathbf{\Psi}) \le 1$ [6]. The next theorem shows that the same bounds hold in the analog case.

Theorem 2: Let $\{\phi_\ell(t), 1 \le \ell \le N\}$ and $\{\psi_\ell(t), 1 \le \ell \le N\}$ denote two orthonormal generators of a SI subspace $\mathcal{A} \subset L_2$ and let $\mu(\Phi,\Psi) = \max_{\ell,r} \operatorname{ess\,sup} \left|R_{\phi_\ell\psi_r}(e^{j\omega})\right|$, where $R_{\phi\psi}(e^{j\omega})$ is defined by (12). Then

$$\frac{1}{\sqrt{N}} \le \mu(\Phi,\Psi) \le 1. \quad (29)$$

Proof: We begin by proving the upper bound, which follows immediately from the Cauchy-Schwarz inequality and the orthonormality of the bases:

$$\left|R_{\phi_\ell\psi_r}(e^{j\omega})\right| \le \left(R_{\phi_\ell\phi_\ell}(e^{j\omega}) R_{\psi_r\psi_r}(e^{j\omega})\right)^{1/2} = 1, \quad (30)$$

where the last equality is a result of (17). Therefore, $\mu(\Phi,\Psi) \le 1$.

To prove the lower bound, note that since $\phi_\ell(t)$ is in $\mathcal{A}$ for each ℓ, we can express it as

$$\phi_\ell(t) = \sum_{r=1}^{N} \sum_{n\in\mathbb{Z}} a_r^\ell[n]\psi_r(t-nT) \quad (31)$$

for some coefficients $a_r^\ell[n]$, or in the Fourier domain,

$$\Phi_\ell(\omega) = \sum_{r=1}^{N} A_r^\ell(e^{j\omega T})\Psi_r(\omega). \quad (32)$$

Since $\|\phi_\ell(t)\| = 1$ and $\{\psi_r(t)\}$ are orthonormal, we have from Proposition 1 that

$$\frac{T}{2\pi}\int_0^{\frac{2\pi}{T}} \sum_{r=1}^{N} \left|A_r^\ell(e^{j\omega T})\right|^2 d\omega = 1, \quad 1 \le \ell \le N. \quad (33)$$

Now, using (32) and the orthonormality condition (17) it follows that

$$\begin{aligned}\mu(\Phi,\Psi) &\geq \left|R_{\phi_\ell\psi_r}(e^{j\omega})\right| \\ &= \left|\sum_{s=1}^{N}\overline{A}_s^{\ell}(e^{j\omega})R_{\psi_s\psi_r}(e^{j\omega})\right| = \left|A_r^{\ell}(e^{j\omega})\right|. \end{aligned} \quad (34)$$

Therefore,

$$\begin{aligned}&\int_0^{2\pi}\sum_{\ell,r=1}^{N}\left|R_{\phi_\ell\psi_r}(e^{j\omega})\right|^2 d\omega = \\ &= \sum_{\ell=1}^{N}\int_0^{2\pi}\sum_{r=1}^{N}\left|A_r^{\ell}(e^{j\omega})\right|^2 d\omega = 2\pi N, \end{aligned} \quad (35)$$

where the last equality follows from (33) by performing a change of variables $\omega' = \omega T$ in the integral. If $\mu(\Phi,\Psi) < 1/\sqrt{N}$, then, $|R_{\phi_\ell\psi_r}(e^{j\omega})| < 1/\sqrt{N}$ a.e. on ω and

$$\int_0^{2\pi}\sum_{\ell,r=1}^{N}\left|R_{\phi_\ell\psi_r}(e^{j\omega})\right|^2 d\omega < 2\pi N, \quad (36)$$

which contradicts (35). ∎

It is easy to see that the lower bound in (29) is achieved if $R_{\phi_\ell\psi_r}(e^{j\omega}) = 1/\sqrt{N}$ for all ℓ, r and ω. In this case the uncertainty relation (22) becomes

$$A + B \geq 2\sqrt{AB} \geq 2\sqrt{N}. \quad (37)$$

As discussed in Section II, in the discrete setting with $\sqrt{N}$ an integer, the inequalities in (37) are achieved using the spike-Fourier basis and $\mathbf{x}$ equal to a spike train. In the next section we show that equality in (37) can be satisfied in the analog case as well using a pair of bases that is analogous to the spike-Fourier pair, and a bandlimited signal $x(t)$ equal to a lowpass train.

IV. Achieving the Uncertainty Principle

A. Minimal Coherence

Consider the space $\mathcal{A}$ of real signals bandlimited to $(-\pi N/T, \pi N/T]$. As we show below, any signal in $\mathcal{A}$ can be expressed in terms of N SI generators. We would like to choose two orthonormal bases, analogous to the spike-Fourier pair in the finite setting, for which the coherence achieves its lower limit of $1/\sqrt{N}$. To this end, we first highlight the essential properties of the finite spike-Fourier bases in $\mathbb{C}^N$, and then choose an analog pair with similar characteristics.

The basic properties of the spike-Fourier pair are illustrated in Fig. 1. The first element of the spike basis, $\boldsymbol{\phi}_1$, is equal to a constant in the discrete Fourier domain, as illustrated in the left-hand side of Fig. 1. The remaining basis vectors are generated by shifts in time, or modulations in frequency, as depicted in the bottom part of the figure. In contrast, the first vector of the Fourier basis is sparse in frequency: it is represented by a single frequency component as illustrated in the right-hand side of the figure. The rest of the basis elements are obtained by shifts in frequency.

We now construct two orthonormal bases for $\mathcal{A}$ with minimal coherence by mimicking these properties in the continuous-time Fourier domain. Since we are considering the class of signals bandlimited to $\pi N/T$, we only treat this frequency range. As we have seen, the basic element of the spike basis occupies the entire frequency spectrum. Therefore, we choose our first analog generator $\phi_1(t)$ to be constant over the frequency range $(-\pi N/T, \pi N/T]$. The remaining generators are obtained by shifts in time of $\phi_1(t)$ or modulations in frequency:

$$\Phi_\ell(\omega) = \begin{cases} \sqrt{\frac{T}{N}}e^{-j\omega(\ell-1)T/N}, & \omega \in (-\pi N/T, \pi N/T]; \\ 0, & \text{otherwise}, \end{cases} \quad (38)$$

corresponding to

$$\phi_\ell(t) = \sqrt{\frac{N}{T}}\,\mathrm{sinc}((t-(\ell-1)T')/T'), \quad (39)$$

with $T' = T/N$. The normalization constant is chosen to ensure that the basis vectors have unit norm. With slight abuse of terminology, we refer to the set $\{\phi_\ell(t), 1 \leq \ell \leq N\}$ as the analog spike basis (the basis is actually constructed by shifts of this set with period T). Note that the samples of $\phi_\ell(t)$ at times nT' create a shifted spike sequence, further justifying the analogy. The Fourier transform of the analog spike basis is illustrated in the left-hand side of Fig. 2.

To construct the second orthonormal basis, we choose $\psi_1(t)$ to be sparse in frequency, as in the finite case. The remaining generators are obtained by shifts in frequency. To ensure that the generators are real we must have that $\Psi_\ell(\omega) = \overline{\Psi}_\ell(-\omega)$. Therefore, we consider only the interval $[0, \pi N/T]$. Since we have N real generators, we divide this interval into equal sections of length π/T, and choose each $\Psi_\ell(\omega)$ to be constant over the corresponding interval, as illustrated in Fig. 2. More specifically, let

$$\mathcal{I}_\ell = \{\omega : |\omega| \in (\pi(\ell-1)/T, \pi\ell/T]\}, \quad (40)$$

be the ℓth interval. Then

$$\Psi_\ell(\omega) = \begin{cases} \sqrt{T}, & \omega \in \mathcal{I}_\ell; \\ 0, & \text{otherwise}. \end{cases} \quad (41)$$

The analog pair of bases generated by $\{\Phi_\ell(\omega), \Psi_\ell(\omega), 1 \leq \ell \leq N\}$ is referred to as the analog spike-Fourier pair. In order to complete the analogy with the discrete spike-Fourier bases we need to show that both analog sets are orthonormal and generate $\mathcal{A}$, and that their coherence is equal to $1/\sqrt{N}$. The latter follows immediately by noting that

$$\overline{\Phi}_\ell(\omega)\Psi_r(\omega) = \begin{cases} \frac{T}{\sqrt{N}}e^{j\omega(\ell-1)T/N}, & \omega \in \mathcal{I}_r; \\ 0, & \text{otherwise}. \end{cases} \quad (42)$$

It is easy to see that replicas of $\mathcal{I}_r$ at distance $2\pi/T$ will not overlap. Furthermore, these replicas tile the entire frequency axis; therefore, $|R_{\phi_\ell\psi_r}(e^{j\omega})| = 1/\sqrt{N}$, and $\mu(\Phi,\Psi) = 1/\sqrt{N}$.

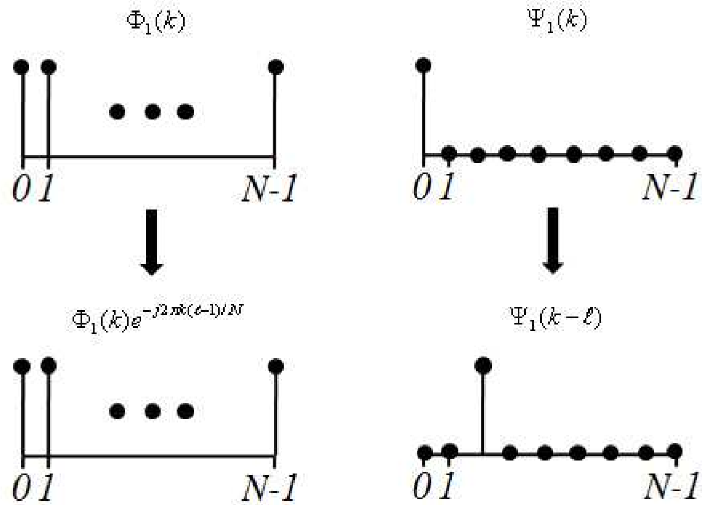

Fig. 1. Discrete Fourier-domain representation of the spike-Fourier bases in $\mathbb{C}^N$. The left-hand side is the discrete Fourier transform of the spike basis. The right-hand side represents the discrete Fourier transform of the Fourier basis. The top row corresponds to the first basis function, while the bottom row represents the ℓth basis function.

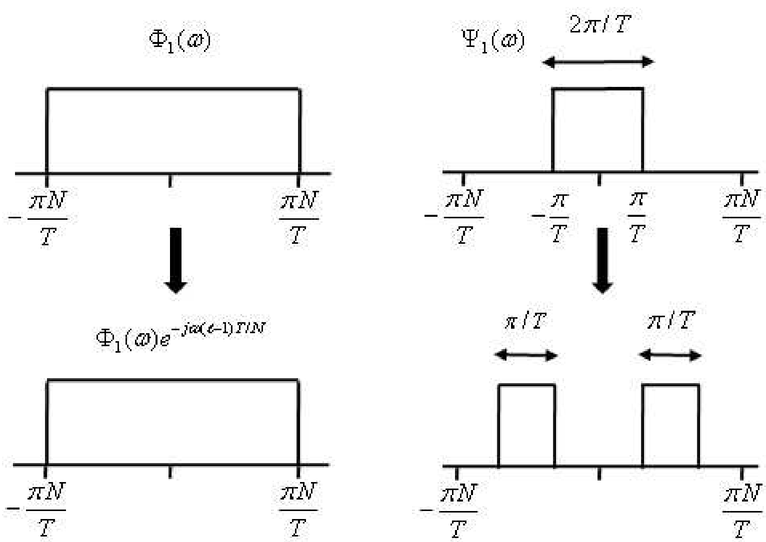

Fig. 2. Continuous Fourier-domain representation of the analog spike-Fourier bases in $\mathcal{A}$. The left-hand side is the Fourier transform of the spike basis. The right-hand side represents the Fourier transform of the Fourier basis. The top row corresponds to the first generator, while the bottom row represents the ℓth generator.

To show that $\{\psi_\ell(t), 1 \leq \ell \leq N\}$ generate $\mathcal{A}$, note that any $x(t) \in \mathcal{A}$ can be expressed in the form (6) (or (7)) by choosing $A_\ell(e^{j\omega T}) = X(\omega)$ for $\omega \in \mathcal{I}_\ell$. If $X(\omega)$ is zero on one of the intervals $\mathcal{I}_\ell$, then $A_\ell(e^{j\omega})$ will also be zero, leading to the multiband structure studied in [20], [23]. Since the intervals on which $\Psi_\ell(\omega)$ are non-zero do not overlap, the basis is also orthogonal. Finally, orthonormality follows from our choice of scaling.

Proving that $\{\phi_\ell(t), 1 \leq \ell \leq N\}$ generate an orthonormal basis is a bit more tricky. To see that these functions span $\mathcal{A}$ note that from Shannon's sampling theorem, any function $x(t)$ bandlimited to π/T' with $T' = T/N$ can be written as

$$x(t) = \sum_{n \in \mathbb{Z}} x(nT') \operatorname{sinc}((t - nT')/T'). \tag{43}$$

Substituting $n = mN + \ell - 1$, we can replace the sum over n by the double sum over $m \in \mathbb{Z}$ and $1 \leq \ell \leq N$, resulting in

$$\begin{aligned} x(t) &= \sum_{\ell=1}^{N} \sum_{m \in \mathbb{Z}} a_\ell[m] \operatorname{sinc}((t - (\ell-1)T' - mT))/T') \\ &= \sqrt{\frac{T}{N}} \sum_{\ell=1}^{N} \sum_{n \in \mathbb{Z}} a_\ell[n] \phi_\ell(t - nT), \end{aligned} \tag{44}$$

with $a_\ell[n] = x((\ell-1)T' + nT)$, proving that $\{\phi_\ell(t)\}$ generate $\mathcal{A}$. Orthonormality of the basis follows from

$$R_{\phi_\ell\phi_r}(e^{j\omega}) = \frac{1}{N} e^{j\omega(\ell-r)/N} \sum_{k=0}^{N-1} e^{-j2\pi k(\ell-r)/N} = \delta_{r\ell}, \tag{45}$$

where we used the relation

$$\sum_{k=0}^{N-1} e^{-j2\pi k(\ell-r)/N} = N\delta_{r\ell}. \tag{46}$$

B. Tightness of the Uncertainty Relation

Given any signal $x(t)$ in $\mathcal{A}$, the uncertainty relation for the analog spike-Fourier pair states that the number of non-zero sequences in the spike and Fourier bases must satisfy (37). We now show that when $\sqrt{N}$ is an integer, these inequalities can be achieved with equality with an appropriate choice of $x(t)$, so that the uncertainty principle is tight. To determine such a signal $x(t)$, we again mimic the construction in the discrete case.

As we discussed in Section II, when using the finite Fourier-spike pair, we have equalities in (37) when $\mathbf{x} \in \mathbb{R}^N$ is a spike train with $\sqrt{N}$ non-zero values, equally spaced, as illustrated in the left-hand side of Fig. 3. This follows from the fact that the spike train has the same form in both time and frequency. To construct a signal in $\mathcal{A}$ satisfying the analog uncertainty relation, we replace each Fourier-domain spike in the discrete setting by a shifted LPF of width $2\pi/T$ in the analog Fourier domain. To ensure that there are $\sqrt{N}$ non-zero intervals of length $2\pi/T$ in $(-\pi N/T, \pi N/T]$, the frequency spacing between the LPFs is set to $2\pi\sqrt{N}/T$, as depicted in the right-hand side of Fig. 3. This signal can be represented in frequency by $\sqrt{N}$ basis functions $\Psi_m(\omega)$, with $m = 2\sqrt{N}\ell, 1 \le \ell \le \lfloor\sqrt{N}/2\rfloor$, and $m = 2\sqrt{N}(\ell-1)+1, 1 \le \ell \le \lceil\sqrt{N}/2\rceil$. It therefore remains to be shown that $x(t)$ can also be expanded in time using $\sqrt{N}$ signals $\phi_m(t)$.

Since $x(t)$ is bandlimited to $\pi N/T$,

$$x(t) = \sum_{\ell=1}^{N} \sum_{n\in\mathbb{Z}} a_\ell[n]\phi_\ell(t-nT), \tag{47}$$

where $a_\ell[n] = \langle \phi_\ell(t-nT), x(t)\rangle$. In the Fourier domain we have

$$A_\ell(e^{j\omega}) = \frac{e^{j\omega(\ell-1)/N}}{\sqrt{NT}} \sum_{k\in\mathbb{Z}} e^{-j\omega 2\pi k(\ell-1)/N} X\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right). \tag{48}$$

Due to the fact that $a_\ell[n]$ is a real sequence, $A_\ell(e^{j\omega}) = \overline{A}_\ell(e^{-j\omega})$. Therefore we consider $A_\ell(e^{j\omega})$ on the interval $[0,\pi]$. For values of ω in this interval, $X(\omega/T - 2\pi k/T)$ is non-zero only for indices $k = m\sqrt{N}$ with $\lfloor -\sqrt{N}/2+1\rfloor \le m \le \lfloor\sqrt{N}/2\rfloor$. Thus,

$$\begin{aligned} A_\ell(e^{j\omega}) &= \frac{e^{j\omega(\ell-1)/N}}{\sqrt{NT}} \sum_{m=\lfloor-\sqrt{N}/2+1\rfloor}^{\lfloor\sqrt{N}/2\rfloor} e^{-j\omega 2\pi m(\ell-1)/\sqrt{N}} \\ &= \frac{e^{j\omega(\ell-1)/N}}{\sqrt{T}} \delta_{\ell-1, r\sqrt{N}}, \end{aligned} \tag{49}$$

where r is an arbitrary integer. The last equality follows from (46) and the fact that the sum is over $\sqrt{N}$ consecutive values. Since $1 \le \ell \le N$, $A_\ell(e^{j\omega})$ is nonzero for $\sqrt{N}$ indices ℓ, so that $x(t)$ can be expanded in terms of $\sqrt{N}$ generators $\phi_\ell(t)$.

V. Recovery of Sparse Representations

A. Discrete Representations

One of the important implications of the discrete uncertainty principle is its relation to sparse approximations [6], [7], [13], [14]. Given two orthonormal bases $\mathbf{\Phi}, \mathbf{\Psi}$ for $\mathbb{R}^N$ an interesting question is whether one can reduce the number of non-zero expansion coefficients required to represent a vector $\mathbf{x} \in \mathbb{R}^N$ by decomposing it in terms of the concatenated dictionary

$$\mathbf{D} = \left[\begin{array}{cc} \mathbf{\Phi} & \mathbf{\Psi} \end{array}\right]. \tag{50}$$

In many cases such a representation can be much sparser than the decomposition in either of the bases alone. The difficulty is in actually finding a sparse expansion $\mathbf{x} = \mathbf{D}\boldsymbol{\gamma}$ in which $\boldsymbol{\gamma}$ has as few non-zero components as possible. Since $\mathbf{D}$ has more columns than rows, the set of equations $\mathbf{x} = \mathbf{D}\boldsymbol{\gamma}$ is underdetermined and therefore $\mathbf{x}$ can have multiple representations $\boldsymbol{\gamma}$. Finding the sparsest choice can be translated into the combinatorial optimization problem

$$\min_{\boldsymbol{\gamma}} \|\boldsymbol{\gamma}\|_0 \quad \text{s. t. } \mathbf{x} = \mathbf{D}\boldsymbol{\gamma}. \tag{51}$$

Problem (51) is NP-complete in general and cannot be solved efficiently. A surprising result of [6], [7], [11], summarized below in Proposition 2, is that if the coherence $\mu(\mathbf{\Phi}, \mathbf{\Psi})$ between the two bases is small enough with respect to the sparsity of $\boldsymbol{\gamma}$, then the sparsest possible $\boldsymbol{\gamma}$ is unique and can be found by the basis pursuit algorithm. This algorithm is a result of replacing the non-convex ℓ_0 norm by the convex ℓ_1 norm:

$$\min_{\boldsymbol{\gamma}} \|\boldsymbol{\gamma}\|_1 \quad \text{s. t. } \mathbf{x} = \mathbf{D}\boldsymbol{\gamma}. \tag{52}$$

Proposition 2: Let $\mathbf{D} = [\mathbf{\Phi}\ \mathbf{\Psi}]$ be a dictionary consisting of two orthonormal bases with coherence $\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \max_{\ell,r} |\boldsymbol{\phi}_\ell^T \boldsymbol{\psi}_r|$. If a vector $\mathbf{x}$ has a sparse decomposition in $\mathbf{D}$ such that $\mathbf{x} = \mathbf{D}\boldsymbol{\gamma}$ and $\|\boldsymbol{\gamma}\|_0 < 1/\mu(\mathbf{\Phi}, \mathbf{\Psi})$ then this representation is unique, namely there cannot be another $\boldsymbol{\gamma}'$ with $\|\boldsymbol{\gamma}'\|_0 < 1/\mu(\mathbf{\Phi}, \mathbf{\Psi})$ and $\mathbf{x} = \mathbf{D}\boldsymbol{\gamma}'$. Furthermore, if

$$\|\boldsymbol{\gamma}\|_0 < \frac{\sqrt{2} - 0.5}{\mu(\mathbf{\Phi}, \mathbf{\Psi})}, \tag{53}$$

then the unique sparse representation can be found by solving the ℓ_1 optimization problem (52).

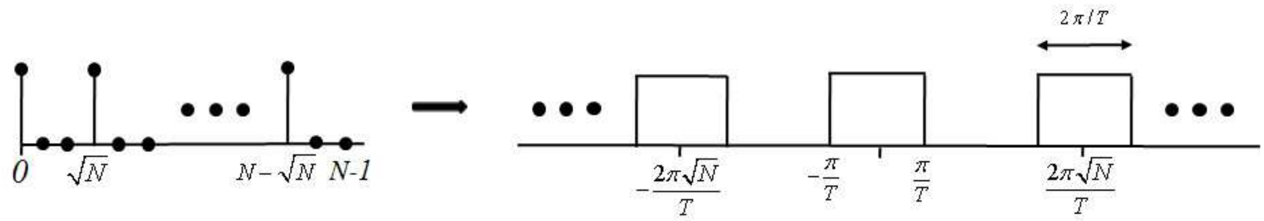

Fig. 3. Discrete and analog signals satisfying the uncertainty principle with equality. The left-hand side is the discrete Fourier transform of the spike train. The right-hand side represents the analog Fourier transform of the LPF train.

As detailed in [6], [7], the proof of Proposition 2 follows from the generalized discrete uncertainty principle.

Another useful result on dictionaries with low coherence is that every set of $k \leq 2/\mu(\mathbf{\Phi}, \mathbf{\Psi}) - 1$ columns are linearly independent [13, Theorem 6]. This result can be stated in terms of the Kruskal-rank of $\mathbf{D}$ [30], which is the maximal number q such that every set of q columns of $\mathbf{D}$ is linearly independent.

Proposition 3: [13, Theorem 6] Let $\mathbf{D} = [\mathbf{\Phi}\ \mathbf{\Psi}]$ be a dictionary consisting of two orthonormal bases with coherence $\mu(\mathbf{\Phi}, \mathbf{\Psi})$. Then $\sigma(\mathbf{D}) \geq 2/\mu(\mathbf{\Phi}, \mathbf{\Psi}) - 1$ where $\sigma(\mathbf{D})$ is the Kruskal rank of $\mathbf{D}$.

B. Analog Representations

We would now like to generalize these recovery results to the analog setup. However, it is not immediately clear how to extend the finite ℓ_1 basis pursuit algorithm of (52) to the analog domain.

To set up the analog sparse decomposition problem, suppose we have a signal $x(t)$ that lies in a space $\mathcal{A}$, and let $\{\phi_\ell(t), 1 \leq \ell \leq N\}, \{\psi_\ell(t), 1 \leq \ell \leq N\}$ be two orthonormal generators of $\mathcal{A}$. Our goal is to represent $x(t)$ in terms of the joint dictionary $\{d_\ell(t - nT), 1 \leq \ell \leq 2N\}$ with

$$d_\ell(t) = \begin{cases} \phi_\ell(t), & 1 \leq \ell \leq N; \\ \psi_{\ell-N}(t), & N+1 \leq \ell \leq 2N, \end{cases} \tag{54}$$

using as few non-zero sequences as possible. Denoting by $\boldsymbol{\gamma}[n]$ the vector at point-n whose elements are $\gamma_\ell[n]$, our problem is to choose the vector sequence $\boldsymbol{\gamma}[n]$ such that

$$x(t) = \sum_{\ell=1}^{2N} \sum_{n \in \mathbb{Z}} \gamma_\ell[n] d_\ell(t - nT), \tag{55}$$

and $\gamma_\ell[n]$ is identically zero for the largest possible number of indices ℓ.

We can count the number of non-zero sequences by first computing the ℓ_2-norm of each sequence. Clearly, $\gamma_\ell[n]$ is equal 0 for all n if and only if its ℓ_2 norm $\|\gamma_\ell[n]\|_2 = (\sum_n |\gamma_\ell^2[n]|)^{1/2}$ is zero. Therefore, the number of non-zero sequences $\gamma_\ell[n]$ is equal to $\|\mathbf{c}\|_0$ where $c_\ell = \|\gamma_\ell[n]\|_2$. For ease of notation, we denote $\|\boldsymbol{\gamma}\|_{2,0} = \|\mathbf{c}\|_0$, and similarly $\|\boldsymbol{\gamma}\|_{2,1} = \|\mathbf{c}\|_1$. Finding the sparsest decomposition (55) can then be written as

$$\min_{\boldsymbol{\gamma}} \|\boldsymbol{\gamma}\|_{2,0} \quad \text{s. t. } x(t) = \sum_{\ell=1}^{2N} \sum_{n \in \mathbb{Z}} \gamma_\ell[n] d_\ell(t - nT). \tag{56}$$

Problem (56) is the analog version of (51). However, in addition to being combinatorial as its finite counterpart, (56) also has infinitely many variables and constraints.

In order to extend the finite-dimensional decomposition results to the analog domain, there are two main questions we need to address:

1) Is there a unique sparse representation for any input signal in a given dictionary?
2) How can we compute a sparse expansion in practice, namely solve (56), despite the combinatorial complexity and infinite dimensions?

The first problem is easy to answer. Indeed, the uniqueness condition of Proposition 2 can be readily extended to the analog case. This is due to the fact that its proof is based on the uncertainty relation (3) which is identical to (22), with the appropriate modification to the coherence measure.

Proposition 4: Suppose that a signal $x(t) \in \mathcal{A}$ has a sparse representation in the joint dictionary $\{d_\ell(t - nT), n \in \mathbb{Z}, 1 \leq \ell \leq 2N\}$ of (54) which consists of two orthonormal bases $\{\phi_\ell(t - nT), \psi_\ell(t - nT), n \in \mathbb{Z}, 1 \leq \ell \leq N\}$. If the coefficient sequences $\gamma_\ell[n]$ of (55) satisfy $\|\boldsymbol{\gamma}\|_{2,0} < 1/\mu(\Phi, \Psi)$ where $\mu(\Phi, \Psi)$ is the coherence defined by (23), then this representation is unique.

The second, more difficult question, is how to find a unique sparse representation when it exists. We may attempt to develop a solution by replacing the ℓ_0 norm in (56) by an ℓ_1 norm, as in the finite-dimensional case. This leads to the convex program

$$\min_{\boldsymbol{\gamma}} \|\boldsymbol{\gamma}\|_{2,1} \quad \text{s. t. } x(t) = \sum_{\ell=1}^{2N} \sum_{n \in \mathbb{Z}} \gamma_\ell[n] d_\ell(t - nT). \tag{57}$$

However, in practice, it is not clear how to solve (57) since it is defined over an infinite set of variables $\gamma_\ell[n]$, and has infinitely many constraints (for all t).

Our approach to treating the analog decomposition problem is to first sample the signal $x(t)$ at a high enough rate, so that $x(t)$ can be determined from the given samples. We will then show that the decomposition problem can be recast in the Fourier domain as that of recovering a set of sparse vectors that share a joint sparsity pattern, from the given sequences of samples. The importance of this reformulation is that under appropriate conditions, it allows to determine the joint support set (or the active generators) by solving a finite-dimensional optimization problem. Once the active generators are determined, the corresponding coefficient sequences can be readily found.

We begin by noting that since $\{\phi_\ell(t)\}$ generate an orthonormal basis for $\mathcal{A}$, $x(t)$ is uniquely determined by the N sequences of samples

$$c_\ell[n] = \langle \phi_\ell(t-nT), x(t)\rangle = r_\ell(nT), \tag{58}$$

where $r_\ell(t)$ is the convolution $r_\ell(t) = \phi_\ell(-t)*x(t)$. Indeed, orthonormality of $\{\phi_\ell(t)\}$ immediately implies that

$$x(t) = \sum_{\ell=1}^{N}\sum_{n\in\mathbb{Z}} c_\ell[n]\phi_\ell(t-nT). \tag{59}$$

Therefore, constraining $x(t)$ is equivalent to imposing restrictions on the expansion coefficients $c_\ell[n]$. Taking the inner products on both sides of (55) with respect to $\phi_r(t-mT)$ leads to

$$\begin{aligned} c_r[m] &= \sum_{\ell=1}^{2N}\sum_{n\in\mathbb{Z}} \gamma_\ell[n]\langle \phi_r(t-mT), d_\ell(t-nT)\rangle \\ &= \sum_{\ell=1}^{2N}\sum_{n\in\mathbb{Z}} \gamma_\ell[n] a_{r\ell}[m-n], \end{aligned} \tag{60}$$

where $a_{r\ell}[n] = \langle \phi_r(t-nT), d_\ell(t)\rangle$. In the Fourier domain, (60) can be written as

$$C_r(e^{j\omega}) = \sum_{\ell=1}^{2N} \Gamma_\ell(e^{j\omega}) A_{r\ell}(e^{j\omega}), \quad 1\le r\le N. \tag{61}$$

Thus, instead of finding $\gamma_\ell[n]$ satisfying the constraints in (56) we can alternatively seek the smallest number of functions $\Gamma_\ell(e^{j\omega})$ that satisfy (61).

To simplify (61) we use the definition (54) of $d_\ell(t)$. Since $\langle \phi_r(t-nT), \phi_\ell(t)\rangle = \delta_{r\ell}\delta_{n0}$ and the Fourier transform of $\langle \phi_r(t-nT), \psi_\ell(t)\rangle$ is equal to $R_{\phi_r\psi_\ell}(e^{j\omega})$, (61) can be written as

$$C_r(e^{j\omega}) = \Gamma_r(e^{j\omega}) + \sum_{\ell=N+1}^{2N} \Gamma_\ell(e^{j\omega}) R_{\phi_r\psi_\ell}(e^{j\omega}). \tag{62}$$

Denoting by $\mathbf{c}(e^{j\omega}), \boldsymbol{\gamma}(e^{j\omega})$ the vectors with elements $C_\ell(e^{j\omega}), \Gamma_\ell(e^{j\omega})$ respectively, we can express (62) as

$$\mathbf{c}(e^{j\omega}) = \left[\begin{array}{cc} \mathbf{I} & \mathbf{M}_{\phi\psi}(e^{j\omega}) \end{array}\right] \boldsymbol{\gamma}(e^{j\omega}), \tag{63}$$

where $\mathbf{M}_{\phi\psi}(e^{j\omega})$ is the sampled cross correlation matrix

$$\mathbf{M}_{\phi\psi}(e^{j\omega}) = \begin{bmatrix} R_{\phi_1\psi_1}(e^{j\omega}) & \dots & R_{\phi_1\psi_N}(e^{j\omega}) \\ \vdots & \vdots & \vdots \\ R_{\phi_N\psi_1}(e^{j\omega}) & \dots & R_{\phi_N\psi_N}(e^{j\omega}) \end{bmatrix}, \tag{64}$$

with $R_{\phi\psi}$ defined by (12). Our sparse recovery problem (56) is therefore equivalent to

$$\begin{array}{ll} \min_{\boldsymbol{\gamma}} & \|\boldsymbol{\gamma}(e^{j\omega})\|_{2,0} \\ \text{s.t.} & \mathbf{c}(e^{j\omega}) = \left[\begin{array}{cc} \mathbf{I} & \mathbf{M}_{\phi\psi}(e^{j\omega}) \end{array}\right] \boldsymbol{\gamma}(e^{j\omega}). \end{array} \tag{65}$$

Problem (65) resembles the multiple measurement vector (MMV) problem, in which the goal is to jointly decompose m vectors $\mathbf{x}_i, 1\le i\le m$ in a dictionary $\mathbf{D}$ [25], [26], [24], [31]. In the next section we review the MMV model and a recently developed generalization to the case in which it is desirable to jointly decompose infinitely many vectors $\mathbf{x}_i$ in terms of a given dictionary $\mathbf{D}$. This extension is referred to as the infinite measurement model (IMV) [21]. In Section V-D we show how these ideas can be used to solve (65).

As we will show, the ability to sparsely decompose a set of signals in the IMV and MMV settings depends on the properties of the corresponding dictionary. In our formulation (65), the dictionary is given by

$$\mathbf{D}(e^{j\omega}) = \left[\begin{array}{cc} \mathbf{I} & \mathbf{M}_{\phi\psi}(e^{j\omega}) \end{array}\right]. \tag{66}$$

The next proposition establishes some properties of $\mathbf{D}(e^{j\omega})$ that will be used in Section V-D in order to solve (65).

Proposition 5: Let $\{\phi_\ell(t-nT), \psi_\ell(t-nT), n\in\mathbb{Z}, 1\le \ell\le N\}$ denote two orthonormal bases for a SI space $\mathcal{A}$. Let $\mathbf{M}_{\phi\psi}(e^{j\omega})$ denote the cross-correlation matrix defined by (64), and let $\mu(\Phi,\Psi)$,$\mu(\mathbf{\Phi},\mathbf{\Psi})$ be the analog and discrete coherence measures defined by (23), (4). Then, for each ω:

1) $\mathbf{M}_{\phi\psi}(e^{j\omega})$ is a unitary matrix;
2) $\mu(\mathbf{I}, \mathbf{M}_{\phi\psi}(e^{j\omega})) \le \mu(\Phi,\Psi)$.

Proof: See Appendix II. ■

C. MMV and IMV Models

The basic results of [7], [12], [13] on expansions in dictionaries consisting of two orthonormal bases can be generalized to the MMV problem in which we would like to jointly decompose m vectors $\mathbf{x}_i, 1\le i\le m$ in a dictionary $\mathbf{D}$. Denoting by $\mathbf{X}$ the matrix with columns $\mathbf{x}_i$, our goal is to seek a matrix $\mathbf{\Gamma}$ with columns $\boldsymbol{\gamma}_i$ such that $\mathbf{X} = \mathbf{D}\mathbf{\Gamma}$ and $\mathbf{\Gamma}$ has as few non-zero rows as possible. In this model, not only is each representation vector $\boldsymbol{\gamma}_i$ sparse, but in addition the vectors share a joint sparsity pattern. The results in [25], [26], [24] establish that under the same conditions as Proposition 2, the unique $\mathbf{\Gamma}$ can be found by solving an extension of the ℓ_1 program:

$$\min_{\mathbf{\Gamma}} \|\mathbf{s}(\mathbf{\Gamma})\|_1 \quad \text{s.t. } \mathbf{X} = \mathbf{D}\mathbf{\Gamma}. \tag{67}$$

Here $\mathbf{s}(\mathbf{\Gamma})$ is a vector whose ℓth element is equal to $\|\mathbf{\Gamma}^\ell\|$ where $\mathbf{\Gamma}^\ell$ is the ℓth row of $\mathbf{\Gamma}$, and the norm is an arbitrary vector norm. When $\mathbf{\Gamma}$ is equal to a single vector $\boldsymbol{\gamma}$, $\|\mathbf{\Gamma}^\ell\| = |\gamma_\ell|$ for any choice of norm and (67) reduces to the standard ℓ_1 optimization problem (52).

Proposition 6: Let $\mathbf{X}$ be an $N\times m$ matrix with columns $\mathbf{x}_i, 1\le i\le m$ that have a joint sparse representation in the dictionary $\mathbf{D} = [\mathbf{\Phi}\ \mathbf{\Psi}]$ consisting of two orthonormal bases, so that $\mathbf{X} = \mathbf{D}\mathbf{\Gamma}$ with $\|\mathbf{s}(\mathbf{\Gamma})\|_0 = k$. If $k < 1/\mu(\mathbf{\Phi}, \mathbf{\Psi})$ where $\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \max_{\ell,r} |\boldsymbol{\phi}_\ell^H \boldsymbol{\psi}_r|$, then this representation is unique. Furthermore, if

$$k < \frac{\sqrt{2} - 0.5}{\mu(\mathbf{\Phi}, \mathbf{\Psi})}, \tag{68}$$

then the unique sparse representation can be found by solving (67) with any vector norm.

The MMV model has been recently generalized to the IMV case in which there are infinitely many vectors $\mathbf{x}$ of length N, and infinitely many coefficient vectors $\boldsymbol{\gamma}$:

$$\mathbf{x}(\lambda) = \mathbf{D}\boldsymbol{\gamma}(\lambda), \quad \lambda \in \Lambda, \tag{69}$$

where Λ is some set whose cardinality can be infinite. In particular, Λ may be uncountable, such as the set of frequencies $\omega \in (-\pi, \pi]$. The k-sparse IMV model assumes that the vectors $\{\boldsymbol{\gamma}(\lambda)\}$, which we denote for brevity by $\boldsymbol{\gamma}(\Lambda)$, share a joint sparsity pattern, so that the non-zero elements are all supported on a fixed location set of size k [21]. This model was first introduced in [20] in the context of blind sampling of multiband signals, and later analyzed in more detail in [21].

A major difficulty with the IMV model is that it is not clear in practice how to determine the entire solution set $\boldsymbol{\gamma}(\Lambda)$ since there are infinitely many equations to solve. Thus, using an ℓ_1 optimization, or a greedy approach, are not immediately relevant here. In [21] it was shown that (69) can be converted to a finite MMV without loosing any information by a set of operations that are grouped under a block refereed to as the continuous-to-finite (CTF) block. The essential idea is to first recover the support of $\boldsymbol{\gamma}(\Lambda)$, namely the non-zero location set, by solving a finite MMV. We then reconstruct $\boldsymbol{\gamma}(\Lambda)$ from the data $\mathbf{x}(\Lambda)$ and the knowledge of the support, which we denote by S. The reason for this separation is that once S is known, the linear relation of (69) becomes invertible when the coherence is low enough.

To see this, let $\mathbf{D}_S$ denote the matrix containing the subset of the columns of $\mathbf{D}$ whose indices belong to S. The system of (69) can then be written as

$$\mathbf{x}(\lambda) = \mathbf{D}_S \boldsymbol{\gamma}^S(\lambda), \quad \lambda \in \Lambda, \tag{70}$$

where the superscript $\boldsymbol{\gamma}^S(\lambda)$ is the vector that consists of the entries of $\boldsymbol{\gamma}(\lambda)$ in the locations S. Since $\boldsymbol{\gamma}(\Lambda)$ is k-sparse, $|S| \leq k$. In addition, from Proposition 3 it follows that if $\mu(\mathbf{\Phi}, \mathbf{\Psi}) < 1/k$ then every k columns of $\mathbf{D}$ are linearly independent. Therefore $\mathbf{D}_S$ consists of linearly independent columns implying that $\mathbf{D}_S^\dagger \mathbf{D}_S = \mathbf{I}$, where $\mathbf{D}_S^\dagger = \left(\mathbf{D}_S^H \mathbf{D}_S\right)^{-1} \mathbf{D}_S^H$ is the Moore-Penrose pseudo-inverse of $\mathbf{D}_S$. Multiplying (70) by $\mathbf{D}_S^\dagger$ on the left gives

$$\boldsymbol{\gamma}^S(\lambda) = \mathbf{D}_S^\dagger \mathbf{x}(\lambda), \quad \lambda \in \Lambda. \tag{71}$$

The elements in $\boldsymbol{\gamma}(\lambda)$ not supported on S are all zero. Therefore (71) allows for exact recovery of $\boldsymbol{\gamma}(\Lambda)$ once the finite set S is correctly identified.

In order to determine S by solving a finite-dimensional problem we exploit the fact that $\mathrm{span}(\mathbf{x}(\Lambda))$ is finite, since $\mathbf{x}(\lambda)$ is of length N. Therefore, $\mathrm{span}(\mathbf{x}(\Lambda))$ has dimension at most N. In addition, it is shown in [21] that if there exists a solution set $\boldsymbol{\gamma}(\Lambda)$ with sparsity k, and the matrix $\mathbf{D}$ has Kruskal rank $\sigma(\mathbf{D}) \geq 2k$, then every finite collection of vectors spanning the subspace $\mathrm{span}(\mathbf{x}(\Lambda))$ contains sufficient information to recover S exactly. Therefore, to find S all we need is to construct a matrix $\mathbf{V}$ whose range space is equal to $\mathrm{span}(\mathbf{x}(\Lambda))$. We are then guaranteed that the linear system

$$\mathbf{V} = \mathbf{D}\mathbf{U} \tag{72}$$

has a unique k-sparse solution $\mathbf{U}$ whose row support is equal S. This result allows to avoid the infinite structure of (69) and to concentrate on finding the finite set S by solving the single MMV system of (72). The solution can be determined using an ℓ_1 relaxation of the form (67) with $\mathbf{V}$ replacing $\mathbf{X}$, as long as the conditions of Proposition 6 hold, namely the coherence is small enough with respect to the sparsity.

In practice, a matrix $\mathbf{V}$ with column span equal to $\mathrm{span}(\mathbf{x}(\Lambda))$ can be constructed by first forming the matrix $\mathbf{Q} = \int_{\lambda \in \Lambda} \mathbf{x}(\lambda)\mathbf{x}^H(\lambda) d\lambda$, assuming that the integral exists. Every $\mathbf{V}$ satisfying $\mathbf{Q} = \mathbf{V}\mathbf{V}^H$ will then have a column span equal to $\mathrm{span}(\mathbf{x}(\Lambda))$ [21]. In particular, the columns of $\mathbf{V}$ can be chosen as the eigenvectors of $\mathbf{Q}$ multiplied by the square-root of the corresponding eigenvalues.

We summarize the steps enabling a finite-dimensional solution to the IMV problem in the following theorem.

Theorem 3: Consider the system of equations (69) where $\mathbf{D} = [\mathbf{\Phi}\ \mathbf{\Psi}]$ is a dictionary consisting of two orthonormal bases with coherence $\mu(\mathbf{\Phi}, \mathbf{\Psi}) = \max_{\ell,r} |\boldsymbol{\phi}_\ell^H \boldsymbol{\psi}_r|$. Suppose (69) has a k-sparse solution set $\boldsymbol{\gamma}(\Lambda)$ with support set S. If the Kruskal rank $\sigma(\mathbf{D}) \geq 2k$, then $\boldsymbol{\gamma}(\Lambda)$ is unique. In addition, let $\mathbf{V}$ be a matrix whose column-space is equal to $\mathrm{span}(\mathbf{x}(\Lambda))$. Then, the linear system $\mathbf{V} = \mathbf{D}\mathbf{U}$ has a unique k-sparse solution $\mathbf{U}$ whose row support is equal to S. Denoting by $\mathbf{D}_S$ the columns of $\mathbf{D}$ whose indices belong to S, the non-zero elements $\boldsymbol{\gamma}^S(\lambda)$ are given by $\boldsymbol{\gamma}^S(\lambda) = \mathbf{D}_S^\dagger \mathbf{x}(\lambda)$. Finally, if

$$k < \frac{\sqrt{2} - 0.5}{\mu(\mathbf{\Phi}, \mathbf{\Psi})}, \tag{73}$$

then $\sigma(\mathbf{D}) \geq 2k$ and the unique sparse $\mathbf{U}$ can be found by solving (67) with any vector norm.

D. Analog Dictionaries

In Section V-B we showed that the analog decomposition problem (56) is equivalent to (65). The later is very similar to the IMV problem (69). Indeed, we seek a continuous set of vectors $\boldsymbol{\gamma}$ with joint sparsity that have the smallest number of non-zero rows, and satisfy an infinite set of linear equations. However, in contrast to (69), the matrix in (65) depends on ω. Therefore, Theorem 3 cannot be applied since it is not clear what matrix figures in the finite MMV representation.

Nonetheless, the essential idea of separating the support recovery from that of the actual values of $\boldsymbol{\gamma}(e^{j\omega})$ is still valid. In particular, we can solve (65) by first determining the support set of $\boldsymbol{\gamma}(e^{j\omega})$. Once the support is known, we have that

$$\boldsymbol{\gamma}^S(e^{j\omega}) = (\mathbf{D}_S^H(e^{j\omega})\mathbf{D}_S(e^{j\omega}))^{-1}\mathbf{D}_S^H(e^{j\omega})\mathbf{c}(e^{j\omega}), \quad (74)$$

where $\mathbf{D}(e^{j\omega})$ is defined by (66). The inverse in (74) exists if $\mu(\mathbf{I}, \mathbf{M}_{\phi\psi}(e^{j\omega}))$ is smaller than $1/k$. From Proposition 5, it is sufficient to require that $\mu(\Phi, \Psi) < 1/k$.

To find the support set S we distinguish between two different cases:

1) The constant case in which $\mathbf{M}_{\phi\psi}(e^{j\omega})$ of (64) can be written as
$$\mathbf{M}_{\phi\psi}(e^{j\omega}) = \mathbf{A}\mathbf{Z}(e^{j\omega}). \quad (75)$$
Here $\mathbf{A}$ is a fixed matrix independent of ω, and $\mathbf{Z}(e^{j\omega})$ is an invertible diagonal matrix with diagonal elements $Z_\ell(e^{j\omega})$; the columns of $\mathbf{A}$ are normalized such that $\operatorname{ess\,sup}|Z_\ell(e^{j\omega})| = 1$ for all ℓ.
2) The rich case in which the support of every subset of $\boldsymbol{\gamma}(e^{j\omega})$ of a given size M, is equal to the support S of the entire set.

The first case involves a condition on the dictionary. The second allows for arbitrary dictionaries, but imposes a constraint on the expansion sequences. This restriction is quite mild, and satisfied for a large class of dictionaries and signals. In both cases we show that the support can be found by solving a finite-dimensional optimization problem.

Constant case: We begin by treating the setting in which the sampled cross correlation matrix can be written as in (75). For example, consider the case in which $\mathcal{A}$ is the space of real signals bandlimited to $\pi N/T$, as in Section IV. Then $\phi_\ell(t), \psi_\ell(t)$ defined by (38), (41) satisfy (75) (for $\omega \geq 0$) with $\mathbf{A} = (1/\sqrt{N})\overline{\mathbf{F}}$, where $\mathbf{F}$ denotes the $N \times N$ Fourier matrix and $Z_\ell(e^{j\omega}) = \exp\{j\omega(\ell-1)/N\}$.

The unitarity of $\mathbf{M}_{\phi\psi}(e^{j\omega})$, which follows from Proposition 5, implies that $\mathbf{A} = \mathbf{M}_{\phi\psi}(e^{j\omega})\mathbf{Z}^{-1}(e^{j\omega})$ must be unitary as well. Indeed, for all ω, we have

$$\mathbf{A}^H\mathbf{A} = (\mathbf{Z}(e^{j\omega})\mathbf{Z}^H(e^{j\omega}))^{-1}. \quad (76)$$

Therefore, $|Z_\ell(e^{j\omega})|$ is independent of ω. Since $\max_\omega |Z_\ell(e^{j\omega})| = 1$, we conclude that $|Z_\ell(e^{j\omega})| = 1$ for all ω so that $\mathbf{Z}(e^{j\omega})\mathbf{Z}^H(e^{j\omega}) = \mathbf{I}$, which together with (76) proves the unitarity of $\mathbf{A}$.

To obtain a correlation structure of the form (75) we may start with a given orthonormal basis $\{\psi_\ell(t-nT)\}$, and then create another orthonormal basis $\{\phi_\ell(t-nT)\}$ by choosing

$$\phi_\ell(t) = \sum_{r=1}^{N}\sum_{n\in\mathbb{Z}} a_r^\ell[n]\psi_r(t-nT). \quad (77)$$

Here $a_r^\ell[n]$ is any set of sequences for which $A_r^\ell(e^{j\omega}) = [\mathbf{A}]_{\ell r}Z_r(e^{j\omega})$ with $\mathbf{A}$ an arbitrary unitary matrix, and $\mathbf{Z}$ is an arbitrary diagonal unitary matrix. This is a direct consequence of the proof of Proposition 5.

Under the condition (75) we now show that we can convert (65) to a finite MMV problem. Indeed, let the first N elements of $\boldsymbol{\gamma}(e^{j\omega})$ be denoted by $\mathbf{a}(e^{j\omega})$ and the remaining N elements by $\mathbf{b}(e^{j\omega})$. Then (65) becomes

$$\begin{array}{ll} \min_{\mathbf{a},\mathbf{d}} & \|\mathbf{a}(e^{j\omega})\|_{2,0} + \|\mathbf{d}(e^{j\omega})\|_{2,0} \\ \text{s.t.} & \mathbf{c}(e^{j\omega}) = \begin{bmatrix} \mathbf{I} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{a}(e^{j\omega}) \\ \mathbf{d}(e^{j\omega}) \end{bmatrix}, \end{array} \quad (78)$$

where $\mathbf{d}(e^{j\omega}) = \mathbf{Z}(e^{j\omega})\mathbf{b}(e^{j\omega})$, and we used the fact that since $\mathbf{Z}(e^{j\omega})$ is diagonal and invertible, $\|\mathbf{b}(e^{j\omega})\|_{2,0} = \|\mathbf{d}(e^{j\omega})\|_{2,0}$ so that the two vector sequences have the same sparsity. Problem (78) has the required IMV form. It can be solved by first finding the sparsest matrix $\mathbf{U}$ that satisfies $\mathbf{C} = [\mathbf{I}\ \mathbf{A}]\mathbf{U}$ where the columns of $\mathbf{C}$ form a basis for the span of $\{\mathbf{c}(e^{j\omega}), -\pi \leq \omega \leq \pi\}$. As we have seen, a basis can be determined in frequency by first forming the correlation matrix

$$\mathbf{Q} = \int_{-\pi}^{\pi} \mathbf{c}(e^{j\omega})\mathbf{c}^H(e^{j\omega})d\omega. \quad (79)$$

Alternatively, we can find a basis in time by creating

$$\mathbf{Q}' = \sum_{n=-\infty}^{\infty} \mathbf{c}[n]\mathbf{c}^H[n]. \quad (80)$$

The basis can then be chosen as the eigenvectors corresponding to nonzero eigenvalues of $\mathbf{Q}$ or $\mathbf{Q}'$, which we denote by $\mathbf{C}$. To find $\mathbf{U}$ we consider the convex program

$$\min_{\mathbf{U}} \|\mathbf{s}(\mathbf{U})\|_1 \quad \text{s.t. } \mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{A} \end{bmatrix}\mathbf{U}. \quad (81)$$

Let S denote the rows in $\mathbf{U}$ that are not identically zero and let $\boldsymbol{\gamma}^S[n]$ be the corresponding sequences $\gamma_\ell[n], \ell \in S$. Then

$$\boldsymbol{\gamma}^S(e^{j\omega}) = \begin{bmatrix} \mathbf{I} \\ \mathbf{Z}_{S'}^{-1}(e^{j\omega}) \end{bmatrix} (\mathbf{D}_S^H\mathbf{D}_S)^{-1}\mathbf{D}_S^H\mathbf{c}(e^{j\omega}), \quad (82)$$

where $\mathbf{D} = [\mathbf{I}\ \mathbf{A}]$, and S' denotes the rows in S between $N+1$ and $2N$. The remaining sequences $\gamma_\ell, \ell \notin S$ are identically zero. Proposition 6 provides conditions under which (81) will find the sparsest representation in terms of the coherence $\mu(\mathbf{I}, \mathbf{A})$ (where we rely on the fact that $\mathbf{A}$ is unitary). Since $|Z_\ell(e^{j\omega})| = 1$, we have that $|[\mathbf{A}\mathbf{Z}(e^{j\omega})]_{ij}| = |\mathbf{A}_{ij}|$ and $\mu(\mathbf{I}, \mathbf{A}) = \mu(\Phi, \Psi)$.

We summarize our results on analog sparse decompositions in the following theorem.

Theorem 4: Let $\{\phi_\ell(t), 1 \leq \ell \leq N\}$ and $\{\psi_\ell(t), 1 \leq \ell \leq N\}$ denote two orthonormal generators of a SI subspace $\mathcal{A}$ of L_2 with coherence $\mu(\Phi, \Psi)$. Let $x(t)$ be a signal in $\mathcal{A}$ and suppose there exists sequences $a_\ell[n], b_\ell[n]$ such that

$$x(t) = \sum_{\ell=1}^{N}\sum_{n\in\mathbb{Z}}(a_\ell[n]\phi_\ell(t-nT) + b_\ell[n]\psi_\ell(t-nT)) \quad (83)$$

with $k = \|\mathbf{a}\|_{2,0} + \|\mathbf{b}\|_{2,0}$ satisfying $k < (\sqrt{2} - 0.5)/\mu(\Phi, \Psi)$. Let $\mathbf{M}_{\phi\psi}(e^{j\omega})$ be the cross-correlation matrix defined by (64) and suppose that it can be written as $\mathbf{M}_{\phi\psi}(e^{j\omega}) = \mathbf{A}\mathbf{Z}(e^{j\omega})$, where $\mathbf{A}$ is unitary and $\mathbf{Z}(e^{j\omega})$ is a diagonal unitary matrix. Then, the sequences $a_\ell[n]$ and $b_\ell[n]$ can be found by solving

$$\begin{array}{ll}\min_{\mathbf{\Gamma}_1,\mathbf{\Gamma}_2} & \|\mathbf{s}(\mathbf{\Gamma}_1)\|_1+\|\mathbf{s}(\mathbf{\Gamma}_2)\|_1 \\ \text{s.t.} & \mathbf{C}=\left[\begin{array}{cc}\mathbf{I} & \mathbf{A}\end{array}\right]\left[\begin{array}{c}\mathbf{\Gamma}_1\\ \mathbf{\Gamma}_2\end{array}\right].\end{array} \tag{84}$$

Here $\mathbf{C}$ is chosen such that its columns form a basis for the range of $\{\mathbf{c}(e^{j\omega}),\omega\in(-\pi,\pi]\}$ where the ℓth component of $\mathbf{c}(e^{j\omega})$ is the Fourier transform at frequency ω of $c_\ell[n]=\langle\phi_\ell(t-nT),x(t)\rangle$, and $\mathbf{s}(\mathbf{\Gamma}_i)$ is a vector whose ℓth element is equal to $\|\mathbf{\Gamma}_i^\ell\|$ where the norm is arbitrary. Let S_1,S_2 denote the rows of $\mathbf{\Gamma}_1,\mathbf{\Gamma}_2$ that are not identically equal 0, and define $\mathbf{D}_S=[\mathbf{I}_{S_1}\ \mathbf{A}_{S_2}]$. Then the non-zero sequences $a_\ell[n],b_\ell[n],\ell\in S$ are given in the Fourier domain by

$$\left[\begin{array}{c}\mathbf{a}_S(e^{j\omega})\\ \mathbf{b}_S(e^{j\omega})\end{array}\right]=\left[\begin{array}{c}\mathbf{I}\\ \mathbf{Z}_{S_2}^{-1}(e^{j\omega})\end{array}\right](\mathbf{D}_S^H\mathbf{D}_S)^{-1}\mathbf{D}_S^H\mathbf{c}(e^{j\omega}). \tag{85}$$

In Theorem 4 the sparse decomposition is determined from the samples $c_\ell[n]=\langle\phi_\ell(t-nT),x(t)\rangle$. However, the theorem also holds when $c_\ell[n]$ is replaced by any sequence of samples $\langle h_\ell(t-nT),x(t)\rangle$ with $h_\ell(t)$ being an orthonormal basis for $\mathcal{A}$ such that both $\mathbf{M}_{h\phi}(e^{j\omega})$ and $\mathbf{M}_{h\psi}(e^{j\omega})$ are constant up to a diagonal matrix:

$$\mathbf{M}_{h\phi}(e^{j\omega})=\mathbf{A}_1\mathbf{Z}_1(e^{j\omega}),\quad \mathbf{M}_{h\psi}(e^{j\omega})=\mathbf{A}_2\mathbf{Z}_2(e^{j\omega}). \tag{86}$$

In this case the matrix $[\mathbf{I}\ \mathbf{A}]$ in (84) should be replaced by the matrix $[\mathbf{A}_1\ \mathbf{A}_2]$. Once we find the sparsity set S, the sequences that are not zero can be found as in (85) with the identity in the first matrix replaced by the appropriate rows of $\mathbf{Z}_1^{-1}(e^{j\omega})$.

Rich case: We next consider the case of an arbitrary $\mathbf{D}(e^{j\omega})$, and impose a condition on the sequences $\gamma_\ell[n]$. Specifically, we assume that there exists a finite number M such that the support set of $\{\boldsymbol{\gamma}(e^{j\omega_i}),|i|=M\}$ is equal S. In other words, the joint support of any M vectors $\boldsymbol{\gamma}(e^{j\omega_i})$ is equal to the support of the entire set. Under this assumption, the support recovery problem reduces to an MMV model and can therefore be solved efficiently using MMV techniques. Specifically, we select a set of M frequencies ω_i, and seek the matrix $\mathbf{\Gamma}$ with columns $\boldsymbol{\gamma}_i$ that is the solution to

$$\begin{array}{ll}\min_{\mathbf{\Gamma}} & \|\mathbf{s}(\mathbf{\Gamma})\|_1 \\ \text{s.t.} & \mathbf{c}(e^{j\omega_i})=\left[\begin{array}{cc}\mathbf{I} & \mathbf{M}_{\phi\psi}(e^{j\omega_i})\end{array}\right]\boldsymbol{\gamma}_i,\quad 1\le i\le M.\end{array} \tag{87}$$

If we choose $\mathbf{s}(\mathbf{\Gamma})$ as the ℓ_1 norm, then (87) is equivalent to M separate problems, each of the form

$$\min_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_1\quad \text{s.t.}\ \mathbf{c}=\left[\begin{array}{cc}\mathbf{I} & \mathbf{U}\end{array}\right]\boldsymbol{\gamma}, \tag{88}$$

were $\mathbf{c}=\mathbf{c}(e^{j\omega_i})$ and $\mathbf{U}=\mathbf{M}_{\phi\psi}(e^{j\omega_i})$ is a unitary matrix (see Proposition 5). From Proposition 2, the correct sparsity pattern will be recovered if $\mu(\mathbf{I},\mathbf{U})$ is low enough, which due to Proposition 5 can be guaranteed by upper bounding $\mu(\Phi,\Psi)$.

In some cases, even one frequency ω_i may be sufficient in order to determine the correct sparsity pattern; this happens when the support of $\boldsymbol{\gamma}(e^{j\omega_i})$ is equal to the support of the entire set of sequences $\boldsymbol{\gamma}(e^{j\omega})$. In practice, we can solve for an increasing number of frequencies, with the hope of recovering the entire support in a finite number of steps. Although we can always construct a set of signals whose joint support cannot be detected in a finite number of steps, this class of signals is small. Therefore, if the sequences are generated at random, then with high probability choosing a finite number of frequencies will be sufficient to recover the entire support set.

VI. Extension to Arbitrary Dictionaries

Until now we discussed the case of a dictionary comprised of two orthonormal bases. The theory we developed can easily be extended to treat the case of an arbitrary dictionary comprised of sequences $d_\ell(t)$ that form a frame (14) for $\mathcal{A}$. These results follow from combining the approach of the previous section with the corresponding statements in the discrete setting developed in [12], [13], [14].

Specifically, suppose we would like to decompose a vector $\mathbf{x}\in\mathbb{R}^N$ in terms of a dictionary $\mathbf{D}$ with columns $\mathbf{d}_\ell$ using as few vectors as possible. This corresponds to solving

$$\min_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_0\quad \text{s.t.}\ \mathbf{x}=\mathbf{D}\boldsymbol{\gamma}. \tag{89}$$

Since (89) has combinatorial complexity, we would like to replace it with a computationally efficient algorithm. If $\mathbf{D}$ has low coherence, where in this case the coherence is defined by

$$\mu(\mathbf{D})=\max_{\ell\neq r}\frac{|\mathbf{d}_\ell^H\mathbf{d}_r|}{\|\mathbf{d}_\ell\|\|\mathbf{d}_r\|}, \tag{90}$$

then we can determine the sparsest solution $\boldsymbol{\gamma}$ by solving the ℓ_1 problem

$$\min_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_1\quad \text{s.t.}\ \mathbf{x}=\mathbf{D}\boldsymbol{\gamma}. \tag{91}$$

The coherence of a dictionary measures the similarity between its elements and is equal to 0 only if the dictionary consists of orthonormal vectors. A general lower bound on the coherence of a matrix $\mathbf{D}$ of size $N\times m$ is [14] $\mu(\mathbf{D})\ge[(m-N)/(N(m-1))]^{1/2}$. The same results hold true for the corresponding MMV model, and are incorporated in the following proposition [13], [12], [14], [25]:

Proposition 7: Let $\mathbf{D}$ be an arbitrary dictionary with coherence $\mu(\mathbf{D})$ given by (90). Then the Kruskal rank satisfies $\sigma(\mathbf{D})>1/\mu(\mathbf{D})-1$. Furthermore, if there exists a choice of coefficients $\mathbf{\Gamma}$ such that $\mathbf{X}=\mathbf{D}\mathbf{\Gamma}$ and

$$\|\mathbf{s}(\mathbf{\Gamma})\|_0<\frac{1}{2}\left(1+\frac{1}{\mu(\mathbf{D})}\right), \tag{92}$$

then the unique sparse representation can be found by solving (67).

We now apply Proposition 7 to the analog design problem. Suppose we have a signal $x(t)$ that lies in a SI space $\mathcal{A}$, and let $\{d_\ell(t-nT),1\le\ell\le m\}$ denote an arbitrary frame for $\mathcal{A}$ with $m>N$. As an example, consider the space $\mathcal{A}$ of real signals bandlimited to $(-\pi N/T,\pi N/T]$, which was introduced in Section IV. As we have seen, this space can be generated by the N functions

$$\phi_\ell(t)=\frac{1}{\sqrt{T'}}\operatorname{sinc}((t-(\ell-1)T')/T'),\quad 1\le \ell\le N, \tag{93}$$

with $T'=T/N$. Suppose now that we define the functions

$$\tilde{\phi}_\ell(t)=\frac{1}{\sqrt{\tilde{T}}}\operatorname{sinc}((t-(\ell-1)\tilde{T})/\tilde{T}),\quad 1\le \ell\le m, \tag{94}$$

where $\tilde{T}=T/m$ and $m>N$. Using similar reasoning as that used to establish the basis properties of the generators (39), it is easy to see that $\{\tilde{\phi}_\ell(t)\}$ constitute an orthonormal basis for the space of signals bandlimited to $(-\pi m/T,\pi m/T]$ which is larger than $\mathcal{A}$. Filtering each one of the basis signals with a (scaled) LPF with cut-off π/T' will result in a redundant set of functions

$$d_\ell(t)=\frac{1}{\sqrt{T'}}\operatorname{sinc}((t-(\ell-1)\tilde{T})/T'),\quad 1\le \ell\le m, \tag{95}$$

that form a frame for $\mathcal{A}$ [32], [33].

Our goal is to represent a signal $x(t)$ in $\mathcal{A}$ using as few sequences $d_\ell(t)$ as possible. More specifically, our problem is to choose the vector sequence $\boldsymbol{\gamma}[n]$ such that

$$x(t)=\sum_{\ell=1}^{m}\sum_{n\in\mathbb{Z}}\gamma_\ell[n]d_\ell(t-nT), \tag{96}$$

and $\|\boldsymbol{\gamma}\|_{2,0}$ is minimized.

To derive an infinite-dimensional alternative to (91) let $\{h_\ell(t)\}$ generate a basis for $\mathcal{A}$. Then $x(t)$ is uniquely determined by the N sampling sequences

$$c_\ell[n]=\langle h_\ell(t-nT),x(t)\rangle=r_\ell(nT), \tag{97}$$

where $r_\ell(t)$ is the convolution $r_\ell(t)=h(-t)*x(t)$. Therefore, $x(t)$ satisfies (96) only if

$$c_r[m]=\sum_{\ell=1}^{m}\sum_{n\in\mathbb{Z}}\gamma_\ell[n]a_{r\ell}[n], \tag{98}$$

where $a_{r\ell}[n]=\langle h_r(t-nT),d_\ell(t)\rangle$. In the Fourier domain (98) becomes

$$C_r(e^{j\omega})=\sum_{\ell=1}^{m}\Gamma_\ell(e^{j\omega})A_{r\ell}(e^{j\omega})=\sum_{\ell=1}^{m}\Gamma_\ell(e^{j\omega})R_{h_r d_\ell}(e^{j\omega}). \tag{99}$$

Denoting by $\mathbf{c}(e^{j\omega}),\boldsymbol{\gamma}(e^{j\omega})$ the vectors with elements $C_\ell(e^{j\omega}),\Gamma_\ell(e^{j\omega})$ respectively we can write (99) as

$$\mathbf{c}(e^{j\omega})=\mathbf{M}_{hd}(e^{j\omega})\boldsymbol{\gamma}(e^{j\omega}). \tag{100}$$

Therefore, our problem is to find the sparsest set of $\boldsymbol{\gamma}(e^{j\omega})$ that satisfies (100).

In order to solve the sparse decomposition problem we first treat the case in which $\{h_\ell(t)\}$ are chosen such that

$$\mathbf{M}_{hd}(e^{j\omega})=\mathbf{W}(e^{j\omega})\mathbf{A}\mathbf{Z}(e^{j\omega}), \tag{101}$$

where $\mathbf{A}$ is a fixed matrix independent of ω, $\mathbf{Z}(e^{j\omega})$ is an invertible diagonal matrix with diagonal elements $Z_\ell(e^{j\omega})$ satisfying $\operatorname{ess\,sup}|Z_\ell(e^{j\omega})|=1$, and $\mathbf{W}(e^{j\omega})$ is an arbitrary invertible matrix. Going back to the bandlimited frame (95) it can be easily seen that with $h_\ell(t)=\phi_\ell(t)$, (101) is satisfied. Indeed,

$$\overline{H}_\ell(\omega)D_r(\omega)=\begin{cases}\frac{T}{N}e^{j\omega(\ell-1)T/N}e^{-j\omega(r-1)T/m}, & \omega\in(-\pi N/T,\pi N/T];\\ 0, & \text{otherwise.}\end{cases} \tag{102}$$

Therefore,

$$R_{h_\ell d_r}(e^{j\omega})=e^{j\omega(\ell-1)/N}e^{-j\omega(r-1)/m}f(\ell,r), \tag{103}$$

where $f(\ell,r)$ is a function only of the indices ℓ,r and not the frequency ω. Choosing $Z_r(e^{j\omega})=e^{-j\omega(r-1)/m}$ and $\mathbf{W}(e^{j\omega})$ as a diagonal matrix with diagonal elements $W_\ell(e^{j\omega})=e^{j\omega(\ell-1)/N}$ leads to the representation (101).

When $\mathbf{M}_{hd}(e^{j\omega})$ has the form (101), the system of equations (100) becomes

$$\mathbf{d}(e^{j\omega})=\mathbf{A}\mathbf{Z}(e^{j\omega})\boldsymbol{\gamma}(e^{j\omega})=\mathbf{A}\mathbf{a}(e^{j\omega}), \tag{104}$$

where we denoted $\mathbf{d}(e^{j\omega})=\mathbf{W}^{-1}(e^{j\omega})\mathbf{c}(e^{j\omega})$, $\mathbf{a}(e^{j\omega})=\mathbf{Z}(e^{j\omega})\boldsymbol{\gamma}(e^{j\omega})$ and used (101). Clearly, $\|\mathbf{a}(e^{j\omega})\|_{2,0}=\|\boldsymbol{\gamma}(e^{j\omega})\|_{2,0}$ because $\mathbf{Z}(e^{j\omega})$ is invertible and diagonal. Therefore, the sparse decomposition problem is equivalent to finding $\mathbf{a}(e^{j\omega})$ satisfying (104) and such that $\|\mathbf{a}(e^{j\omega})\|_{2,0}$ is minimized.

As in the previous section, the sparsest $\mathbf{a}(e^{j\omega})$ can be determined by first converting (104) to a finite MMV problem, in which we seek the sparsest matrix $\mathbf{U}$ that satisfies $\mathbf{C}=\mathbf{A}\mathbf{U}$ where the columns of $\mathbf{C}$ form a basis for the span of $\{\mathbf{W}^{-1}(e^{j\omega})\mathbf{c}(e^{j\omega}),-\pi\le\omega\le\pi\}$. The matrix $\mathbf{U}$ can be determined by solving the convex problem

$$\min_{\mathbf{U}}\|\mathbf{s}(\mathbf{U})\|_1\quad \text{s.t. } \mathbf{C}=\mathbf{A}\mathbf{U}. \tag{105}$$

From Proposition 7 it follows that the unique sparse matrix $\mathbf{U}$ can be recovered as long as $\mu(\mathbf{A})$ satisfies (92). Once we determine the non-zero rows S in $\mathbf{U}$, we can find the non-zero sequences $\boldsymbol{\gamma}^S[n]$ by noting that from Proposition 7 the columns $\mathbf{A}_S$ of $\mathbf{A}$ corresponding to S are linearly independent. Therefore,

$$\boldsymbol{\gamma}^S(e^{j\omega})=\mathbf{Z}_S^{-1}(e^{j\omega})(\mathbf{A}_S^H\mathbf{A}_S)^{-1}\mathbf{A}_S^H\mathbf{W}^{-1}(e^{j\omega})\mathbf{c}(e^{j\omega}). \tag{106}$$

If (101) is not satisfied, but instead $\boldsymbol{\gamma}(e^{j\omega})$ is rich, so that the support of every M set of vectors (for M different frequencies) is equal to the span of the entire set, then we can still convert the problem into an MMV. To do this, we choose M frequency values and seek the set of vectors $\boldsymbol{\gamma}_i, 1\le i\le M$ with the sparsest joint support that satisfy

$$\mathbf{c}(e^{j\omega_i})=\mathbf{M}_{hd}(e^{j\omega_i})\boldsymbol{\gamma}_i,\quad 1\le i\le M. \tag{107}$$

Once the support is determined, we can find the non-zero sequences $\boldsymbol{\gamma}^S[n]$ using (106).

We have outlined a concrete method to find the sparsest representation of a signal $x(t)$ in $\mathcal{A}$ in terms of an arbitrary dictionary. In our proposed approach, the reconstruction is performed with respect to the samples $c_\ell[n] = \langle h_\ell(t-nT), x(t)\rangle$. We may alternatively view our algorithm as a method to reconstruct $x(t)$ from these samples assuming the knowledge that $x(t)$ has a sparse decomposition in the given dictionary. Thus, our results can also be interpreted as a reconstruction method from a given set of samples, and in that sense complements the results of [22].

VII. Conclusion

In this paper, we extended the recent line of work on generalized uncertainty principles to the analog domain, by considering sparse representations in SI bases. We showed that there is a fundamental limit on the ability to sparsely represent an analog signal in an infinite-dimensional SI space in two orthonormal bases. The sparsity bound is similar to that obtained in the finite-dimensional discrete setting: In both cases the joint sparsity is limited by the inverse coherence of the bases. However, while in the finite setting, the coherence is defined as the maximal absolute inner product between elements from each basis, in the analog problem the coherence is the maximal absolute value of the sampled cross-spectrum between the signals.

As in the finite domain, we can show that the proposed uncertainty relation is tight by providing a concrete example in which it is achieved. Our example mimics the finite setting by considering the class of bandlimited signals as the signal space. This leads to a Fourier representation that is defined over a finite, albeit continuous, interval. Within this space we can achieve the uncertainty limit by considering a bandlimited train of LPFs. This choice of signal resembles the spike train which is known to achieve the uncertainty principle in the discrete setting.

Finally, we treated the problem of sparsely representing an analog signal in an overcomplete dictionary. Building upon the uncertainty principle and recent works in the area of compressed sensing for analog signals, we showed that under certain conditions on the Fourier domain representation of the dictionary, the sparsest representation can be found by solving a finite-dimensional convex optimization problem. The fact that sparse decompositions can be found by solving a convex optimization problem has been established in many previous works in compressed sensing in the finite setting. The additional twist here is that even though the problem has infinite dimensions, it can be solved exactly by a finite-dimensional program in many interesting cases.

In this paper we have focused on analog signals in SI spaces. A very interesting further line of research is to extend these ideas and notions to a larger class of analog signals, leading to a broader notion of analog sparsity and analog compressed sensing.

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Appendix I
Proof of Proposition 1

To prove the proposition, note that

$$\int_{-\infty}^{\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\sum_{\ell=1}^{N}A_\ell(e^{j\omega T})\Phi_\ell(\omega)\right|^2 d\omega, \quad (108)$$

where the last equality follows from (7). To simplify (108) we rewrite the integral over the entire real line, as the sum of integrals over intervals of length $2\pi/T$:

$$\int_{-\infty}^{\infty}X(\omega)d\omega = \int_0^{\frac{2\pi}{T}}\sum_{k=-\infty}^{\infty}X\left(\omega-\frac{2\pi}{T}k\right)d\omega, \quad (109)$$

for all $X(\omega)$. Substituting into (108) and using the fact that $A_\ell(e^{j\omega T})$ is $2\pi/T$-periodic, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|x(t)|^2dt &= \\ &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{T}}\sum_{k=-\infty}^{\infty}\left|\sum_{\ell=1}^{N}A_\ell(e^{j\omega T})\Phi_\ell\left(\omega-\frac{2\pi}{T}k\right)\right|^2 d\omega \\ &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}}\sum_{\ell=1}^{N}\sum_{r=1}^{N}\overline{A_\ell}(e^{j\omega T})A_r(e^{j\omega T})R_{\phi_\ell\phi_r}(e^{j\omega})d\omega \\ &= \frac{T}{2\pi}\int_0^{\frac{2\pi}{T}}\sum_{\ell=1}^{N}\left|A_\ell(e^{j\omega T})\right|^2 d\omega, \quad (110)\end{aligned}$$

where we used (17).

Appendix II
Proof of Proposition 5

To prove the proposition, we first note that since $\phi_\ell(t)$ is in $\mathcal{A}$ for each ℓ, we can express it as

$$\phi_\ell(t) = \sum_{r=1}^{N}\sum_{n\in\mathbb{Z}}a_r^\ell[n]\psi_r(t-nT) \quad (111)$$

for some coefficients $a_r^\ell[n]$ with Fourier transform $A_r^\ell(e^{j\omega})$. We have shown in the proof of Theorem 2 that the orthonormality condition (17) of $\psi_\ell(t)$ implies that

$$A_r^\ell(e^{j\omega}) = \overline{R}_{\phi_\ell\psi_r}(e^{j\omega}). \quad (112)$$

Now, since $\{\phi_\ell(t-nT)\}$ is an orthonormal basis for $\mathcal{A}$, $R_{\phi_\ell\phi_r}(e^{j\omega}) = \delta_{\ell,r}$. From (111),

$$\begin{aligned}R_{\phi_\ell\phi_r}(e^{j\omega}) &= \sum_{m=1}^{N}\sum_{s=1}^{N}\overline{A}_m^\ell(e^{j\omega})A_s^r(e^{j\omega})R_{\psi_m\psi_s} \\ &= \sum_{m=1}^{N}\overline{A}_m^\ell(e^{j\omega})A_m^r(e^{j\omega}) \\ &= [\mathbf{M}_{\phi\psi}(e^{j\omega})]_\ell[\mathbf{M}_{\phi\psi}(e^{j\omega})]_r^H, \quad (113)\end{aligned}$$

where $[\mathbf{C}]_r$ denotes the rth row of $\mathbf{C}$. The second equality in (113) follows from the orthonormality of $\{\psi_\ell(t-nT)\}$, and the last equality is a result of (112). Since $R_{\phi_\ell \phi_r}(e^{j\omega}) = \delta_{\ell,r}$, it follows from (113) that the matrix $\mathbf{M}_{\phi\psi}(e^{j\omega})$ is unitary for all ω.

Since $\mathbf{M}_{\phi\psi}(e^{j\omega})$ is unitary, the coherence $\mu(\mathbf{I}, \mathbf{M}_{\phi\psi}(e^{j\omega}))$ is well defined. Now for any unitary $\mathbf{U}$, $\mu(\mathbf{I}, \mathbf{U}) = \max_{i,j} |U_{ij}|$. In addition, $\mu(\Phi, \Psi) = \max_{i,j} \sup_\omega |[\mathbf{M}_{\phi\psi}(e^{j\omega})]_{ij}|$, so that $\mu(\mathbf{I}, \mathbf{M}_{\phi\psi}(e^{j\omega})) \le \mu(\Phi, \Psi)$, completing the proof.

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