ANALOG COMPRESSED SENSING

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ABSTRACT

A traditional assumption underlying most data converters is that the signal should be sampled at a rate exceeding twice the highest frequency. Practical signals often posses a sparse structure so that a large part of the bandwidth is not exploited. In this paper, we consider a framework for utilizing this sparsity in order to sample such analog signals at a low rate. By relying on results developed in the context of compressed sensing (CS) of finite-length vectors, we develop a general framework for low-rate sampling of signals in shift-invariant spaces. In contrast to the problems treated in the context of CS, here we explicitly consider sampling of analog signals for which no underlying finite-dimensional model exists.

Index terms – Sampling methods, Compressed sensing.

1. INTRODUCTION

Digital applications rely on sampling a continuous-time signal to obtain a discrete-time representation. A class of signals that play an important role in sampling theory are shift-invariant (SI) spaces [1, 2]. Such functions can be expressed as linear combinations of shifts of a set of generators with period T. This model encompasses many signals used in communication and signal processing.

Any signal $x(t)$ in a SI space generated by m functions shifted with period T can be perfectly recovered from m sequences, obtained by filtering $x(t)$ with a bank of m filters and uniformly sampling their outputs at times $t = nT$. The overall sampling rate of this scheme is m/T. In Section 2 we show explicitly how to recover $x(t)$ from these samples using an appropriate filter bank. If $x(t)$ is generated by a known subset k of the m generators, then it suffices to sample at an average rate of k/T. However, a more difficult question is whether the rate can be reduced if we know that only k of the generators are active, but we do not know in advance which ones. Since in principle $x(t)$ may be comprised of any of the generators, it may seem at first that the rate cannot be lower than m/T.

This question is a special case of sampling a signal in a union of subspaces [3, 4]. In [3] necessary and sufficient conditions where derived to ensure invertibility of a sampling operator over such a union. In our problem this essentially reduces to the requirement that the sampling rate is at least $2k/T$. However, no concrete algorithms where provided to recover such a signal from a given set of samples.

One case of sampling on a union of spaces that has been studied extensively is compressed sensing (CS). In this setting, the goal is to recover a length-m vector $\mathbf{x}$ from $p < m$ linear measurements, where $\mathbf{x}$ has at most k non-zero elements [5, 6]. Many methods have been proposed to efficiently recover $\mathbf{x}$ [5]. Algorithms for signal recovery in an arbitrary finite union of subspaces where developed in [4]. A fundamental difference between our problem and these results is that we aim to reconstruct a continuous signal, while the classical CS problem is the recovery of discrete and finite vectors. The methods developed in the context of CS rely on the finite nature of the problem and cannot be immediately adopted to infinite-dimensional settings without discretization or heuristics.

The model we treat here is inherently infinite-dimensional as it involves an infinite sequence of samples from which we would like to recover an analog signal with infinitely many parameters. A special case is when $x(t)$ has a multiband structure [7]. Explicit sub-Nyquist reconstruction schemes that ensure perfect recovery of $x(t)$ at the minimal possible rate were recently developed in [7]. The important aspect of these results is that they do not require knowledge of the band locations. The cornerstone of these algorithms is a set of operations grouped under a block named continuous-to-finite (CTF). The CTF allows to transform the continuous reconstruction problem into a finite dimensional equivalent, without discretization or heuristics. The resulting problem is formulated within the CS framework, and thus can be solved efficiently using known tractable algorithms from this area. The CTF block has been further studied in [8].

Here, we combine ideas from standard sampling theory and CS via the CTF block in order to develop a more general framework for analog compressed sensing. In particular, we show explicitly how signals in a SI union created by m generators with period T, can be sampled and stably recovered at a rate much lower than m/T using CS algorithms. Essentially, we use a front-end borrowed from analog sampling theory, together with a mixing matrix that satisfies the CS requirements. Combining these two components via the CTF block allows to sample analog signals at a low rate, while still ensuring stable and efficient recovery for a large class of problems.

2. SHIFT-INVARIANT SUBSPACE SAMPLING

Traditional sampling theory deals with the recovery of a function $x(t)$ in L_2 from its uniform samples. To this end, it is typically assumed that $x(t)$ lies in a subspace $\mathcal{A}$. A common choice is a finitely-generated SI subspace, generated by m functions $a_\ell(t)$ [1, 2]:

$$\mathcal{A} = \left\{ x(t) = \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}} d_\ell[n] a_\ell(t - nT) : d_\ell[n] \in \ell_2 \right\}, \quad (1)$$

where T is the sampling period. In the Fourier domain,

$$X(\omega) = \sum_{\ell=1}^{m} D_\ell(e^{j\omega T}) A_\ell(\omega), \quad (2)$$

where

$$D_\ell(e^{j\omega T}) = \sum_{n\in\mathbb{Z}} d_\ell[n] e^{j\omega nT}, \quad (3)$$

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and is $2\pi/T$ periodic. To emphasize the fact that the discrete-time Fourier transform (DTFT) is 2π-periodic we use the notation $D(e^{j\omega})$.

To guarantee a unique stable representation of any $x(t) \in \mathcal{A}$, the generators $a_\ell(t)$ are chosen to form a Riesz basis for L_2, so that

$$\alpha\mathbf{I} \preceq \mathbf{M}_{AA}(e^{j\omega}) \preceq \beta\mathbf{I}, \quad \text{a.e. } \omega, \tag{4}$$

for some constants $0 < \alpha \leq \beta < \infty$, where

$$\mathbf{M}_{AA}(e^{j\omega}) = \begin{bmatrix} \phi_{A_1A_1}(e^{j\omega}) & \dots & \phi_{A_1A_m}(e^{j\omega}) \\ \vdots & \vdots & \vdots \\ \phi_{A_mA_1}(e^{j\omega}) & \dots & \phi_{A_mA_m}(e^{j\omega}) \end{bmatrix}. \tag{5}$$

Here

$$\phi_{SA}(e^{j\omega}) = \frac{1}{T}\sum_{k\in\mathbb{Z}} S^*\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right) A\left(\frac{\omega}{T} - \frac{2\pi}{T}k\right), \tag{6}$$

and is the DTFT of the cross correlation sequence $r_{sa}[n] = \langle s(t-nT), a(t)\rangle$, where

$$\langle s(t), a(t)\rangle = \int_{t=-\infty}^{\infty} s^*(t)a(t)dt, \tag{7}$$

and $(\cdot)^*$ denotes the complex conjugate. Throughout the paper we assume that (4) is satisfied.

Since $x(t)$ lies in a space generated by m functions, we sample it by pre-filtering the signal with m filters $s_\ell^*(-t)$ [9], and then sampling their outputs at times $t = nT$. The sequence of samples $c_\ell[n]$ can then be represented as the inner products

$$c_\ell[n] = \langle s_\ell(t-nT), x(t)\rangle, \quad 1 \leq \ell \leq m. \tag{8}$$

Taking the Fourier transform of $c_\ell[n]$ and using (1) [9]

$$\mathbf{c}(e^{j\omega}) = \mathbf{M}_{SA}(e^{j\omega})\mathbf{d}(e^{j\omega}), \tag{9}$$

where $\mathbf{c}(e^{j\omega}), \mathbf{d}(e^{j\omega})$ are the vectors whose ℓth elements $C_\ell(e^{j\omega}), D_\ell(e^{j\omega})$ are the DTFTs of $c_\ell[n], d_\ell[n]$ respectively, and $\mathbf{M}_{SA}(e^{j\omega})$ is the matrix with elements $\phi_{S_iA_\ell}(e^{j\omega})$, defined by (6). Consequently, as long as $\mathbf{M}_{SA}(e^{j\omega})$ is invertible a.e. in ω, $\mathbf{d}(e^{j\omega})$ can be recovered from $\mathbf{c}(e^{j\omega})$ by $\mathbf{d}(e^{j\omega}) = \mathbf{M}_{SA}^{-1}(e^{j\omega})\mathbf{c}(e^{j\omega})$. The signal $x(t)$ is then constructed by modulating each of the outputs $d_\ell[n]$ by a sequence of impulses $\sum_n \delta(t-nT)$ with period T, and filtering with the corresponding analog filter $a_\ell(t)$. In practice, we may choose any set of sampling functions $\{s_\ell(t)\}$ for which $\mathbf{M}_{SA}(e^{j\omega})$ is stably invertible in order to guarantee perfect reconstruction of $x(t)$. This approach consists of m sequences of samples, each at rate $1/T$, leading to an average sampling rate of m/T.

The resulting sampling scheme can be interpreted as a basis expansion of any $x(t)$ in $\mathcal{A}$. To see this, let

$$\mathbf{v}(\omega) = \mathbf{M}_{SA}^{-*}(e^{j\omega T})\mathbf{s}(\omega), \tag{10}$$

where $\mathbf{v}(\omega), \mathbf{s}(\omega)$ are the vectors with ℓth elements $V_\ell(\omega), S_\ell(\omega)$, with $s_\ell(t)$ being an arbitrary set of functions such that $\mathbf{M}_{SA}(e^{j\omega})$ is invertible, and $(\cdot)^{-*}$ denoting the inverse of the matrix conjugate. Then it can be easily seen that

$$x(t) = \sum_{\ell=1}^{m}\sum_{n\in\mathbb{Z}} \langle v_\ell(t-nT), x(t)\rangle a_\ell(t-nT), \tag{11}$$

which is a basis expansion of $x(t)$. The functions $\{v_\ell(t-nT)\}$ have the property that they are biorthogonal to $\{a_\ell(t-nT)\}$, namely,

$$\langle v_\ell(t-nT), a_i(t-rT)\rangle = \delta_{\ell i}\delta_{nr}, \tag{12}$$

where $\delta_{\ell i} = 1$ if $\ell = i$, and 0 otherwise. The inner products in (12) can be obtained by filtering $x(t)$ with the filters $v_\ell^*(-t)$, and uniform sampling the outputs at times nT. Note that although $s_\ell(t)$ are arbitrary, the biorthogonal vectors in $\mathcal{A}$ are unique.

2.1. Union of Shift-Invariant Subspaces

Evidently, when subspace information is available, perfect reconstruction is possible using a simple filter bank. A more interesting scenario is when $x(t)$ lies in a union of SI subspaces. Here we consider the case in which

$$x(t) = \sum_{|\ell|=k}\sum_{n\in\mathbb{Z}} d_\ell[n]a_\ell(t-nT), \tag{13}$$

where the notation $|\ell| = k$ means a sum over k elements, so that only k out of the m sequences $d_\ell[n]$ in the sum (13) are not identically zero. Our goal is to recover $x(t)$ of (13) from a given set of samples.

In principle, if we know which k sequences are non-zero, then $x(t)$ can be recovered from samples at the output of k filters. The average sampling rate in this case is k/T since we have k sequences of samples, each at rate $1/T$. Alternatively, even without knowledge of the active subspaces, we can recover $x(t)$ from samples at the output of m filters resulting in a sampling rate of m/T. Although this strategy does not require knowledge of the active subspaces, the price we pay is an increase in sampling rate.

The goal in this paper is to develop algorithms for recovering $x(t)$ from a set of $2k \leq p < m$ sampling sequences, obtained by sampling the outputs of p filters at rate $1/T$. To this end we rely on ideas developed in the CS literature. Specifically, we show that by proper choice of the sampling filters $s_i(t), 1 \leq i \leq p$ recovery of $x(t)$ can be translated into an infinite measurement vector (IMV) model [8, 7], which is a broad framework for many CS-type problems. We may then rely on results obtained in that context to recover $x(t)$ from the given samples. As we show, if we are not concerned with computational complexity and stability issues, then $p = 2k$ sampling sequences suffice to recover $x(t)$ by brute-force solving an optimization problem with combinatorial complexity. By slightly increasing the number of filters, efficient reconstruction can be obtained by solving a finite-dimensional convex optimization problem.

3. COMPRESSED SENSING

A central part of our paradigm relies on CS of finite vectors. The standard CS problem is to recover a k-sparse vector $\mathbf{x}$ of length m from $p < m$ measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a $p \times m$ matrix. A sufficient condition to ensure a unique k-sparse solution is that $\mathbf{A}$ has Kruskal-rank $\sigma(\mathbf{A}) \geq 2k$, where $\sigma(\mathbf{A})$ is the maximal number q such that every q columns are linearly independent. The unique $\mathbf{x}$ can be found by solving the problem [5]:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}, \tag{14}$$

where $\|\mathbf{x}\|_0$ denotes the number of non-zero elements in $\|\mathbf{x}\|$. Since (14) is NP-hard, several alternative algorithms have been proposed that have polynomial complexity, such as ℓ_1 optimization [5, 6]. For a given sparsity, these techniques are guaranteed to recover $\mathbf{x}$ as long as certain conditions on $\mathbf{A}$ are satisfied. These efficient methods require a number of measurements p that is larger than $2k$, however still considerably smaller than m. For example, if $\mathbf{A}$ is chosen as p random rows from the Fourier matrix, then the ℓ_1 program will recover $\mathbf{x}$ with high probability as long as $p \geq ck\log m$ where c is a constant.

These results have been generalized to the multiple-measurement vector (MMV) problem in which the goal is to recover an unknown matrix $\mathbf{X}$ from measurements $\mathbf{Y} = \mathbf{AX}$, where $\mathbf{X}$ has at most k non-zero rows. Here again, if $\sigma(\mathbf{A}) \geq 2k$, then there is a unique $\mathbf{X}$ consistent with $\mathbf{Y}$, which can be found by combinatorial optimization. Efficient algorithms that recover $\mathbf{X}$ under certain conditions on $\mathbf{A}$ have also been proposed for this problem [10],[8]. In the sequel, when we refer to a matrix $\mathbf{A}$ satisfying the CS requirements, we mean that with high probability it may be used to sense a sparse vector $\mathbf{x}$ in such a way that $\mathbf{x}$ can be recovered efficiently from the given measurements.

3.1. Infinite Model

A recent extension of the MMV model that will be instrumental in our sampling framework, is to the IMV case in which there are infinitely-many unknown vectors $\mathbf{x}$ and measurement vectors $\mathbf{y}$:

$$\mathbf{y}(\lambda) = \mathbf{A}\mathbf{x}(\lambda), \quad \lambda \in \Lambda. \tag{15}$$

Here Λ is a possibly infinite set, which may be uncountable, such as the frequencies $\omega \in [-\pi, \pi)$. The k-sparse IMV model assumes that the vectors $\{\mathbf{x}(\lambda)\}$, denoted $\mathbf{x}(\Lambda)$, are jointly sparse, so that the non-zero elements are supported on a fixed location set of size k [8].

As in the MMV case, $\sigma(\mathbf{A}) \geq 2k$ ensures a unique solution. However, a major difficulty with the IMV model is that it is not clear how to recover the entire set $\mathbf{x}(\Lambda)$ since there are infinitely many equations to solve. In [8] it was shown that (15) can be converted to a finite MMV without loosing any information by a set of operations referred to as the continuous-to-finite (CTF) block. The essential idea is to first recover the support S of $\mathbf{x}(\Lambda)$ by solving a finite MMV, and then reconstruct $\mathbf{x}(\Lambda)$ from the data $\mathbf{y}(\Lambda)$ and knowledge of S. The reason for this separation is that once S is known, (15) becomes invertible.

To see this, let $\mathbf{A}_S$ denote the matrix containing the columns of $\mathbf{A}$ with indices in S. Since $\mathbf{x}(\Lambda)$ is k-sparse, $|S| \leq k$. Because $\sigma(\mathbf{A}) \geq k$, $\mathbf{A}_S$ consists of linearly independent columns implying that $\mathbf{A}_S^\dagger \mathbf{A}_S = \mathbf{I}$, where $\mathbf{A}_S^\dagger = (\mathbf{A}_S^* \mathbf{A}_S)^{-1} \mathbf{A}_S^*$ is the pseudo-inverse of $\mathbf{A}_S$. Using S, (15) can be written as

$$\mathbf{y}(\lambda) = \mathbf{A}_S \mathbf{x}^S(\lambda), \quad \lambda \in \Lambda, \tag{16}$$

where the superscript $\mathbf{x}^S(\lambda)$ is the vector that consists of the entries of $\mathbf{x}(\lambda)$ in the locations S. Multiplying (15) by $\mathbf{A}_S^\dagger$ gives

$$\mathbf{x}^S(\lambda) = \mathbf{A}_S^\dagger \mathbf{y}(\lambda), \quad \lambda \in \Lambda. \tag{17}$$

Since the elements in $\mathbf{x}(\lambda)$ not supported on S are all zero, (17) leads to exact recovery of $\mathbf{x}(\Lambda)$ once the finite set S is correctly identified.

To find S we rely on the following theorem from [8] which is based on the fact that $\operatorname{span}(\mathbf{y}(\Lambda))$ is finite since $\mathbf{y}(\Lambda)$ has finite length for every λ:

Theorem 1 *Suppose (15) has a unique k-sparse solution set $\mathbf{x}(\Lambda)$ with support S and that $\sigma(\mathbf{A}) \geq 2k$. Let $\mathbf{V}$ be a matrix with column span equal to $\operatorname{span}(\mathbf{y}(\Lambda))$. Then, the linear system $\mathbf{V} = \mathbf{AU}$ has a unique k-sparse solution $\mathbf{U}$ whose support is equal to S.*

Theorem 1 allows to avoid the infinite structure of (15), and instead find the set S by solving the single MMV system $\mathbf{V} = \mathbf{AU}$.

To apply Theorem 1 we need to construct a matrix $\mathbf{V}$ with column span equal to $\operatorname{span}(\mathbf{y}(\Lambda))$. A possible method is to form the matrix $\mathbf{Q} = \int \mathbf{y}(\lambda)\mathbf{y}^*(\lambda)d\lambda$ if Λ is continuous, or $\mathbf{Q} = \sum \mathbf{y}(\lambda)\mathbf{y}^*(\lambda)$ if Λ is a discrete set. Then every matrix $\mathbf{V}$ satisfying $\mathbf{Q} = \mathbf{V}\mathbf{V}^*$ has a column span equal to $\operatorname{span}(\mathbf{y}(\Lambda))$. In particular, the columns of $\mathbf{V}$ can be chosen as the eigenvectors of $\mathbf{Q}$ multiplied by the square-root of the corresponding eigenvalues. The CTF block is depicted in Fig. 1.

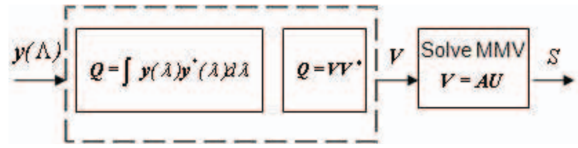

Fig. 1. Continuous-to-finite (CTF) block.

4. COMPRESSED SENSING OF SI SIGNALS

We now combine the ideas of Sections 2 and 3 in order to recover $x(t)$ of the form (13), where we do not know which k subspaces are chosen. Our sampling strategy is to filter $x(t)$ with $p < m$ sampling filters $s_i(t)$, and uniformly sample their outputs at rate $1/T$.

To design $s_i(t)$ we first choose a $p \times m$ matrix $\mathbf{A}$ to sample a k-sparse vector $\mathbf{x}$ of length m. Using standard CS results we can select $\mathbf{A}$ to either guarantee exact recovery with combinatorial optimization assuming $p \geq 2k$, or lead to recovery (possibly only with high probability) using efficient algorithms with $p > 2k$. We show below that the same $\mathbf{A}$ chosen for this finite problem can be used for analog CS. Specifically, our design relies on two ingredients:

1. A matrix $\mathbf{A}$ that solves a discrete CS problem in the dimensions m (vector length) and k (sparsity).
2. A set of functions $h_i(t), 1 \leq i \leq m$ which can be used to sample and reconstruct the entire set of generators $a_i(t), 1 \leq i \leq m$, namely such that $\mathbf{M}_{HA}(e^{j\omega})$ is stably invertible.

Note that $h_i(t)$ can be used to recover $x(t)$; however, since there are m functions this results in more measurements than actually needed.

4.1. Union of Discrete Sequences

In order to motivate our choice of sampling functions, we begin by considering the simpler scenario in which our goal is to recover the discrete-time vector sequence $\mathbf{d}[n]$ with ℓth component $d_\ell[n]$, where only k out of the m sequences $d_\ell[n]$ are non-zero. We then show that using sampling functions $h_\ell(t)$ for which $\mathbf{M}_{HA}(e^{j\omega})$ is invertible, we can convert our problem to this discrete counterpart.

The problem of recovering a union of discrete sequences can be solved by using the IMV model of Section 3.1. Indeed, suppose we measure $\mathbf{d}[n]$ using a matrix $\mathbf{A}$ of size $p \times m$ that allows for CS of k-sparse vectors of length m. Then, at each n, we have

$$\mathbf{y}[n] = \mathbf{A}\mathbf{d}[n], \quad n \in \mathbb{Z}. \tag{18}$$

This system is an IMV model: The infinite set of vectors $\{\mathbf{d}[n], n \in \mathbb{Z}\}$ have a joint sparsity pattern since at most k sequences $d_\ell[n]$ are non-zero. As we described in Section 3.1, such a system of equations can be solved by transforming it into an equivalent MMV, whose recovery properties are determined by those of $\mathbf{A}$. Since $\mathbf{A}$ was designed such that CS techniques will work, we are guaranteed that $\mathbf{d}[n]$ can be recovered for each n using the CTF block. Instead of solving (18) we may also consider the Frequency-domain equations:

$$\mathbf{y}(e^{j\omega}) = \mathbf{A}\mathbf{d}(e^{j\omega}), \quad 0 \leq \omega < 2\pi, \tag{19}$$

where $\mathbf{y}(e^{j\omega}), \mathbf{d}(e^{j\omega})$ are the vectors with components $Y_\ell(e^{j\omega}), D_\ell(e^{j\omega})$. In principle, we may apply the CTF block to either representations, depending on which choice offers a simpler method for determining a basis $\mathbf{V}$ for the range of $\{\mathbf{y}(\Lambda)\}$.

4.2. Biorthogonal Expansion

We have shown that given the ability to sample the m sequences $d_\ell[n]$ we can recover them exactly from $p < m$ discrete-time sequences. We now utilize these results to compressively sample $x(t)$.

Our approach is based on two conceptual steps: We first use a biorthogonal set of vectors in order to obtain the m coefficient sequences $d_\ell[n]$. Specifically, we sample $x(t)$ with the filters $v_\ell^*(-t)$ given in the Fourier domain by

$$\mathbf{v}(\omega) = \mathbf{M}_{HA}^{-*}(e^{j\omega T})\mathbf{h}(\omega), \tag{20}$$

where $h_i(t)$ are any sampling functions for which $\mathbf{M}_{HA}(e^{j\omega})$ is stably invertible. In the next step, we measure $d_\ell[n]$ compressively. This is equivalent to sensing m discrete-time sequences, where only k are non-zero which can be achieved by multiplying the sequences by an $m \times p$ matrix $\mathbf{A}$ that satisfies the requirements of CS. Combining these two steps, the compressed measurement sequences $y_\ell[n]$ can be obtained directly from $x(t)$, by filtering $x(t)$ with p analog filters $s_\ell(t)$ and uniformly sampling their outputs at time nT. The sampling functions $s_\ell(t)$ are the result of concatenating the biorthogonal functions (20) with the discrete-time filter bank $\mathbf{A}$, as depicted in Fig. 2, and summarized in the following theorem.

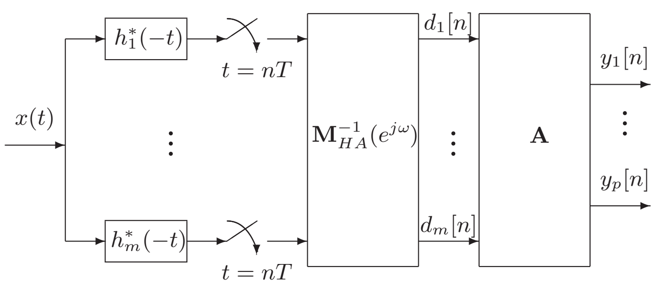

Fig. 2. Analog compressed sampling with arbitrary filters $h_i(t)$.

Theorem 2 *Let the compressed measurement sequences* $y_\ell[n], 1 \leq \ell \leq p$ *be the output of the hybrid filter bank in Fig. 2. Then* $\{y_\ell[n]\}$ *can be obtained by filtering* $x(t)$ *with* p *filters* $\{s_\ell^*(-t)\}$ *and sampling the outputs at rate* $1/T$ *with*

$$\mathbf{s}(\omega) = \mathbf{A}^*\mathbf{v}(\omega) = \mathbf{A}^*\mathbf{M}_{HA}^{-*}(e^{j\omega T})\mathbf{h}(\omega), \tag{21}$$

where $\mathbf{s}(\omega), \mathbf{h}(\omega)$ *are the vectors with* ℓ*th elements* $S_\ell(\omega), H_\ell(\omega)$.

Theorem 2 is the main result which allows for compressive sampling of analog signals. Specifically, starting from any matrix $\mathbf{A}$ that satisfies the CS requirements of finite vectors, and a set of functions $h_i(t)$ for which $\mathbf{M}_{HA}(e^{j\omega})$ is invertible, we can create sampling functions $s_i(t)$ to compressively sample $x(t)$. The sensing is performed by filtering $x(t)$ with the $p < m$ corresponding filters, and sampling their outputs at rate $1/T$. Reconstruction from the compressed measurements $y_i[n], 1 \leq i \leq p$ is obtained by applying the CTF block in order to recover the sequences $d_i[n]$. The original signal $x(t)$ is then constructed by modulating appropriate impulse trains and filtering with $a_i(t)$, as depicted in Fig. 3.

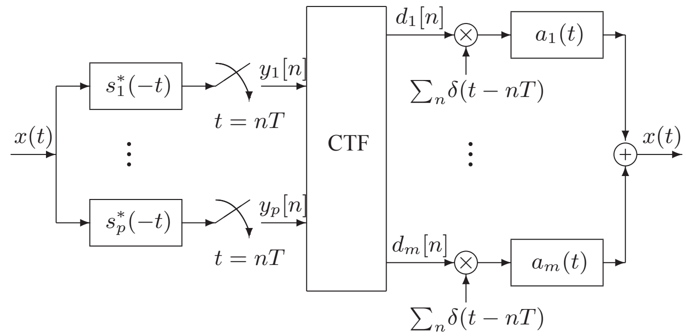

Fig. 3. Compressed sensing of analog signals.

5. CONCLUSION

We developed a general framework to treat sampling of sparse analog signals. We focused on signals that lie in a SI space generated by m kernels, where only k out of the m generators are active. The difficulty arises from the fact that we do not know which k are chosen. Our approach was based on merging ideas from standard analog sampling, with results from CS. The latter focuses on sensing sparse finite dimensional vectors. Although our problem is inherently infinite-dimensional, we showed that by using the notion of biorthogonal sampling sets and the CTF block [8, 7], we can convert our problem to a finite-dimensional CS counterpart.

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