EFFICIENT GABOR EXPANSION USING NON MINIMAL DUAL GABOR WINDOWS

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ABSTRACT

In this paper, we illustrate the computational advantages of a class of non minimum norm dual Gabor frames over the traditional minimum norm dual Gabor frame. Specifically, we demonstrate that using non minimum norm dual Gabor frames can lead to Gabor expansions that can be computed far more efficiently than the traditional expansions. Several properties of the non minimum norm Gabor frame are mentioned and some necessary and sufficient conditions are derived for the existence of this class of non minimum norm dual Gabor frames.

1. INTRODUCTION

In many applications, it is important to represent a signal in the time-frequency domain, in order to be able to localise the frequencies. Localisation of frequencies is important since many recognition techniques, both biological and computer based, are dependent on the detection of change of frequencies in time. One of the most popular ways to represent a signal in the combined time-frequency domain is the Gabor representation, since it has the important advantages uniform resolution and best resolution in the combined space (with the Gaussian window function). Gabor representation is widely used in many applications ranging from speech processing and texture segmentation to texture segmentation and object recognition among others.

Given a discrete time, finite energy signal $f[k]$ of length L, with Gabor coefficients $c_{m,n}$ given by:

$$c_{m,n} = \sum_{k=0}^{L-1} f[k] g^*[k - na] e^{-j2\pi mbk/L}, \quad (1)$$

where a and b are the shifts along the time axis and frequency axis respectively and $g[k]$ is the window function used. Essentially, we project the signal into the combined space using a translated and modulated window function. The signal $f[k]$ is reconstructed from the coefficients $c_{m,n}$ using a dual function [1]

$$f[k] = \sum_{m=0}^{\overline{b}-1} \sum_{n=0}^{\overline{a}-1} c_{m,n} \gamma_{m,n}[k], \quad (2)$$

where $\gamma_{m,n}[k] = \gamma[k - na]e^{j2\pi mbk/L}$ is a dual Gabor window, $\overline{a} = L/a \in N$ is the number of shifts along the time axis, $\overline{b} = L/b \in N$ is the number of shifts along the frequency axis and N denotes the set of natural numbers. The task is to efficiently compute a dual Gabor window $\gamma[k]$.

In order for a representation of form (2) to exist for all sequences $f[k]$ of length L, a and b must satisfy the condition $ab \leq L$ [1], [2], [3]. If $ab = L$, then a representation of form (2) exists but is unstable [2] and localisation properties are lost [1], [2]. Hence we assume that $ab < L$, in which case, the representation is overcomplete, meaning that the vectors $\{g_{m,n}[k] = g[k - na]e^{j2\pi mbk/L}\}$ are linearly dependent. As a result, there are infinite possible choices available for the dual Gabor window $\gamma[k]$. The most popular (and traditional) choice is the minimum norm dual window [3] of the window function $g[k]$ used. There are, however, two main drawbacks to this approach. The first problem is that, given an arbitrary window function, finding the minimum norm dual window is often computationally very demanding. The second is that there are many cases wherein attempting to find the minimum norm dual leads to computational instability - in Section 5 of this paper, this will be further elaborated upon.

The paper is organised as follows: in Section 2, we focus on the formulation of the problem and non minimum norm duals, in Section 3, we investigate conditions for the existence of the non minimum norm duals and in Section 4, we present our technique for reducing computational complexity. In Section 5, we present our simulations and discuss them.

2. NON MINIMUM NORM DUAL GABOR WINDOWS

We consider the expansion of a discrete time signals $f[k]$ of finite length L. Given a set of Gabor vectors $g_{m,n}[k] =$

$g[k-na]e^{j2\pi mbk/L}, k=0,1,\ldots,L-1$ that constitute an overcomplete basis for C^L, our problem is to reconstruct the signal f from the coefficients (1) which can be written in matrix form as

$$c = G^* f. \tag{3}$$

Here $f = f[k]$ is a vector of length L, c is a vector of length $\overline{a}\overline{b}$ with the elements of $\{c_{m,n}\}$, and G is a matrix of size $L \times \overline{a}\overline{b}$, given by

$$G = \begin{bmatrix} g_{0,0}[0] & \cdots & g_{\overline{a}-1,\overline{b}-1}[0] \\ g_{0,0}[1] & \cdots & g_{\overline{a}-1,\overline{b}-1}[1] \\ \vdots & \vdots & \vdots \\ g_{0,0}[L-1] & \cdots & g_{\overline{a}-1,\overline{b}-1}[L-1] \end{bmatrix}. \tag{4}$$

We reconstruct the signal f from the coefficients c using (2), where the dual Gabor matrix satisfies the equation

$$\Gamma G^* = I. \tag{5}$$

Here Γ is the matrix with columns $\gamma_{m,n}[k] = \gamma[k-na]e^{j2\pi mbk/L}$. The minimum norm dual is given by $\Gamma = G^{*\dagger} = (GG^*)^{-1}G$, where $(\cdot)^\dagger$ denotes the minimum norm pseudoinverse. Since the vectors $g_{m,n}[k]$ span the space C^L, the matrix GG^* is always invertible. However, the creation of the matrix GG^* and the computation of its inverse are non-trivial tasks. In order to overcome this problem, we propose a dual of the form $\{d_{m,n} = d[k-na]e^{j2\pi mbk/L}\}$ whose vectors span C^L and are columns of the matrix

$$D = (HG^*)^{-1}H, \tag{6}$$

where H is a matrix that corresponds to an overcomplete basis for C^L, with vectors $h_{m,n}[k] = h[k-na]e^{j2\pi mbk/L}$ and has the same dimensions as G. The advantage here is that since we control H the computational task can be eased out considerably and the properties of the dual window can be tailored to the needs of the user, as we show in Section 4. Further, in Section 5, we show that a correct choice of H can also improve the stability of the expansion.

3. PROPERTIES OF NON MINIMUM NORM DUAL

We first discuss conditions under which a dual of the form (6) exists. A necessary and sufficient condition is given by the following theorem.

Theorem 1 The non minimum norm dual Gabor frame of the form $D = (HG^*)^{-1}H$ exists iff $\mathcal{R}(G^*) \cap \mathcal{N}(H) = \{0\}$, where $\mathcal{R}(\cdot)$ is the range and $\mathcal{N}(\cdot)$ is the null space.

Proof: It can be easily seen that if the columns of H span the space C^L, then the columns of D also span C^L provided the matrix HG^* is invertible. So it is sufficient to show that the matrix HG^* is invertible or equivalently, that if $HG^*f = 0$, then $f = 0$. Let us assume that $f \neq 0$. Since the matrix G has a rank of L, we know that $c = G^*f \neq 0$. Therefore, $HG^*f = 0$ if and only if $Hc = 0$, which implies that $c \in \mathcal{N}(H)$. In addition, $c \in \mathcal{R}(G^*)$, so that if $\mathcal{N}(H) \cup \mathcal{R}(G^*) = \{0\}$, then $Hc \neq 0$, and HG^* is invertible. The reverse implication is obvious. □

Theorem 1 provides a sufficient condition for the existence of the dual Gabor window of the kind shown in (6). The following theorem gives a necessary condition for the existence of the dual, the proof of which can be found in [4].

Theorem 2 A necessary condition for the matrix HG^* to be invertible is given by $\sum_{n=0}^{\overline{a}-1} g^*[k-na+q\overline{b}]h[k-na] \neq 0, q = 0,\ldots,b-1, k = 0,1,\ldots,L-1$.

Some properties of the matrix $P = (HG^*)$, are mentioned below. All these properties are also valid for the matrix $S = (GG^*)$. Owing to paucity of space, many properties and their proofs have not been given here. Some of the proofs are parallel to the ones found in [3] and others are found in [4].

1. **Banded structure of the matrix:** The nonzero entries of P are located on the main diagonal and every k-th sub-diagonal of P for $k = \pm\overline{b}, \pm 2\overline{b}, \cdots, \pm(b-1)\overline{b}$. The $P_{k,l}$-th entry is given by $P_{k,l} = \overline{b}\sum_{n=0}^{\bar{a}-1} h[k-na]g^*[l-na]$ if $\mid k-l \mid$ is divisible by $\overline{b}$ and 0 otherwise . This shows the extreme sparsity of the matrix.

2. **Periodicity of the matrix:** Every diagonal and $l\overline{b}, l \in 0,\ldots,b-1$ sub-diagonal of the matrix P is periodic with period a. This property has very important implications in calculating the dual for certain choices of a and b as will be shown later in this section.

3. **Block Circulant Structure:** If the first $a \times L$ block of the matrix P is A, then the k-th block is the first block rotated by $(k-1)a$ for $k = 1, 2, \cdots, \overline{a}-1$. This property shows that the entire matrix can be created by rotating the first $a \times L$ block so that the matrix is block circulant.

Using the above mentioned properties, another necessary and sufficient condition can be formulated for the frame of the form $D = (HG^*)^{-1}H$ to exist. We showed in Theorem 1 that it is sufficient to prove that HG^* is invertible for the frame to exist. Here we give another condition for the invertibility of the matrix HG^* based on the block circulant property of the matrix.

The block circulant matrix $P = HG^*$ is given by $\mathcal{C}(A_0, A_1, \ldots, A_{\overline{a}-1})$, where $A_i \in R^{a \times a}$ and

$$P = \mathcal{C}(A_0, A_1, \ldots, A_{\overline{a}-1}) = \begin{bmatrix} A_0 & A_1 & \cdots & A_{\overline{a}-1} \\ A_{\overline{a}-1} & A_0 & \cdots & A_{\overline{a}-2} \\ \vdots & \vdots & \cdots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix}, \tag{7}$$

where $\mathcal{C}$ stands for the rotation of the blocks $A_k, k \in 0, \ldots, \overline{a}-1$.

The discrete Fourier transform of P is given by $\mathcal{F}(P) = \mathcal{C}(\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A_{\overline{a}-1}})$ where

$$\tilde{A}_s = \sum_{r=0}^{\overline{a}-1} e^{-j2\pi rs/\overline{a}} A_r. \tag{8}$$

Lemma 1: [5] The matrix P is invertible if and only if $\tilde{A}_s, s \in 0, 1, \ldots, \overline{a}-1$ is invertible for every s, where $\tilde{A}_s$ is as defined in (8).

The inverse of the matrix $P = HG^* = \mathcal{C}(A_0, A_1, \ldots, A_{\overline{a}-1}$ is given by $B = P^{-1}$, also a block circulant matrix, where

$$B = \mathcal{C}(B_0, B_1, \ldots, B_{\overline{a}-1}) B_r = \frac{1}{\overline{a}} \sum_{s=0}^{\overline{a}-1} e^{j2\pi rs/\overline{a}} (\tilde{A}_s)^{-1}, \tag{9}$$

where $\tilde{A}_s$ and $0 \leq r \leq \overline{a}-1$ are the block Fourier components generated as shown in (8).

4. COMPUTATIONAL ISSUES

The minimum dual requires computing the matrix $S = GG^*$ whose kj-th element is given by [3] $S_{k,j} = \overline{b}\sum_{n=0}^{\overline{a}-1} g[k-na]g^*[j-na]$ if $\mid k-j \mid = q\overline{b}, q = 0, \ldots, b-1$ and 0 otherwise. The number of operations necessary to carry out the entire generation of the matrix is $(\overline{a}+1)ab$ multiplications and $ba\overline{a}$ additions. In many cases, $\overline{a}$ is very large since we do not shift the window too far. The shift needs to be small enough to be able to capture all the details of the signal accurately. Therefore, anything that removes this component from the multiplications is going to simplify the procedure greatly. As we will show, this can be achieved by a proper choice of H. The window $g[k]$ is usually chosen to be the Gaussian function $g[k] = e^{-k^2/\sigma_1^2}, \sigma_1 > 0$.

The elements of HG^* are theoretically generated as shown in property 1, but because we have the freedom to choose $h[k]$, we control H and have a far faster technique to generate HG^*. In this case, $h[k] = e^{-k^2/\sigma_2^2}$ can also be chosen as a Gaussian function. We select σ_2 such that the effective spread of the $h[k]$ is a. In this it is to be understood that outside this effective spread, the value of the window is not equal to zero. Rather it is made negligibly small (less than a thousandth or even less of the value of the window at the peak).

Theorem 4: [6] If $g[k] = e^{-k^2/\sigma_1^2}$ and $h[k] = e^{-k^2/\sigma_2^2}$ and $\overline{b}$ is divisible by a, then the matrix $P = HG^*$ is invertible.

Proof: From equation (8), we have

$$\tilde{A}_s = 1.A_0 + e^{-j2\pi\overline{b}s/a\overline{a}} A_{\overline{b}/a} + \cdots + e^{-j2\pi(b-1)\overline{b}s/a\overline{a}} A_{(b-1)\overline{b}/a} \tag{10}$$

Now if we can show that the diagonal elements of the Fourier block matrix is not zero out proof is complete. The diagonal elements of $\tilde{A}_s$ are given by

$$\begin{aligned} A_s(\tilde{k}, k) &= \overline{b} \sum_{q=0}^{b-1} \sum_{n=0}^{\overline{a}-1} e^{-\frac{(k-na)^2}{\sigma_2^2}} \cdot e^{-\frac{(k-na+q\overline{b})^2}{\sigma_1^2}} \cdot e^{-\frac{j2\pi q\overline{b}s}{a\overline{a}}} \\ &= \overline{b} \sum_{q=0}^{b-1} \sum_{n=0}^{\overline{a}-1} e^{-\frac{(k-na)^2}{\sigma_2^2}} \cdot e^{-\frac{(k-na+q\overline{b})^2}{\sigma_1^2}} \cdot e^{\frac{-j2\pi qs}{b}} \\ &= \overline{b} \sum_{q=0}^{b-1} e^{-\frac{j2\pi qs}{b}} \sum_{n=0}^{\overline{a}-1} e^{-\frac{(k-na)^2}{/}\sigma_2^2} \cdot e^{-(k-na+q\overline{b})^2/\sigma_1^2} \end{aligned} \tag{11}$$

It can easily be seen that (11) cannot be zero. Therefore, the statement is proved. □

The effective spread of the window $h[k]$ is from $-a/2$ to $a/2$. Therefore, the matrix H is made block diagonal for all practical purposes and looks as shown below.

$$H = \begin{bmatrix} B & \varepsilon & \cdots & \varepsilon \\ \varepsilon & B & \cdots & \varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ \varepsilon & \varepsilon & \cdots & B \end{bmatrix} \tag{12}$$

The ε is used to denote that the value of the elements in these blocks is extremely small. Each block B is of size $a \times \overline{b}$ and this is true for the ε blocks as well.

In the creation of the matrix $P = HG^*$ it can be seen that, because we limited the spread of $h[k]$ the $k, k+q\overline{b}$-th elements of P are generated by$P_{k,k+q\overline{b}} = \overline{b}e^{-(k-na)^2/\sigma_2^2}e^{-(k+q\overline{b}-na)/\sigma_1^2}, q \in 0, \ldots, b-1$and $k-na \in 0, 1, \ldots, a$ and 0 otherwise.

The number of operations to generate this matrix is $2ab$ and the number of additions necessary is 0. This is the greatest advantage in this method.

Here it is not necessary to actually compute the inverse of HG^*. From [3], we know the

$$P\gamma = h, \tag{13}$$

where h and γ are the first columns of the matrices H and Γ respectively. Since it is sufficient to compute the first column (the others are generated by shifting and modulating the first column), we can solve (13) we utilise the sparse structure of the matrix and the Conjugate Gradients method does a very good job of solving for γ (it has a computational complexity of $nlog(n)$).

5. SIMULATIONS AND DISCUSSION

The simulations show a Gaussian window and two of its possible dual windows. Figure 1 shows a typical Gaussian window that has $\sigma_1 = 64$ The lattice constants (a, b) were set to 4 and 8 respectively and the length of the signal was

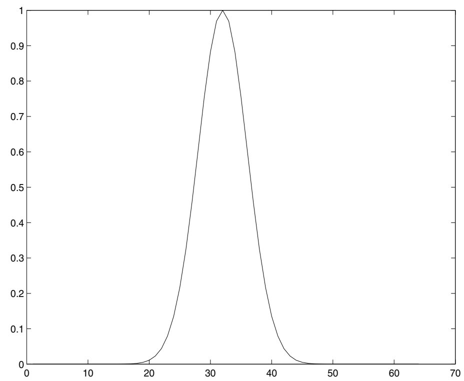

Fig. 1. The typical Gaussian window with $l = 64, \sigma_1 = 4$

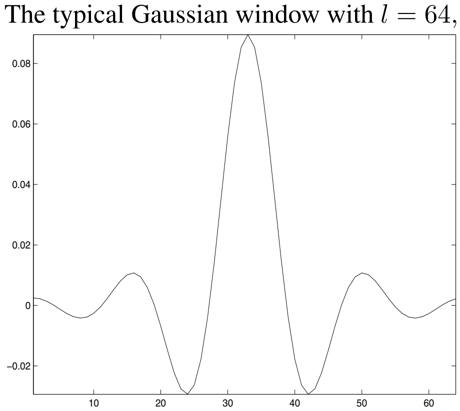

Fig. 2. The minimum norm dual window for lattice constants $(a = 4, b = 8, \sigma_1 = \sigma_2 = 4)$

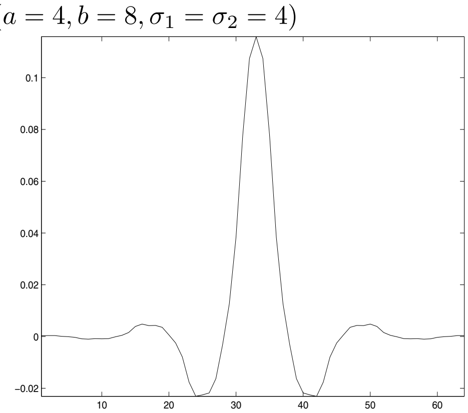

Fig. 3. The non minimum norm dual window for lattice constants ($a = 4, b = 8, \sigma_1 = 4, \sigma_2 = 2$)

64. Figure 2 shows the minimum norm dual and Figure 3 shows the non minimum dual with $\sigma_2 = 2$. As can be seen, even the non minimal dual window gives a good localisation. The number of operations necessary to compute the two duals can be computed as given in Section 4. Due to our choice of H in place of the traditional G, the number of flops (multiplication operations only) necessary to compute the dual is about 1/3 of what would be necessary to compute the minimum dual.

5.1. Value of the Determinant

There are many cases where the choice of parameters is such that computing the inverse of GG^* will be extremely expensive in terms of the precision of the arithmetic required. In many pattern recognition applications, when it is necessary to recognise low frequencies, the shift a has to be very small and the spread of the function σ_1 has to be very large. It has been observed that when the shift and the spread are too different, the value of the determinant of GG^* will be very close to 0 (although it is technically never equal to 0). For instance, if $a = 16$ and $\sigma_1 = 80$, the determinant will be very small (of the order of 10^{-59}). Our method provides a means to alleviate this trap. By selecting σ_2 (the spread of $h[\cdot]$) different from σ_1 (the spread of $g[\cdot]$), it is possible to obtain a larger determinant and consequently have a dual function for the window function with the set of parameters given. Therefore, it is of considerable interest to see how the determinant changes with respect to the variations in the spreads of $g[\cdot]$ and $h[\cdot]$. The full graphs are shown in [4].

6. REFERENCES

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