Hidden Convexity Based Near Maximum-Likelihood CDMA Detection

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Abstract — **We propose a computationally efficient approximation of the maximum-likelihood (ML) multiuser detector based on a nonconvex relaxation of the ML optimization problem. Using the hidden convexity methodology we obtain an explicit solution to the relaxed problem, which has the same form as the linear minimum mean-squared error (MMSE) receiver, where the constant diagonal loading in the MMSE receiver is replaced by a data-dependent constant that can be found efficiently by a simple bisection algorithm. Combining this relaxation with a local search algorithm results in a detector whose performance is close to that of the ML receiver, with a computational complexity on the same order as that of the linear multiuser receivers.**

I. Introduction

The optimal method for detecting symbols transmitted by different users in a CDMA system is the maximum-likelihood (ML) detector, which minimizes the joint error probability. Unfortunately, the ML detector requires solving a combinatorial optimization problem that has exponential complexity, rendering it impractical for systems of more than a few users. Efficient approximations of the ML solution are therefore of great practical importance.

The simplest approach to multiuser detection is to design a linear receiver, followed by single user detection. Examples are the minimum mean-squared error (MMSE) receiver, the decorrelator [1], the orthogonal multiuser receiver [2] and the covariance shaping receiver [3]. Linear receivers are computationally simple to implement, however, their performance can be far from the ML performance. Therefore, there has been considerable interest in nonlinear ML approximations. As we discuss in Section II, these methods can be broadly characterized into two groups: Relaxation methods, in which the discrete ML problem is approximated by a tractable continuous optimization problem, and heuristic methods in which the ML problem is solved as a discrete optimization problem using known heuristics. The drawback of the heuristic approach is that the quality of the solution depends largely on the initial point. A good strategy, therefore, which we follow in this paper, is to begin with a solution of a relaxation-based algorithm and then apply a heuristic method to improve the solution.

The most popular relaxation method is semidefinite programming (SDP) relaxation [4, 5], which is a convex optimization problem that can be solved in polynomial time [6]. However, as we show in Section IV, the main drawback of this approach is that in practice its computational load is very high and therefore the SDP relaxation is not practical for large systems. Nonetheless, it provides a good approximation of the ML solution. An alternative relaxation approach is to replace the nonconvex discrete constraint set by a convex continuous constraint set [7, 8]. As we discuss in Sections II and IV, these methods result in relaxations that are computationally less demanding than the SDP relaxation, however, their performance is often similar to that of the linear receivers. Furthermore, these algorithms are still computationally demanding, and are far more complex than the linear receivers.

In Section III, we propose a relaxation whose computational complexity is on the same order as that of the linear receivers, but combined with a local search algorithm it achieves almost ML performance. The receiver we propose has the same structure as the linear MMSE receiver, where the constant diagonal loading in the MMSE receiver is replaced by a nonlinear constant (independent of the noise level). This constant is determined by solving a relaxed optimization problem in which the discrete constraint set is replaced by a continuous *nonconvex* constraint set. Using the hidden convexity methodology [9], we show that our problem can be transformed into a convex problem, and then solved by using Lagrange duality theory. The resulting detector depends on a single parameter which can be found very efficiently by bisection. To further improve the performance of the proposed relaxation, we apply a simple local search algorithm, which is guaranteed to decrease the objective value and therefore lead to an improved solution. Since the complexity per iteration of both the bisection method and the local search algorithm is linear in the size of the problem, the overall computational complexity of the proposed receiver is on the same order of magnitude as that of the linear receivers.

II. Known Multiuser Detection Methods

Consider an n-user white Gaussian synchronous CDMA system where each user transmits information

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by modulating a signature sequence. The discrete-time model for the received signal $\mathbf{y}$ is

$$\mathbf{y} = \mathbf{SAx} + \mathbf{n} = \mathbf{Hx} + \mathbf{w}, \tag{1}$$

where $\mathbf{S}$ is the $m \times n$ matrix of signatures, $\mathbf{A} = \mathrm{diag}(A_1, \ldots, A_n)$ is the matrix of received amplitudes, $\mathbf{H} = \mathbf{SA}$, $\mathbf{x}$ is the data vector with $x_i \in \{1, -1\}$ being the ith user's transmitted symbol, and $\mathbf{w} \in \mathbb{R}^m$ is a noise vector whose elements are independent $\mathcal{N}(0, \sigma^2)$. We assume that all data vectors are equally likely with covariance $\mathbf{I}$.

Based on the observed signal $\mathbf{y} \in \mathbb{R}^m$, we design a receiver to detect the information transmitted by each user, where we assume that $\mathbf{H}$ is known. The ML sequence detector declares as the detected bit vector the vector $\hat{\mathbf{x}} \in \{1, -1\}^n$ which maximizes the likelihood function $p(\mathbf{y}|\mathbf{x})$ [10]. Since $\mathbf{w}$ is Gaussian and white,

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \{1,-1\}^n} p(\mathbf{y}|\mathbf{x}) = \arg \min_{\mathbf{x} \in \{1,-1\}^n} \|\mathbf{y} - \mathbf{Hx}\|^2. \tag{2}$$

Problem (2) is a quadratic binary problem, which is NP-complete in general [10]. Two main approaches to developing polynomial-time algorithms which approximate the ML solution are the continuous approach (relaxation methods) and the discrete approach.

The Continuous Approach: In this approach we consider a relaxation of the discrete problem (2). The relaxation is a continuous optimization problem, that is typically chosen to be a tractable optimization problem (*e.g.,* a convex optimization problem) whose optimal value is a lower bound of the discrete problem. One of the most popular relaxation methods is the SDP relaxation [4, 5, 6], in which the binary problem (2) is replaced by the SDP

$$\begin{cases} \min_{\mathbf{X}} & \mathrm{Tr}(\mathbf{ZX}) \\ \text{s.t.} & \mathbf{X} \succeq \mathbf{0}, \mathbf{X}_{i,i} = 1, 1 \le i \le n+1, \end{cases} \tag{3}$$

where $\mathbf{X} \succeq \mathbf{0}$ means that $\mathbf{X}$ is positive semidefinite. The problem (3) is a convex problem, which can be solved with polynomial complexity [5, 11]. However, as we show in Section IV, in practice the computational complexity of the SDP relaxation can be quite high, rendering the SDP relaxation impractical for large systems.

Two alternative relaxation methods which are computationally less demanding than the SDP relaxation where proposed in [7, 8], based on replacing the nonconvex set $\{\mathbf{x} : \mathbf{x} \in \{-1, 1\}^n\}$ by a convex set. Specifically, the Norm Relaxation (NR),

$$(\mathrm{NR}): \min_{\|\mathbf{x}\|^2 \le n} \|\mathbf{y} - \mathbf{Hx}\|^2, \tag{4}$$

and the Bound Relaxation (BR),

$$(\mathrm{BR}): \min_{-1 \le x_i \le 1} \|\mathbf{y} - \mathbf{Hx}\|^2. \tag{5}$$

Both NR and BR are convex problems, which can be solved, *e.g.,* using the gradient projection algorithm [7]. The discrete solution is obtained by taking the signs of the continuous solution.

In Section IV we show that the detectors based on the SDP, NR and BR relaxations are computationally much more demanding than linear detectors and thus are not attractive for large systems.

The Discrete Approach: In this approach, (2) is solved as a discrete optimization problem using known heuristics such as local search, simulated annealing, and tabu search (see [11] and references therein). The quality of the solution depends largely on the quality of the initial point and thus a good strategy might be to combine the two approaches, *i.e.,* begin with a relaxation-based algorithm and then apply a heuristic method that will improve the relaxation-based solution. We would further like both the relaxation-based and heuristic algorithms to require the same magnitude of operations as the linear detectors, but to lead to a smaller bit-error rate (BER).

In the next section we develop a detector that satisfies these properties. Specifically, we propose a relaxation of the ML problem that is theoretically tighter than the NR relaxation of (4), and its solution involves only one spectral decomposition and a bisection procedure aimed to find a root of an equation with one variable. The signs vector of the solution is the initial vector of a local search algorithm that produces the resulting detector.

III. Hidden Convexity-Based Relaxation

Following the general approach of [7], we develop a relaxation based on replacing the set $\mathbf{x} \in \{-1, 1\}^n$ by a relaxed constraint set. However, contrary to the relaxations of [7], our relaxed constraint set is not convex. Nonetheless, we develop an explicit expression for the relaxed optimal solution that depends on a *single* parameter, which can be found efficiently with bisection.

The constraint set we consider is $\{\mathbf{x} : \|\mathbf{x}\|^2 = n\}$. Thus, we suggest the following relaxation of (2):

$$(\mathrm{NER}): \min_{\|\mathbf{x}\|^2 = n} \|\mathbf{y} - \mathbf{Hx}\|^2 = \min_{\|\mathbf{x}\|^2 = n} \left\{ \mathbf{x}^T \mathbf{Qx} - 2\mathbf{f}^T \mathbf{x} + c \right\}, \tag{6}$$

which we call the NER (Norm Equality Relaxation). Here $\mathbf{Q} = \mathbf{H}^T \mathbf{H}, \mathbf{f} = \mathbf{H}^T \mathbf{y}$ and $c = \mathbf{y}^T \mathbf{y}$.

Clearly, the relaxation (6) is tighter than the relaxation of (4). In contrast with (4), the problem (6) *is not* convex since the constraint $\|\mathbf{x}\|^2 = n$ does not define a convex set. However, as we show, by using the *hidden convexity* methodology [9], we can transform this problem into a *convex* optimization problem.

To develop a solution to (NER), we note that since $\mathbf{Q}$ is positive semidefinite it is diagonalizable by an orthogonal matrix $\mathbf{U}$, so that $\mathbf{U}^T \mathbf{QU} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i \ge 0, 1 \le i \le n$. Making the change of variables $\mathbf{x} = \mathbf{Uz}$, (NER) is equivalent to

$$\min_{\|\mathbf{z}\|^2 = n} \left\{ \sum_{j=1}^{n} (\lambda_j z_j^2 - 2b_j z_j) + c \right\}, \tag{7}$$

where $\mathbf{b} = \mathbf{U}^T\mathbf{f}$. The following lemma will enable us to transform the problem (7) to a convex optimization problem.

Lemma III.1. *Let* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *be an optimal solution of* $\min_{\|\mathbf{z}\|^2=n} q(\mathbf{z})$ *where* $q(\mathbf{z}) = \sum_{j=1}^n (\lambda_j z_j^2 - 2b_j z_j)$*. Then* $w_j b_j \geq 0$ *for* $1 \leq j \leq n$*.*

Proof: Let $\mathbf{w}_k = (w_1, w_2, \ldots, w_{k-1}, -w_k, w_{k+1}, \ldots, w_n)$. Then $\|\mathbf{w}_k\|^2 = \|\mathbf{w}\|^2 = n$, so that $\mathbf{w}_k$ is feasible. Since $\mathbf{w}$ is optimal, $q(\mathbf{w}) \leq q(\mathbf{w}_k)$ for $1 \leq k \leq n$, which implies that

$$-\sum_{j=1}^{n} 2b_j w_j \leq -\sum_{j=1, j\neq k}^{n} 2b_j w_j + 2b_k w_k. \tag{8}$$

Therefore, $b_k w_k \geq 0$, and the result follows. □

In view of Lemma III.1, we can make the change of variables $z_j = \text{sign}(b_j)\sqrt{v_j}$, where $\text{sign}(b_j) = 1$ if $b_j \geq 0$ and -1 otherwise, and $v_j \geq 0$. Using the variables z_j, (NER) is equivalent to the *convex* optimization problem

$$\min_{v_j \geq 0} \left\{ \sum_{j=1}^{n} \left(\lambda_j v_j - 2|b_j|\sqrt{v_j}\right) + c : \sum_{j=1}^{n} v_j = n \right\}. \tag{9}$$

Since the problem (9) is convex with linear constraints, the optimal value is equal to the value of its dual problem. To develop the dual problem, we use the Lagrangian of (9),

$$L(\mathbf{v}, \eta) = \sum_{j=1}^{n} \left((\lambda_j + \eta)v_j - 2|b_j|\sqrt{v_j}\right) - \eta n + c. \tag{10}$$

Differentiating (10) with respect to v_j and equating to zero,

$$v_j = \frac{b_j^2}{(\lambda_j + \eta)^2}, \quad 1 \leq j \leq n, \tag{11}$$

subject to $\eta \geq -\lambda_j$ for $1 \leq j \leq n$. Thus, the dual function is

$$h(\eta) = \min_{v_j \geq 0} L(\mathbf{v}, \eta) = -\sum_{j=1}^{n} \frac{b_j^2}{\lambda_j + \eta} - \eta n + c, \tag{12}$$

and the dual problem of (9) is

$$(\text{D}): \quad \max_{\eta} \left\{ h(\eta) : \eta \geq \alpha \right\}, \tag{13}$$

where $\alpha \stackrel{\triangle}{=} \max_{1\leq j\leq n}\{-\lambda_j\}$. Since $h(\eta)$ is continuous in $\eta > \alpha$, the maximum of $h(\eta)$ is obtained either at a local maximum or at one of the end points $\eta = \alpha$ or $\eta \to \infty$. Noting that $h(\eta) \to -\infty$ at both of the end points, the solution of (D) is at a local maximum. Differentiating $h(\eta)$ with respect to η and equating to 0, $\sum_{j=1}^n b_j^2/((\lambda_j + \eta)^2) = n$. Denoting

$$\mathcal{G}(\eta) = \sum_{j=1}^{n} \frac{b_j^2}{(\lambda_j + \eta)^2} - n, \tag{14}$$

it follows that the optimal η is the root of $\mathcal{G}(\eta)$. Since $\mathcal{G}(\eta)$ is continuous and monotonically decreasing for $\eta > \alpha$, there is only one root in the domain (α, ∞), which can be easily found by bisection.

We conclude that the solution to (6) is $\mathbf{x} = \mathbf{U}\mathbf{z}$, where $z_j = b_j/(\lambda_j + \eta)$, $\mathbf{b} = \mathbf{U}^T\mathbf{H}^T\mathbf{y}$, and $\eta \geq \alpha$ is the unique root of $\mathcal{G}(\eta)$. Since $\mathbf{H}^T\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^T$, where $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\mathbf{x} = \mathbf{U}\left(\mathbf{D} + \eta\mathbf{I}\right)^{-1}\mathbf{U}^T\mathbf{y} = \left(\mathbf{H}^T\mathbf{H} + \eta\mathbf{I}\right)^{-1}\mathbf{H}^T\mathbf{y}. \tag{15}$$

The solution (15) has the same form as the linear MMSE receiver, where the constant load σ^2 in the MMSE receiver is replaced by the data-dependent constant η. Note, that the receiver of (15) does not require knowledge of σ^2. As we detail further in Section IV, since the complexity per iteration of the bisection method is linear in the size of the problem, the overall computational complexity of the proposed receiver is on the same order of magnitude as that of the MMSE receiver.

We summarize our results in the following theorem.

Theorem III.1. *Let* $\mathbf{H} \in \mathbb{R}^{m\times n}, \mathbf{y} \in \mathbb{R}^m$ *and let* $\mathbf{H}^T\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^T$ *where* $\mathbf{D} = \textit{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$*. Then the solution to*

$$\min_{\|\mathbf{x}\|^2=n} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2$$

is

$$\mathbf{x} = \mathbf{U}\left(\mathbf{D} + \eta\mathbf{I}\right)^{-1}\mathbf{U}^T\mathbf{y} = \left(\mathbf{H}^T\mathbf{H} + \eta\mathbf{I}\right)^{-1}\mathbf{H}^T\mathbf{y},$$

where $\eta \geq \alpha$ *is the unique solution of* $\sum_{j=1}^n b_j^2/(\lambda_j+\eta)^2 = n$ *with* $\mathbf{b} = \mathbf{U}^T\mathbf{H}^T\mathbf{y}$ *and* $\alpha = \max_{1\leq j\leq n}\{-\lambda_j\}$*.*

To obtain a solution to the original ML problem of (2) we need to relax the continuous solution of (NER) to a discrete solution. The simplest approach is to choose the detected bits as the signs of the solution $\mathbf{x}$; we refer to this approach as Algorithm NER1.

We can further improve the relaxation solution by using a heuristic algorithm. In particular, the solution derived from (NER) does not satisfy the necessary optimality conditions [12] $\mathbf{XQXe} - \mathbf{Xf} \leq \text{Diag}(\mathbf{Q})$, where $\mathbf{X} = \text{diag}(\mathbf{x})$ and $\mathbf{e} = (1, 1, \ldots, 1)^T$. We can improve the derived solution and obtain a solution that does satisfy the necessary conditions by applying the *local search algorithm* [13] on the solution obtained by NER1, leading to an improved algorithm which we refer to as algorithm NER2. We note that each iteration of the local search algorithm requires $O(n)$ operations.

IV. Simulation Results

We now demonstrate that the NER2 algorithm achieves almost the same performance as the SDP relaxation and the ML solution, while maintaining a computational complexity similar to that of the MMSE receiver.

In the examples below we assume equal power users with $\mathbf{A} = \mathbf{I}$. For each SNR, the BER is evaluated by counting the number of erroneous decisions in 10000 realizations.

IV.A Comparison of NER with ML

We first compare the performance of the NER1 and NER2 methods with the ML detector. Since the computational complexity of the ML detector is exponential in the number of users, we consider a 6-user system with signatures that are generated using a 21-chip Gold code sequence of length 31.

In Fig. 1 we plot the BER averaged over all users as a function of the SNR, which is defined as $-10 \log \sigma^2$, using the NER1, NER2 and ML detectors. As can be seen in the figure, the NER2 detector has almost the same performance as the ML detector. It is also evident that the local search algorithm can improve the performance, particularly at high SNR values.

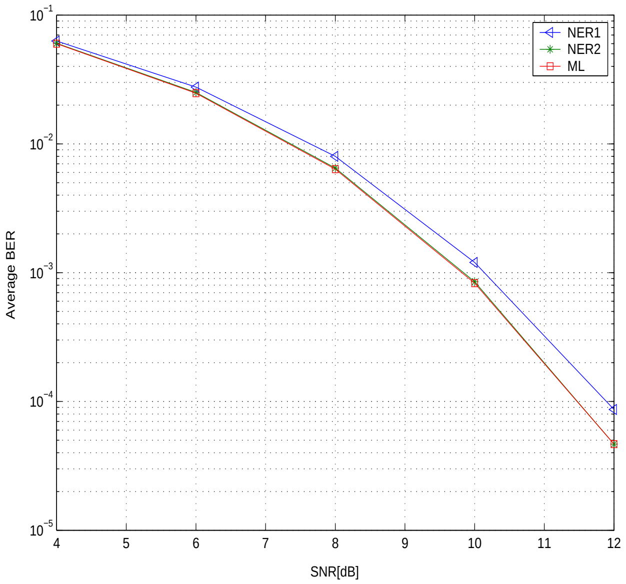

Figure 1: BER as a function of SNR for a 6-user CDMA system.

IV.B Comparison with Other Relaxations

We now compare the NER2 relaxation with other nonlinear relaxations, and the linear MMSE and decorrelator detectors. In this example, we consider a 15-user system with signatures that are generated using a 30-chip Gold code sequence of length 31.

In Fig. 2 we plot the BER averaged over all users as a function of the SNR using the NER2, SDP, NR, BR, decorrelator and MMSE detectors. It is evident from the figure that the SDP and the NER2 detectors have almost the same performance. Furthermore, the other relaxations have essentially the same performance as the linear receivers. In the next example we show that although the performance of the SDP relaxation is almost identical to that of the NER2 detector, the computational complexity of SDP is much heavier. In fact, the complexity of NER2 is roughly the same as the linear detectors, but achieves almost ML performance.

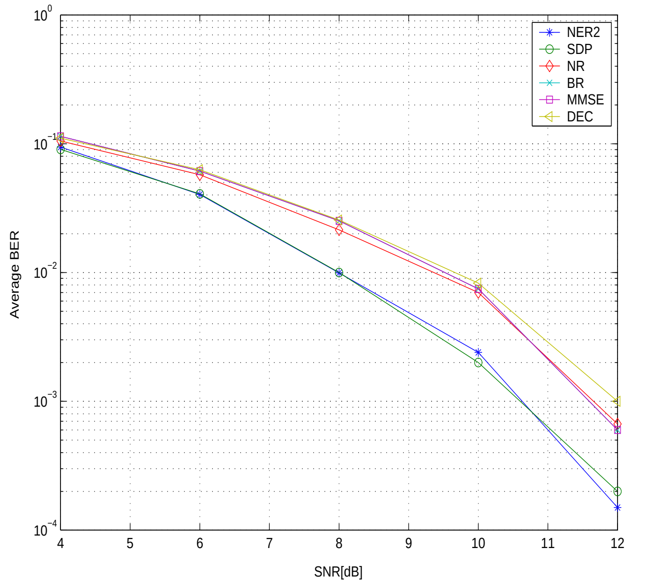

Figure 2: BER as a function of SNR for a 15-user CDMA system.

IV.C Large-System Performance

In this last example we consider a large system with $n = 512$ users. We used 1023-length Gold code sequences and $m = 768$. We compared the NER2 only with the linear receivers, decorrelator and MMSE, since each run of the SDP detector requires approximately 800 seconds on a Pentium 4, 1.8 Ghz computer and thus we cannot approximate the BER of the SDP detector in reasonable time. All the other detectors are also computationally too demanding to be applied thousands of times (see Table 1).

In Fig. 3 we plot the BER averaged over all users as a function of SNR using the NER2, decorrelator and MMSE detectors.

Table 1 illustrates the efficiency of NER2. In the table we present CPU times averaged over 100 realizations for different choices of the number of users n and $m = 1.5n$, with an SNR of 4dB. The CPU time of NER2 is divided

n	*DEC/* MMSE	*NER2*		*NR/BR*	*SDP*
		PRE	POST		
16	0.004	0.0063	0.0066	0.0078	0.24
32	0.01	0.0103	0.0141	0.0225	0.465
64	0.015	0.024	0.0292	0.1232	1.44
128	0.09	0.064	0.0664	1.39	6.038
256	0.28	0.141	0.210	5.19	47.78
512	1.292	1.162	1.23	44.9	796.8
1024	8.54	8.55	5.16	-	-

Table 1: CPU time in seconds on a Pentium 4, 1.8Ghz.

into two parts. The first part is the eigenvalue decomposition of the matrix $\mathbf{H}^T\mathbf{H}$, which we refer to as pre-

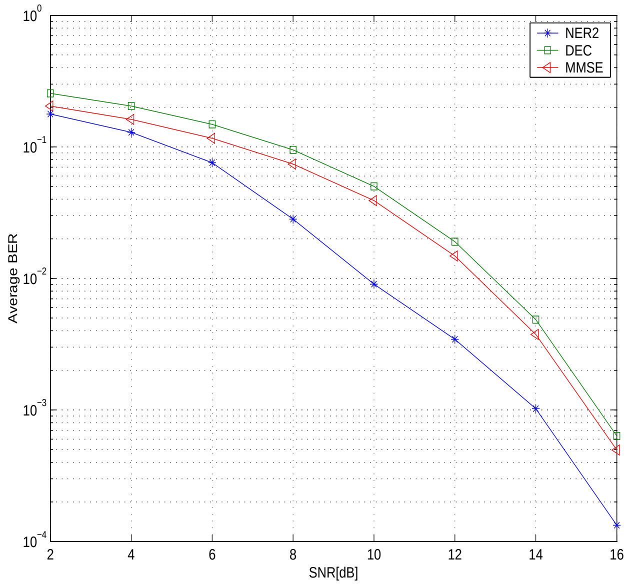

Figure 3: BER as a function of SNR for a 512-user CDMA system.

processing (PRE). The second time (POST) is the CPU time of the rest of the algorithm (solving the NER relaxation and the local search). In applications where the matrix **H** is constant over a large time period, the actual time required per transmutation is the post processing time. Even if we consider the CPU time of all of NER2 (PRE and POST) it is very clear that it is considerably more efficient than all the other relaxation-based detectors. Furthermore, its complexity is on the same order as the linear decorrelator and MMSE detectors.

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